0.1 Homotopic maps induce the same map in cohomology

Let f, $g : M \to N$ be smooth maps between smooth manifolds. A (smooth) *homotopy* between f and g is a smooth map $F : M \times [0,1] \to N$ such that

$$\begin{cases} F(p,0) & = & f(p) \\ F(p,1) & = & g(p) \end{cases}$$

for $p \in M$. If there exists a homotopy between f and g, we say that they f and g are *homotopic*.

0.1.1 Proposition *Let f, g be homotopic maps. Then the induced maps in cohomology*

$$f^*, g^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$

are equal.

The proof of this propositon is given below. First, we need to make some remarks. For $t \in [0,1]$, consider the inclusions i_t given by

$$i_t(p) = (p,t)$$

for $p \in M$, and consider the natural projection $\pi : M \times [0,1] \to M$ given by $\pi(p,t) = p$. Then, obviously,

$$\pi \circ i_t = \mathrm{id}_M$$

implying that

$$i_t^* \pi^* = \mathrm{id} \quad \text{in } \Omega^k(M) \text{ and } H^k_{\mathrm{dR}}(M).$$

We consider the projection $t : M \times [0,1] \to [0,1]$. Then there exists a "vertical" vector field $\frac{\partial}{\partial t}$ and a 1-form dt. Note that $\ker d\pi$ is spanned by $\frac{\partial}{\partial t}$.

0.1.2 Lemma *Let $\omega \in \Omega^k(M \times [0,1])$. Then we can write*

$$\omega = \zeta + dt \wedge \eta \tag{0.1.3}$$

where $\zeta \in \Omega^k(M \times [0,1])$ has the property that it vanishes if some of its arguments belongs to $\ker d\pi$, and $\eta \in \Omega^{k-1}(M \times [0,1])$ has the same property.

Proof. Set $\eta = i_{\frac{\partial}{\partial t}} \omega$ and $\zeta = \omega - dt \wedge \eta$. Since

$$i_{\frac{\partial}{\partial t}} \eta = i_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial t}} \omega = 0,$$

it is clear that η has the claimed property. Similarly,

$$\begin{aligned} i_{\frac{\partial}{\partial t}} \zeta &= i_{\frac{\partial}{\partial t}} \omega - i_{\frac{\partial}{\partial t}} (dt \wedge \eta) \\ &= \eta - i_{\frac{\partial}{\partial t}} dt \wedge \eta + dt \wedge i_{\frac{\partial}{\partial t}} \eta \\ &= \eta - \eta + 0 \\ &= 0, \end{aligned}$$

as desired. ☐

We define the homotopy operator

$$H_k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$$

by the formula

$$(H_k\omega)_p(v_1,\dots,v_{k-1}) = \int_0^1 \eta_{(p,t)}(di_t(v_1),\dots,di_i(v_{k-1}))\,dt,$$

where ω is decomposed as in (0.1.3) and $p \in M$, $v_1,\dots,v_{k-1} \in T_pM$. Note that H_k is "integration along the fiber of π". For simplicity, we henceforth drop the subscript and just write H for the homotopy operator.

Proof of Propostion 0.1.1. Let $\omega \in H^k_{\mathrm{dR}}(M \times [0,1])$. We first claim that

$$dH\omega + Hd\omega = i_1^*\omega - i_0^*\omega. \tag{0.1.4}$$

The proof is by direct computation: since this is a pointwise identity, we can work in a coordinate system. Let $(U, x_1,\dots,x_n)$ be a coordinate system in M. Then $(U\times[0,1], x_1\circ\pi,\dots,x_n\circ\pi, t)$ is a coordinate system in $M\times[0,1]$ and we can write

$$\omega|_{U\times[0,1]} = \sum_I a_I dx_I + dt \wedge \sum_J b_J dx_J$$

where a_i, b_J are smooth functions on $U\times[0,1]$ and I, J are increasing multi-indices. In $U\times[0,1]$, we have:

$$H\omega = \sum_J \left(\int_0^1 b_J dt\right) dx_J,$$

$$dH\omega = \sum_{J,i} \left(\int_0^1 \frac{\partial b_J}{\partial x_i} dt\right) dx_i \wedge dx_J,$$

$$d\omega = \sum_{I,i} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I + \sum_I \frac{\partial a_I}{\partial t} dt \wedge dx_I - dt \wedge \sum_{J,i} \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_J,$$

$$Hd\omega = \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t} dt\right) dx_I - \sum_{J,i} \left(\int_0^1 \frac{\partial b_J}{\partial x_i} dt\right) dx_i \wedge dx_J.$$

It follows that

$$\begin{aligned} dH\omega + Hd\omega|_p &= \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t}(p,t)\,dt\right) dx_I \\ &= \sum_I (a_I(p,1) - a_I(p,0)) dx_I \\ &= i_1^*\omega - i_0^*\omega|_p, \end{aligned}$$

as claimed.

Suppose now that $F : M\times[0,1] \to N$ is a homotopy between f and g. Let α be a closed k-form in N representing the cohomology class $[\alpha] \in H^K_{\mathrm{dR}}(N)$. Applying identity (0.1.4) to $\omega = F^*\alpha$ yields

$$dHF^*\alpha + HF^*d\alpha = i_1^*F^*\alpha - i_0^*F^*\alpha.$$

Since $d\alpha = 0$ and $F\circ i_0 = f$, $F\circ i_1 = g$, we get

$$d\,(HF^*\alpha) = g^*\alpha - f^*\alpha.$$

Hence $g^*\alpha$ and $f^*\alpha$ are cohomologous. □

0.2 Hairy ball theorem

Consider Euclidean space $\mathbf{R}^{n+1}$ with coordinates $(x_0, x_1, \ldots, x_n)$ and the unit sphere $\iota : S^n \to \mathbf{R}^{n+1}$. Consider the n-form in $\mathbf{R}^{n+1}$

$$\omega = \sum_{i=0}^{n} (-1)^i \, x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n.$$

Note that ω vanishes only at the origin. In particular,

$$\alpha = \iota^*\omega = \sum_{i=0}^{n} (-1)^i \, (x_i \circ \iota) \, d(x_0 \circ \iota) \wedge \cdots d\widehat{(x_i \circ \iota)} \wedge \cdots d(x_n \circ \iota) \tag{0.2.1}$$

is a nowhere vanishing n-form on S^n, hence it defines an orientation there. Of course,

$$d\omega = (n+1) dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_{S^n} \alpha = \int_{S^n} \omega = \int_{\bar{B}^n} d\omega = (n+1)\mathrm{vol}(\bar{B}^n) > 0$$

(where the orientation of S^n is induced from $\bar{B}^n$) so α is not exact by Stokes theorem. Thus $[\alpha] \neq 0$ in $H^n_{\mathrm{dR}}(S^n)$.

In the sequel, we consider $n = 2m$.

0.2.2 Theorem *Let X be a smooth vector field on S^{2m}. Then there exists $p \in S^{2m}$ such that $X_p = 0$. In other words, every smooth vector field on an even-dimensional sphere has a zero.*

Proof. Suppose, on the contrary, that X never vanishes. By rescaling, we may assume that X is a unit vector field with respect to the metric induced from Euclidean space. Set

$$F_t : S^{2m} \to S^{2m}, \quad F_t(p) = \cos t \, x + \sin t \, X(p).$$

It is clear that F_t defines a homotopy between the identity map and the antipodal map of S^{2m}:

$$F_0 = \mathrm{id}_{S^{2m}} \quad \text{and} \quad F_\pi = -\mathrm{id}_{S^{2m}}.$$

Note that

$$F_\pi^*(x_i \circ \iota) = -x_i \circ \iota.$$

It follows that

$$F_\pi^*\alpha = (-1)^{2m+1}\alpha = -\alpha.$$

On the other hand,

$$F_0^*\alpha = \alpha,$$

and by Proposition 0.1.1, $[F_0^*\alpha] = [F_\pi^*\alpha]$, which contradicts the fact that $[\alpha] \neq 0$. □

0.2.3 Remark The theorem can be extended to the case of continuous vector fields by using an approximation result.

0.3 The smooth Brouwer fixed point theorem

Let $\bar{B}^n$ be the closed ball in $\mathbf{R}^n$, and denote its boundary by $\partial\bar{B}^n$; of course, $\partial\bar{B}^n$ is diffeomorphic to S^{n-1}. We first prove

0.3.1 Lemma *There exists no smooth retraction $r : \bar{B}^n \to \partial\bar{B}^n$ (that is, there exists no smooth map $r : \bar{B}^n \to \partial\bar{B}^n$ whose restriction to $\partial\bar{B}^n$ is the identity).*

Proof. The case $n = 1$ is easy as the closed interval $\bar{B}^1$ is connected and its boundary is disconnected. Assume $n \geq 2$ and suppose, to the contrary, that such a retraction r exists. Recall that n-form α defined in (0.2.1). Since r is the identity along $\partial\bar{B}^n$,

$$\int_{\partial\bar{B}^n} r^*\alpha = \int_{\partial\bar{B}^n} \alpha \neq 0.$$

On the other hand, by Stokes theorem,

$$\int_{\partial\bar{B}^n} r^*\alpha = \int_{\bar{B}^n} dr^*\alpha = \int_{\bar{B}^n} r^*d\alpha = 0,$$

since $d\alpha = 0$, which is a contradiction. □

0.3.2 Theorem *Let $f : \bar{B}^n \to \bar{B}^n$ be a smooth map. Then there exists $p \in \bar{B}^n$ such that $f(p) = p$. In other words, every smooth self-map of the closed n-ball admits a fixed point.*

Proof. Suppose, on the contrary, that $f(x) \neq x$ for all $x \in \bar{B}^n$. The half-line originating at x and going through $f(x)$ meets $\partial\bar{B}^n$ at a unique point; call it $r(x)$. It is easy to see that this defines a smooth retraction $r : \bar{B}^n \to \partial\bar{B}^n$ which is prohibited by Lemma 0.3.1. □

0.3.3 Remark The theorem is not true in the case of the open ball B^n, as is easily seen.