AULA DE EXERCÍCIOS

05/09/23

Ex. 4, Cap. 1

$Gr_k(\mathbb{R}^n) = \{\text{subespaços de dim } k \text{ de } \mathbb{R}^n\}$

(a) $S : \mathcal{U} \longrightarrow Gr_k(\mathbb{R}^n)$ é sobrejetora

$\qquad x \longmapsto$ span das linhas

$\mathcal{U} \underset{ab}{\subset} M(k \times n, \mathbb{R})$: matrizes de posto k

(b) $U_I = \{\sigma \in Gr_k(\mathbb{R}^n) : \det \pi_I(x) \neq 0$ para alguma $x \in S^{-1}(\sigma)\}$

$I \subset \{1, \dots, n\}$
$\# I = k$

$$\begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ x = k & \left(\begin{array}{c|c} \boxed{} & \end{array} \right) \end{array}$$

$$\Rightarrow Gr_k(\mathbb{R}^n) = \bigcup_I U_I$$

(a) e (b) Dado $\sigma \in Gr_k(\mathbb{R}^n)$, seja $v_1, \dots, v_k$ uma base de σ. Então $v_i = \sum_j a_{ij} e_j$ $\quad e_1, \dots, e_n$ base can. de $\mathbb{R}^n$

$$x = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \Rightarrow S(x) = \sigma.$$

posto$(x) = k \Rightarrow \exists\, I \subset \{1, \dots, n\}$, $\# I = k$ t.q.

$\det \pi_I(x) \neq 0 \quad \therefore \quad \sigma \in U_I$.

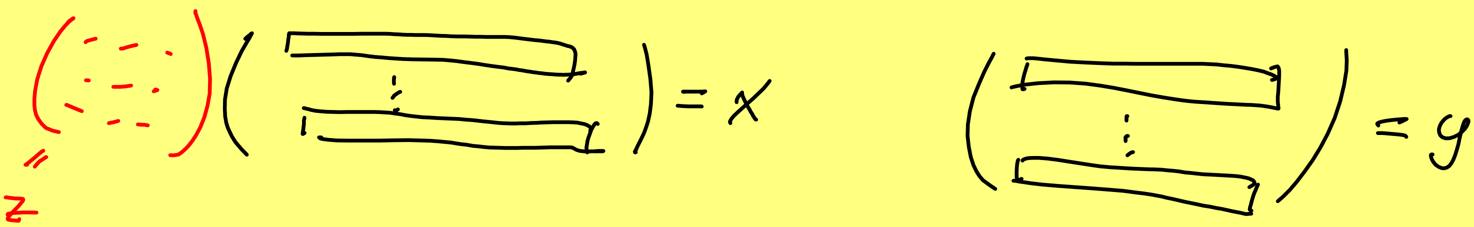
(c) $\varphi_I : U_I \longrightarrow M(k \times (n-k), \mathbb{R}) \cong \mathbb{R}^{k(n-k)}$

$$\sigma \longmapsto \pi_I(x)^{-1}\, \pi_{I'}(x)$$

p/ algum $x \in \phi^{-1}(\sigma)$ $\qquad I \dot\cup I' = \{1, \dots, n\}$.

Mostrar que φ_I está bem def e é bijetora.

De fato: se $x, y \in \phi^{-1}(\sigma)$ então



$\exists\ z \in M(k \times k, \mathbb{R})$ não-singular t.g. $zx = y$ $\qquad z\,\pi_I(x) = \pi_I(y)$

$$\pi_I(y)^{-1}\, \pi_{I'}(y) = \pi_I(zx)^{-1}\, \pi_{I'}(zx) = (\star)$$

$$z\left(\pi_I(x) \,\middle|\, \pi_{I'}(x)\right) = \left(z\pi_I(x) \,\middle|\, z\pi_{I'}(x)\right)$$

$$\Rightarrow \pi_I(zx) = z\pi_I(x)$$

$$(\star) = (z\pi_I(x))^{-1}\, (z\pi_{I'}(x)) = \pi_I(x)^{-1}\, \pi_{I'}(x) \ \checkmark$$

(d) $I, J \subset \{1, \dots, n\}$ Mostrar que

$\varphi_I(U_I \cap U_J)$ é aberto e $\varphi_J \varphi_I^{-1} : \varphi_I(U_I \cap U_J) \longrightarrow \varphi_J(U_I \cap U_J)$

$$U_I \cap U_J = \{ \sigma \in Gr_k(\mathbb{R}^n) \mid \det \pi_I(x),\ \underline{\det \pi_J(x) \neq 0} \text{ p/algum } \alpha \in \delta^{-1}(\sigma) \}.$$

$$\sigma \in U_I \cap U_J \qquad \varphi_I(\sigma) = \underline{\pi_I(x)^{-1}\pi_I'(x)} \in M(k \times (n-k), \mathbb{R})$$

$I = \{1, \dots, k\}$

$$\alpha = k\Big(\underbrace{\boxed{\pi_I(x)}}_{k} \; \vdots \; \underbrace{\pi_I'(x)}_{n-k} \Big)$$

$$\pi_J\Big(\underbrace{\pi_I(x)^{-1}}_{k \times k} x\Big) = \pi_J \begin{pmatrix} 1 & & 0 & \vdots & \\ & \ddots & & \vdots & \varphi_I(\sigma) \\ 0 & & 1 & \vdots & \end{pmatrix}$$

$$\underbrace{\pi_I(x)^{-1}}_{\text{não sing}} \pi_J(x) = \pi_J \begin{pmatrix} 1 & & & \vdots & \\ & \ddots & & \vdots & \varphi_I(\sigma) \\ & & 1 & \vdots & \end{pmatrix}$$

J

$\pi_J(x)$ é não-sing $\Leftrightarrow$ $\pi_J \begin{pmatrix} 1 & & & \vdots & \\ & \ddots & & \vdots & \underline{\varphi_I(\sigma)} \\ & & 1 & \vdots & \end{pmatrix}$ é não-sing.
$x \in U_I$

$\varphi_I : U_I \longrightarrow M(k \times (n-k), \mathbb{R})$ é bijetora

$\sigma \longmapsto \pi_I(x)^{-1}\pi_I'(x)$ p/algum $x \in \delta^{-1}(\sigma)$

$\alpha = \big(\pi_I(x) \mid \pi_I'(x) \big)$ $\qquad I = \{1, \dots, k\}.$

$$\pi_I(x)^{-1} x = \begin{pmatrix} 1 & & 0 & \vdots & \\ & \ddots & & \vdots & \underline{\sigma_I(x)} \\ 0 & & 1 & \vdots & \end{pmatrix}$$

$$\varphi_I^{-1}(a) = \underbrace{\mathcal{S}\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & a \\ 0 & & 1 & \end{array}\right)}_{\in\, \mathcal{U}}$$

$a \in M(k \times (n-k), \mathbb{R})$

$$\varphi_I(U_I \cap U_J) = \left\{ a \in M(k\times(n-k), \mathbb{R}) \;\middle|\; \pi_J\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & a \end{array}\right) \text{ é não-singular} \right\}$$

$$\Longleftrightarrow \det \pi_J \overline{\left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)} \neq 0 \quad \text{é } \underline{\text{aberto}}.$$

$a \mapsto$ cont. $\quad \det$ cont. $\quad \pi_J$ cont.

$$a \mapsto \left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right) \quad \text{cont.} \quad \mapsto \pi_J\left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)$$

$$\downarrow \det$$

$I = \{1, \dots, k\}$ $\qquad \det \pi_J \left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)$

$$\varphi_J \varphi_I^{-1}(a) = \varphi_J\left(\mathcal{S}\left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)\right)$$

$$= \underline{\underline{\pi_J}}\left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)^{-1} \cdot \underline{\underline{\pi_{J'}}}\left(\begin{array}{c|c} 1 \ddots 1 & a \end{array}\right)$$

(e)

$$\begin{array}{ccc} \mathcal{S}^{-1}(U_I) \subset \mathcal{U} & \xrightarrow{\ \mathcal{S}\ } & U_I \subset Gr_k(\mathbb{R}^n) \\ \cap & & \downarrow \varphi_I \\ M(k\times n, \mathbb{R}) & \overset{(\varphi_I \circ \mathcal{S})}{\dashrightarrow} & M(k\times(n-k), \mathbb{R}) \end{array}$$

$$\varphi_I \circ \mathcal{S}(x) = \pi_I(x)^{-1}\, \pi_{I'}(x)$$

$$x = \left(\pi_I(x) \,\middle|\, \pi_{I'}(x) \right) \longmapsto \pi_I(x)^{-1} \pi_{I'}(x)$$

$$f: (A \;\; B) \mapsto A^{-1} \cdot B \qquad f: M(k \times n, \mathbb{R}) \to \underbrace{M(k \times (n-k), \mathbb{R})}_{\mathbb{R}'}$$

$\det A \neq 0$

$$df_{(A,B)}(\underbrace{C}_{k\times k}, \underbrace{D}_{n-k\times k}) = -A^{-1} \underline{\underline{C}} A^{-1} \cdot B + A^{-1} \underline{\underline{D}}$$

$$df_{(A,B)}(0, AD) = D \quad \text{onde } D \in M(k \times (n-k), \mathbb{R})$$

$\therefore df_{(A,B)}$ é sobrej.

$$g(X) = X^{-1} \qquad dg_X(Y) = \frac{d}{dt}\Big|_{t=0} g(X(t))$$

$$X(0) = X \qquad = \frac{d}{dt}\Big|_{t=0} X(t)^{-1}$$

$$\frac{d}{dt}\Big|_{t=0} X(t) = Y \qquad = -X^{-1} Y X^{-1}$$

$$X(t) X(t)^{-1} = I$$

$$\frac{d}{dt}\Big|_{t=0}: \quad Y \cdot X^{-1} + X \frac{d}{dt} X(t)^{-1} = 0$$

(f) $Gr_k(\mathbb{R}^n)$ é compacta.

$\phi : \mathcal{U} \to Gr_k(\mathbb{R}^n)$ sobrejetora

U ← também é sobrej. ✓

$$\mathcal{U}_c = \{ x \in M(k \times n, \mathbb{R}) \mid \text{linhas de } x \text{ são o.n.} \}$$

Dado $\sigma \in Gr_k(\mathbb{R}^n)$ $\exists\, x \in \mathcal{U}$ tq. $\phi(x) = \sigma$

$x \longrightarrow \tilde{x}$, $\mathcal{S}(\tilde{x}) = \sigma$

Gram-Schmidt

$\begin{pmatrix} & \\ & \end{pmatrix}$

$\mathcal{U}_c$ é cpto. : $x \in \mathcal{U}_c \iff \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$\underbrace{xx^t}_{} = id \Rightarrow \|x\|^2 = k$

Condição fechada.

$x \in M(k \times n, \mathbb{R})$

$\|x\|^2 = \sum x_{ij}^2$

$$tr(xx^t) = \sum_i (xx^t)_{ii} = \sum_{ij} x_{ij} \cdot (x^t)_{ji} = \sum_{ij} x_{ij}^2$$

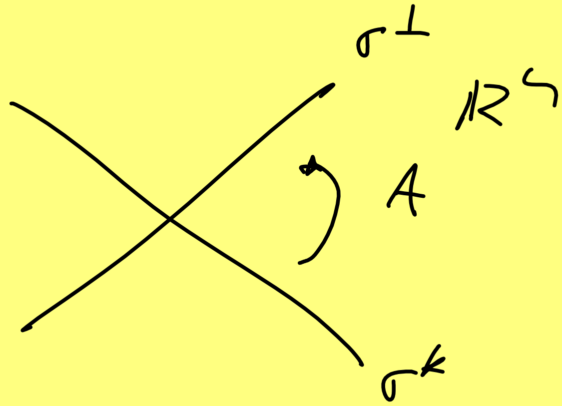
$$= \|x\|^2$$

(g) $\sigma \in Gr_k(\mathbb{R}^n)$

$$\to \boxed{T_\sigma Gr_k(\mathbb{R}^n)} \longleftarrow Hom(\sigma, \sigma^\perp) : \Phi \quad (*)$$

$$\frac{d}{dt}\Big|_{t=0} \sigma_{tA} \longleftarrow A$$

$A : \sigma \to \sigma^\perp$ linear



$tA : \sigma \to \sigma^\perp$, $|t| < \varepsilon$

$\sigma_{tA} := \text{Gráfico}(tA) \subset \sigma \oplus \sigma^\perp = \mathbb{R}^n$

$t \mapsto \sigma_{tA} \in Gr_k(\mathbb{R}^n)$ curva $\qquad \sigma_0 = \sigma$

$\mathcal{S}(x) = \sigma \qquad x = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$

"$x + Ax$" $\quad x_t = \begin{pmatrix} v_1 + tAv_1 \\ \vdots \\ v_k + tAv_k \end{pmatrix}_{\sigma} \qquad \mathcal{S}(x_t) = \sigma_{tA}$

é linear em t

Se $\sigma \in U_I$ então $\sigma_{tA} \in U_I$ para $|t| < \varepsilon$

$I = \{1, \dots, k\}$

$\hookrightarrow \varphi_I(\sigma_{tA}) = \pi_I(x_t)^{-1} \cdot \pi_{I'}(x_t)$ é suave em t

(*) está bem definida ✓.

$$y_t = \begin{pmatrix} v_1 + tBv_1 \\ \vdots \\ v_k + tBv_k \end{pmatrix}$$

(*) é linear ?

(*) é injetora ?

$$z_t = \begin{pmatrix} v_1 + t(A+B)v_1 \\ \vdots \\ v_k + t(A+B)v_k \end{pmatrix}$$

$A, B \in \mathrm{Hom}(\sigma, \sigma^\perp)$

$$\Phi(A+B) = \frac{d}{dt}\Big|_{t=0} \sigma_{t(A+B)}$$

$$z_0 = y_0 = x_0 = x = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

$\sigma_{t(A+B)} = \text{Gráfico}\,(t(A+B))$

$$\varphi_I(\sigma_{t(A+B)}) = \pi_I(z_t)^{-1}\, \pi_{I'}(z_t)$$

$$\frac{d}{dt}\Big|_{t=0} \dots = -\pi_I(x)^{-1} \pi_I\begin{pmatrix} (A+B)v_1 \\ \vdots \\ (A+B)v_k \end{pmatrix} \pi_I(x)^{-1} \pi_{I'}(x)$$

$$+ \pi_I(x)^{-1} \pi_{I'}\begin{pmatrix} (A+B)v_1 \\ \vdots \\ (A+B)v_k \end{pmatrix} = \dots$$

$$\pi_I\begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_k + v_k \end{pmatrix} = \pi_I\begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} + \pi_I\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

$u_i, v_i \in \mathbb{R}^n$

$$\| \quad \frac{d}{dt}\Big|_{t=0} \varphi_I(\sigma_{tA}) + \frac{d}{dt}\Big|_{t=0} \varphi_I(\sigma_{tB})$$

Se $\Phi(A) = 0$ então $\frac{d}{dt}\Big|_{t=0} \sigma_{tA} = 0$

$$\Leftrightarrow \frac{d}{dt}\Big|_{t=0} \varphi_I(\sigma_t A) = 0$$

$$\Leftrightarrow -\pi_I(x)^{-1}\,\pi_I\begin{pmatrix}Av_1\\ \vdots \\ Av_k\end{pmatrix}\pi_I(x)^{-1}\,\pi_{I'}(x)$$

$$+\, \pi_I(x)^{-1}\,\pi_{I'}\begin{pmatrix}Av_1\\ \vdots \\ Av_k\end{pmatrix} = 0$$

$$x = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & a \\ & & 1 & \end{array}\right) = \begin{pmatrix}v_1\\ \vdots \\ v_k\end{pmatrix}$$

$$\pi_I\begin{pmatrix}Av_1\\ \vdots \\ Av_k\end{pmatrix} a = \pi_{I'}\begin{pmatrix}Av_1\\ \vdots \\ Av_k\end{pmatrix}$$

$$\underbrace{\sigma}_{v_1,\dots,v_k} \oplus \underbrace{\sigma^\perp}_{w_{k+1},\dots,w_n} = \mathbb{R}^n \qquad \cancel{Av_j = \sum_{i=k+1}^{n} a_{ij} w_i \quad j=1,\dots,k}$$

$$Av_1,\dots,Av_k \in \sigma^\perp$$

$$u_1,\dots,u_k \in \sigma^\perp \qquad u_j = \sum_{i=1}^{n} b_{ij} e_i \quad j = 1,\dots,k$$

$$\pi_I\begin{pmatrix}u_1\\ \vdots \\ u_k\end{pmatrix} a = \pi_{I'}\begin{pmatrix}u_1\\ \vdots \\ u_k\end{pmatrix}$$

$I = \{1,\dots,k\}$

$$\begin{pmatrix} b_{11} & \dots & b_{k1} \\ \vdots & & \\ b_{k1} & \dots & b_{kk} \end{pmatrix} a = \begin{pmatrix} b_{k+1,1} & \dots & b_{n1} \\ \vdots & & \vdots \\ b_{k+1,k} & \dots & b_{nk} \end{pmatrix}$$