UNIVERSITY OF SÃO PAULO

# *Notes on Smooth Manifolds*

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`Preliminary version 2`

December 2015

# Foreword

The concept of smooth manifold is ubiquitous in Mathematics. Indeed smooth manifolds appear as Riemannian manifolds in differential geometry, space-times in general relativity, phase spaces and energy levels in mechanics, domains of definition of ODE's in dynamical systems, Riemann surfaces in theory of complex analytic functions, Lie groups in algebra and geometry..., to name a few instances.

The notion took some time to evolve until it reached its present form in H. Whitney's celebrated Annals of Mathematics paper in 1936. Whitney's paper in fact represents a culmination of diverse historical developments which took place separately, each in a different domain, all striving to make the passage from the local to the global.

From the modern point of view, the initial goal of introducing smooth manifolds is to generalize the methods and results of differential and integral calculus, in special, the inverse and implicit function theorems, the theorem on existence, uniqueness and regularity of ODE's and Stokes' theorem. As usual in Mathematics, once introduced such objects start to atract interest on their own and new structure is uncovered. The subject of differential topology studies smooth manifolds *per se*. Many important results about the topology of smooth manifolds were obtained in the 1950's and 1960's in the high dimensional range. For instance, there exist topological manifolds admitting several non-diffeomorphic smooth structures (Milnor, 1956, in the case of $S^7$), and there exist topological manifolds admitting no smooth strucuture at all (Kervaire, 1961). Moreover the Poincaré conjecture in dimensions bigger than $4$ was proved independently by Stallings and Smale in the 1960's. On the other hand, the topology of compact surfaces is a classical subject already tackled in the nineteenth century; the very important case of dimension 3 has seen tremendous development after the works of Thurston (late 1970's), Hamilton (1981) and Perelman (2003), and continues to attract a lot of attention; and the case of dimension 4, despite the breakthroughs of Donaldson and Freedman in the 1980's, is largely terra incognita.

The aim of these notes is much more modest. Their contents cover, with some looseness, the syllabus of the course "Differentiable manifolds and Lie groups" that I taught at the Graduate Program of the University of São Paulo in 2001, 2008, 2013 and 2015. Chapter 1 introduces the basic language of smooth manifolds, culminating with the Frobenius theorem. Chapter 2 introduces the basic language of tensors. The most important construction there is perhaps the exterior derivative of differential forms. Chapter 3 is a first encounter with Lie groups and their Lie algebras, in which also homogeneous manifolds are briefly discussed. Finally, Chapter 4 is about integration on manifolds and explains Stokes' theorem, de Rham cohomology and some rudiments of differential topology. Routine exercises are scattered throughout the text, which aim to help the reader digest the material. More elaborate problems can be found in the final section of each chapter. Needless to say, working arduously in problems is a necessary (but not sufficient) condition to advance one's comprehension of a mathematical theory.

I am indebted to the (dozens of) graduate students who took my courses and impelled me to write this set of notes. Special thanks go to Dr. Pedro Zühlke whose careful reading and suggestions has helped improve the text. Any remaining errors are of course my own fault.

*São Paulo, December 2015*

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CHAPTER 1

# Smooth manifolds

In order to motivate the definition of abstract smooth manifold, we first define submanifolds of Euclidean spaces. Recall from vector calculus and differential geometry the ideas of parametrizations and inverse images of regular values.

## 1.1 Submanifolds of Euclidean spaces

A smooth map $f : U \to \mathbf{R}^{n+k}$, where $U \subset \mathbf{R}^n$ is open, is called an *immersion at* $p$, where $p \in U$, if $df_p : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective. $f$ is called simply an *immersion* if it is an immersion everywhere. An injective immersion will be called a *parametrization*.

A smooth map $F : W \to \mathbf{R}^k$, where $W \subset \mathbf{R}^{n+k}$ is open, is called a *submersion at* $p$, where $p \in W$, if $df_p : \mathbf{R}^{n+k} \to \mathbf{R}^k$ is surjective. $F$ is called simply a *submersion* if it is a submersion everywhere. For $z_0 \in \mathbf{R}^k$, if $F$ is a submersion along the level set $F^{-1}(z_0)$, then $z_0$ is called a *regular value* of $F$ (in particular, a point $z_0 \in \mathbf{R}^k$ *not* in the image of $F$ is always a regular value!).

Images of parametrizations and inverse images of regular values are thus candidates to be submanifolds of Euclidean spaces. Next we would like to explain why the second class has stronger properties than the first one. The argument involves the implicit function theorem, and how it is proved to be a consequence of the inverse function theorem.

Assume then $z_0$ is a regular value of $F$ as above and $F^{-1}(z_0)$ is non-empty; write $M$ for this set and consider $p \in M$. Then $dF_p$ is surjective and, up to relabeling the coordinates, we may assume that $(d_2F)_p$, which is the restriction of $dF_p$ to $\{0\} \oplus \mathbf{R}^k \subset \mathbf{R}^{n+k}$, is an isomorphism onto $\mathbf{R}^k$. Write $p = (x_0, y_0)$ where $x_0 \in \mathbf{R}^n$, $y_0 \in \mathbf{R}^k$. Define a smooth map

$$\Phi : W \to \mathbf{R}^{n+k}, \qquad \Phi(x, y) = (x, F(x, y) - z_0)$$

Then $d\Phi_{(x_0,y_0)}$ is easily seen to be an isomorphism, so the inverse function theorem implies that there exist open neighborhoods $U$, $V$ of $x_0$, $y_0$ in $\mathbf{R}^n$,

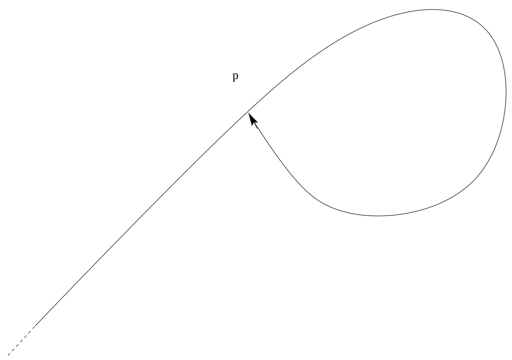


Figure 1.1: A non-embedded submanifold of $\mathbf{R}^2$.

$\mathbf{R}^k$, respectively, such that $\Phi$ is a diffeomorphism of $U \times V$ onto an open subset of $\mathbf{R}^{n+k}$, i.e. $\Phi$ is a smooth bijective map onto its image and the inverse map is also smooth. Now the fundamental fact is that

$$\Phi(M \cap (U \times V)) = (\mathbf{R}^n \times \{0\}) \cap \Phi(U \times V),$$

as it follows from the form of $\Phi$; namely, $\Phi$ "rectifies" $M$.

Let $\varphi : M \cap (U \times V) \to \mathbf{R}^n$ be the restriction of $\Phi$. Then $\varphi^{-1}$ is the restriction of $\Phi$ to $\mathbf{R}^n$ and thus smooth. It also follows from the above calculation that $M \cap (U \times V)$ is exactly the graph of the smooth map $f : U \to V$, satisfying $f(x_0) = y_0$, given by $f = \mathrm{proj}_{\mathbf{R}^k} \circ \varphi^{-1}$. Another way to put it is that $M \cap (U \times V)$ is the image of a parametrization $\varphi^{-1} : \varphi(M \cap (U \times V)) \subset \mathbf{R}^n \to \mathbf{R}^{n+k}$ which is a homeomorphism onto its image, where the latter is equipped with the topology induced from $\mathbf{R}^{n+k}$.

**1.1.1 Definition** (i) A subset $M \subset \mathbf{R}^{n+k}$ will be called a *embedded submanifold of dimension $n$ of* $\mathbf{R}^{n+k}$ if for every $p \in M$, there exists a diffeomorphism $\Phi$ from an open neighborhood $U$ of $p$ in $\mathbf{R}^{n+k}$ onto its image such that $\Phi(M \cap U) = (\mathbf{R}^n \times \{0\}) \cap \Phi(U)$. In this case we will say that $(U, \Phi)$ is a *local chart of* $\mathbf{R}^{n+k}$ *adapted to* $M$.

(ii) A *parametrized submanifold of dimension $n$ of* $\mathbf{R}^{n+k}$ is a pair $(U, f)$ where $U \subset \mathbf{R}^n$ is open and $f : U \to \mathbf{R}^{n+k}$ is an injective immersion.

**1.1.2 Example** Let $(\mathbf{R}, f)$ be a parametrized submanifold of dimension 1 of $\mathbf{R}^2$, where $f : \mathbf{R} \to \mathbf{R}^2$ has image $M$ described in Figure 1.1. Then $M$ is non-embedded. In fact no connected neighborhood of $p$ can be homeomorphic to an interval of $\mathbf{R}$ (restrict such a homeomorphism to the complement of $\{p\}$ to get a contradiction). Note that $f$ is not a homeomorphism onto its image.

**1.1.3 Exercise** Prove that the graph of a smooth map $f : U \to \mathbf{R}^k$, where $U \subset \mathbf{R}^n$ is open, is an embedded submanifold of dimension $n$ of $\mathbf{R}^{n+k}$.

**1.1.4 Exercise** Let $f, g : (0, 2\pi) \to \mathbf{R}^2$ be defined by

$$f(t) = (\sin t, \sin t \cos t), \qquad g(t) = (\sin t, -\sin t \cos t).$$

$a.$ Check that $f$, $g$ are injective immersions with the same image.
$b.$ Sketch a drawing of their image.
$c.$ Write a formula for $g^{-1} \circ f : (0, 2\pi) \to (0, 2\pi)$.
$d.$ Deduce that the identity map $\mathrm{id} : \mathrm{im}\, f \to \mathrm{im}\, g$ is not continuous, where $\mathrm{im}\, f$ and $\mathrm{im}\, g$ are equipped with the topology induced from $\mathbf{R}$ via $f$ and $g$, respectively.

**The algebra $C^\infty(M)$ of real smooth functions on $M$**

Let $M$ be an embedded submanifold of $\mathbf{R}^{n+k}$.

**1.1.5 Definition** A function $f : M \to \mathbf{R}$ is said to be *smooth* at $p \in M$ if $f \circ \Phi^{-1} : \Phi(U) \cap \mathbf{R}^n \to \mathbf{R}$ is a smooth function for some adapted local chart $(U, \Phi)$ around $p$.

**1.1.6 Remark** (i) The condition is independent of the choice of adapted local chart around $p$. Indeed if $(V, \Phi)$ is another one,

$$f \circ \Phi^{-1} = (f \circ \Psi^{-1}) \circ (\Psi \circ \Phi^{-1})$$

where $\Psi \circ \Phi^{-1} : \Phi(U \cap V) \to \Psi(U \cap V)$ is a diffeomorphism and the claim follows from the the chain rule for smooth maps between Euclidean spaces.

(ii) A smooth function on $M$ is automatically continuous.

(iii) Let $F$ be a smooth function defined on an open neighborhood of $p$ in $\mathbf{R}^{n+k}$. The restriction of $F$ to $M$ is smooth at $p$.

## 1.2 Definition of abstract smooth manifold

Let $M$ be a topological space. A *local chart* of $M$ is a pair $(U, \varphi)$, where $U$ is an open subset of $M$ and $\varphi$ is a homeomorphism from $U$ onto an open subset of $\mathbf{R}^n$. A local chart $\varphi : U \to \mathbf{R}^n$ introduces coordinates $(x_1, \ldots, x_n)$ on $U$, namely, the component functions of $\varphi$, and that is why $(U, \varphi)$ is also called a *system of local coordinates* on $M$.

A *(topological) atlas* for $M$ is a family $\{(U_\alpha, \varphi_\alpha)\}$ of local charts of $M$, where the dimension $n$ of the Euclidean space is fixed, whose domains cover $M$, namely, $\bigcup U_\alpha = M$. If $M$ admits an atlas, we say that $M$ is *locally modeled on* $\mathbf{R}^n$ and $M$ is a *topological manifold*.

A *smooth atlas* is an atlas whose local charts satisfy the additional *compatibility condition*:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{1.2.1}$$

is smooth, for all $\alpha$, $\beta$. A smooth atlas $\mathcal{A}$ defines a notion of smooth function on $M$ as above, namely, a function $f : M \to \mathbf{R}$ is *smooth* if $f \circ \varphi^{-1} : \varphi(U) \to \mathbf{R}$ is smooth for all $(U, \varphi) \in \mathcal{A}$. We say that two atlas $\mathcal{A}$, $\mathcal{B}$ for $M$ are *equivalent* if the local charts of one are compatible with those of the other, namely, $\psi \circ \phi^{-1}$ is smooth for all $(U, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$. In this case, it is obvious that $\mathcal{A}$ and $\mathcal{B}$ define the same notion of smooth function on $M$.

A *smooth structure* on $M$ is an equivalence class $[\mathcal{A}]$ of smooth atlases on $M$. Finally, a *smooth manifold* is a topological space $M$ equipped with a smooth structure $[\mathcal{A}]$. In order to be able to do interesting analysis on $M$, we shall assume, as usual, that *the topology of $M$ is Hausdorff and second countable*.

**1.2.2 Remark** (a) It follows from general results in topology that (smooth) manifolds are metrizable. Indeed, manifols are locally Euclidean and thus locally compact. A locally compact Hausdorff space is (completely) regular, and the Urysohn metrization theorem states that a second countable regular space is metrizable.

(b) The condition of second countability also rules out pathologies of the following kind. Consider $\mathbf{R}^2$ with the topology with basis of open sets $\{(a, b) \times \{c\} \mid a, b, c \in \mathbf{R}, a < b\}$. This topology is Hausdorff but not second countable, and it is compatible with a structure of smooth manifold of dimension $1$ (a continuum of real lines)!

**1.2.3 Exercise** Let $M$ be a topological space. Prove that two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if their union $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Deduce that every equivalence class of smooth atlases for $M$ contains a unique representative which is *maximal* (i.e. not properly contained in any other smooth atlas in the same equivalence class).

Let $M$, $N$ be smooth manifolds. A map $f : M \to N$ is called *smooth* if for every $p \in M$, there exist local charts $(U, \varphi)$, $(V, \psi)$ of $M$, $N$ around $p$, $f(p)$, resp., such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

**1.2.4 Remark** (i) The definition is independent of the choice of local charts.

(ii) The definition is local in the sense that $f : M \to N$ is smooth if and only if its restriction to an open subset $U$ of $M$ is smooth (cf. Example 1.2.7(vi)).

(iii) A smooth map $M \to N$ is automatically continuous.

We have completed the definition of the category **DIFF**, whose objects are the smooth manifolds and whose morphisms are the smooth maps. An isomorphism in this category is usually called a *diffeomorphism*.

**1.2.5 Exercise** Let $M$ be a smooth manifold with smooth atlas $\mathcal{A}$. Prove that any local chart $(U, \varphi) \in \mathcal{A}$ is a diffeomorphism onto its image. Conversely, prove any map $\tau : W \to \mathbf{R}^n$, where $n = \dim M$ and $W \subset M$ is open, which is a diffeomorphism onto its image belongs to a smooth atlas equivalent to $\mathcal{A}$; in particular, $(W, \tau) \in \mathcal{A}$ if $\mathcal{A}$ is maximal.

**1.2.6 Remark** In practice, explicitly written down atlases are finite (compare Problem 1 and Example 1.2.9). However, in view of the last assertion in Exercise 1.2.5, it is often convenient to implicitly represent a smooth structure by a maximal atlas, and we shall be doing that.

**1.2.7 Examples** (i) $\mathbf{R}^n$ has a canonical atlas consisting only of one local chart, namely, the identity map, which in fact is a *global* chart. This is the standard smooth structure on $\mathbf{R}^n$ with respect to which all definitions coincide with the usual ones. Unless explicit mention, we will always consider $\mathbf{R}^n$ with this smooth structure.

(ii) Any finite dimensional real vector space $V$ has a canonical structure of smooth manifold. In fact a linear isomorphism $V \cong \mathbf{R}^n$ defines a global chart and thus an atlas, and two such atlases are always equivalent since the transition map between their global charts is a linear isomorphism of $\mathbf{R}^n$ and hence smooth.

(iii) Submanifolds of Euclidean spaces (Definition 1.1.1(i)) are smooth manifolds. Namely, atlases are construted by using restrictions of adapted charts. Note that the compatibility condition (1.2.1) is automatically satisfied.

(iv) Graphs of smooth maps defined on open subsets of $\mathbf{R}^n$ with values on $\mathbf{R}^{n+k}$ are smooth manifolds (cf. Exercise 1.1.3 and (iii)). More generally, a subset $M$ of $\mathbf{R}^{n+k}$ with the property that every one of its points admits an open neighborhood in $M$ which is a graph as above is a smooth manifold.

(v) It follows from (iv) that the $n$*-sphere*

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is a smooth manifold.

(vi) If $\mathcal{A}$ is an atlas for $M$ and $V \subset M$ is open then $\mathcal{A}|_V := \{(V \cap U, \varphi|_{V \cap U}) : (U, \varphi) \in \mathcal{A}\}$ is an atlas for $V$. It follows that any open subset of a smooth manifold is a smooth manifold.

(vii) If $M$, $N$ are smooth manifolds with atlases $\mathcal{A}$, $\mathcal{B}$, resp., then $\mathcal{A} \times \mathcal{B}$ is an atlas for the Cartesian product $M \times N$ with the product topology, and hence $M \times N$ is canonically a smooth manifold of dimension $\dim M + \dim N$.

(viii) It follows from (iv) and (vi) that the $n$*-torus*

$$T^n = S^1 \times \cdots \times S^1 \qquad (n \text{ factors})$$

is a smooth manifold.

(ix) The *general linear group* $\mathbf{GL}(n,\mathbf{R})$ is the set of all $n\times n$ non-singular real matrices. Since the set of $n\times n$ real matrices can be identified with a $\mathbf{R}^{n^2}$ and as such the determinant becomes a continuous function, $\mathbf{GL}(n,\mathbf{R})$ can be viewed as the open subset of $\mathbf{R}^{n^2}$ where the determinant does not vanish and hence acquires the structure of a smooth manifold of dimension $n^2$.

The following two examples deserve a separate discussion.

**1.2.8 Example** The map $f:\mathbf{R}\to\mathbf{R}$ given by $f(x)=x^3$ is a homeomorphism, so it defines a local chart around any point of $\mathbf{R}$ and we can use it to define an atlas $\{f\}$ for $\mathbf{R}$; denote the resulting smooth manifold by $\tilde{\mathbf{R}}$. We claim that $\tilde{\mathbf{R}}\neq\mathbf{R}$ as smooth manifolds, because $C^\infty(\tilde{\mathbf{R}})\neq C^\infty(\mathbf{R})$. In fact, $\mathrm{id}:\mathbf{R}\to\mathbf{R}$ is obviously smooth, but $\mathrm{id}:\tilde{\mathbf{R}}\to\mathbf{R}$ is not, because $\mathrm{id}\circ f^{-1}:\mathbf{R}\to\mathbf{R}$ maps $x$ to $\sqrt[3]{x}$ so it is not differentiable at $0$. On the other hand, $\tilde{\mathbf{R}}$ is diffeomorphic to $\mathbf{R}$. Indeed $f:\tilde{\mathbf{R}}\to\mathbf{R}$ defines a diffeomorphism since its local representation $\mathrm{id}\circ f\circ f^{-1}$ is the identity.

**1.2.9 Example** The *real projective space*, denoted $\mathbf{R}P^n$, as a set consists of all one-dimensional subspaces of $\mathbf{R}^{n+1}$. We introduce a structure of smooth manifold of dimension $n$ on $\mathbf{R}P^n$. Each subspace is spanned by a non-zero vector $v\in\mathbf{R}^{n+1}$. Let $U_i$ be the subset of $\mathbf{R}P^n$ specified by the condition that the $i$-th coordinate of $v$ is not zero. Then $\{U_i\}_{i=1}^{n+1}$ covers $\mathbf{R}P^n$. Each line in $U_i$ meets the hyperplane $x_i=1$ in exactly one point, so there is a bijective map $\varphi_i:U_i\to\mathbf{R}^n\subset\mathbf{R}^{n+1}$. For $i\neq j$, $\varphi_i(U_i\cap U_j)\subset\mathbf{R}^n\subset\mathbf{R}^{n+1}$ is precisely the open subset of the hyperplane $x_i=1$ defined by $x_j\neq 0$, and

$$\varphi_j\circ\varphi_i^{-1}:\{x\in\mathbf{R}^{n+1}:x_i=1,\ x_j\neq 0\}\to\{x\in\mathbf{R}^{n+1}:x_j=1,\ x_i\neq 0\}$$

is the map

$$v\mapsto\frac{1}{x_j}v,$$

thus smooth. So far there is no topology in $\mathbf{R}P^n$, and we introduce one by declaring

$$\cup_{i=1}^{n+1}\{\varphi_i^{-1}(W):W\subset\varphi_i(U_i)=\mathbf{R}^n\text{ is open}\}$$

to be a basis of open sets. It is clear that $\varnothing$ and $M$ are open sets (since each $U_i$ is open) and we have only to check that finite intersections of open sets are open. Let $W_i\subset\varphi_i(U_i)$ and $W_j\subset\varphi_i(U_j)$ be open. Then

$$\varphi_i^{-1}(W_i)\cap\varphi_j^{-1}(W_j)=\varphi_j^{-1}\left(\varphi_j\varphi_i^{-1}(W_i\cap\varphi_i(U_i\cap U_j))\cap W_j\right).$$

Since $\varphi_j\varphi_i^{-1}$ is a homeomorphism, a necessary and sufficient condition for the left hand side to describe an open set for all $i$, $j$, is that $\varphi_i(U_i\cap U_j)$ be open for all $i$, $j$, and this does occur in this example. Now the topology is well defined, second countable, and the $\varphi_i$ are homeomorhisms onto their images. It is also clear that for $\ell \in \mathbf{R}P^n$ the sets

$$\{\ell' \in \mathbf{R}P : \angle(\ell, \ell') < \epsilon\}$$

for $\epsilon > 0$ are open neighborhoods of $\ell$. It follows that the topology is Hausdorff.

The argument in Example 1.2.9 is immediately generalized to prove the following proposition.

**1.2.10 Proposition** *Let $M$ be a set and let $n$ be a non-negative integer. A countable collection $\{(U_\alpha, \varphi_\alpha)\}$ of injective maps $\varphi : U_\alpha \to \mathbf{R}^n$ whose domains cover $M$ satisfying*

*a. $\varphi_\alpha(U_\alpha)$ is open for all $\alpha$;*
*b. $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open for all $\alpha$, $\beta$;*
*c. $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth for all $\alpha$, $\beta$;*

*defines a second countable topology and smooth structure on $M$ (the Hausdorff condition is not automatic and must be checked in each case).*

## 1.3 Tangent space

As a motivation, we first discuss the case of an embedded submanifold $M$ of $\mathbf{R}^{n+k}$. Fix $p \in M$ and take an adapted local chart $(U, \Phi)$ around $p$. Recall that we get a parametrization of $M$ around $p$ by setting $\varphi := \mathrm{proj}_{\mathbf{R}^n} \circ \Phi|_{M \cap U}$ and taking

$$\varphi^{-1} : \mathbf{R}^n \cap \Phi(U) \to \mathbf{R}^{n+k}.$$

It is then natural to define the *tangent space* of $M$ at $p$ to be the image of the differential of the parametrization, namely,

$$T_pM := d(\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n).$$

If $(V, \Psi)$ is another adapted local chart around $p$, $\psi := \mathrm{proj}_{\mathbf{R}^n} \circ \Psi|_{M \cap V}$ and $\psi^{-1} : \mathbf{R}^n \cap \Psi(V) \to \mathbf{R}^{n+k}$ is the associated parametrization, then

$$\begin{aligned} d(\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n) &= d(\psi^{-1})_{\psi(p)} d(\psi\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n) \\ &= d(\psi^{-1})_{\varphi(p)}(\mathbf{R}^n) \end{aligned}$$

since $d(\psi\varphi^{-1})_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. It follows that $T_pM$ is well defined as a subspace of dimension $n$ of $\mathbf{R}^{n+k}$.

Note that we have the following situation:

$$\begin{tikzcd} & v \in T_pM & \\ a \in \mathbf{R}^n \arrow[ur, "d\varphi^{-1}_{\varphi(p)}"] \arrow[rr, "d(\psi\varphi^{-1})_{\varphi(p)}"'] & & b \in \mathbf{R}^n \arrow[ul, "d\psi^{-1}_{\psi(p)}"'] \end{tikzcd}$$

Namely, the tangent vector $v \in T_pM$ is represented by two different vectors $a$, $b \in \mathbf{R}^n$ which are related by the differential of the transition map. We can use this idea to generalize the construction of the tangent space to an abstract smooth manifold.

Let $M$ be a smooth manifold of dimension $n$, and fix $p \in M$. Suppose that $\mathcal{A}$ is an atlas defining the smooth structure of $M$. The *tangent space* of $M$ at $p$ is the set $T_pM$ of all pairs $(a, \varphi)$ — where $a \in \mathbf{R}^n$ and $(U, \varphi) \in \mathcal{A}$ is a local chart around $p$ — quotiented by the equivalence relation

$$(a, \varphi) \sim (b, \psi) \quad \text{if and only if} \quad d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

It follows from the chain rule in $\mathbf{R}^n$ that this is indeed an equivalence relation, and we denote the equivalence class of $(a, \varphi)$ by $[a, \varphi]$. Each such equivalence class is called a *tangent vector* at $p$. For a fixed local chart $(U, \varphi)$ around $p$, the map

$$a \in \mathbf{R}^n \mapsto [a, \varphi] \in T_pM$$

is a bijection, and it follows from the linearity of $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ that we can use it to transfer the vector space structure of $\mathbf{R}^n$ to $T_pM$. Note that $\dim T_pM = \dim M$.

**1.3.1 Exercise** Let $M$ be a smooth manifold and let $V \subset M$ be an open subset. Prove that there is a canonical isomorphism $T_pV \cong T_pM$ for all $p \in V$.

Let $(U, \varphi = (x_1, \ldots, x_n))$ be a local chart of $M$, and denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$. The *coordinate vectors* at $p$ are with respect to this chart are defined to be

$$\left.\frac{\partial}{\partial x_i}\right|_p = [e_i, \varphi].$$

Note that

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \tag{1.3.2}$$

is a basis of $T_pM$.

In the case of $\mathbf{R}^n$, for each $p \in \mathbf{R}^n$ there is a canonical isomorphism $\mathbf{R}^n \to T_p\mathbf{R}^n$ given by

$$a \mapsto [a, \mathrm{id}], \tag{1.3.3}$$

where $\mathrm{id}$ is the identity map of $\mathbf{R}^n$. Usually we will make this identification without further comment. In particular, $T_p\mathbf{R}^n$ and $T_q\mathbf{R}^n$ are canonically isomorphic for every $p$, $q \in \mathbf{R}^n$. In the case of a general smooth manifold $M$, obviously there are no such canonical isomorphisms. Occasionally we shall denote by $(r_1, \ldots, r_n)$ the coordinates on $\mathbf{R}^n$ corresponding to $\mathrm{id}$.

**Tangent vectors as directional derivatives**

Let $M$ be a smooth manifold, and fix a point $p \in M$. For each tangent vector $v \in T_pM$ of the form $v = [a, \varphi]$, where $a \in \mathbf{R}^n$ and $(U, \varphi)$ is a local chart of $M$, and for each $f \in C^\infty(U)$, we define the *directional derivative of $f$ in the direction of $v$* to be the real number

$$\begin{aligned} v(f) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= d(f \circ \varphi^{-1})(a). \end{aligned}$$

It is a simple consequence of the chain rule that this definition does not depend on the choice of representative of $v$.

In the case of $\mathbf{R}^n$, $\frac{\partial}{\partial r_i}\big|_p f$ is simply the partial derivative in the direction $e_i$, the $i$th vector in the canonical basis of $\mathbf{R}^n$. In general, if $\varphi = (x_1, \ldots, x_n)$, then $x_i \circ \varphi^{-1} = r_i$, so

$$v(x_i) = d(r_i)_{\varphi(p)}(a) = a_i,$$

where $a = \sum_{i=1}^n a_i e_i$. Since $v = [a, \varphi] = \sum_{i=1}^n a_i [e_i, \varphi]$, it follows that

$$v = \sum_{i=1}^n v(x_i) \frac{\partial}{\partial x_i}\Big|_p. \tag{1.3.4}$$

If $v$ is a coordinate vector $\frac{\partial}{\partial x_i}$ and $f \in C^\infty(U)$, we also write

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial f}{\partial x_i}\Big|_p.$$

As a particular case of (1.3.4), take now $v$ to be a coordinate vector of another local chart $(V, \psi = (y_1, \ldots, y_n))$ around $p$. Then

$$\frac{\partial}{\partial y_j}\Big|_p = \sum_{i=1}^n \frac{\partial x_i}{\partial y_j}\Big|_p \frac{\partial}{\partial x_i}\Big|_p.$$

Note that the preceding formula shows that even if $x_1 = y_1$ we do not need to have $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}$.

**The differential**

Let $f : M \to N$ be a smooth map between smooth manifolds. Fix a point $p \in M$, and local charts $(U, \varphi)$ of $M$ around $p$, and $(V, \psi)$ of $N$ around $q = f(p)$. The *differential* or *tangent map* of $f$ at $p$ is the linear map

$$df_p : T_pM \to T_qN$$

given by
$$[a,\varphi]\mapsto[d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}(a),\psi].$$
It is easy to check that this definition does not depend on the choices of local charts. Using the identification (1.3.3), one checks that $d\varphi_p : T_pM \to \mathbf{R}^n$ and $d\psi_q : T_pM \to \mathbf{R}^n$ are linear isomorphisms and
$$df_p = (d\psi_q)^{-1}\circ d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}\circ d\varphi_p.$$

**1.3.5 Proposition (Chain rule)** *Let $M$, $N$, $P$ be smooth manifolds. If $f : M \to N$ and $g : N \to P$ are smooth maps, then $g\circ f : M \to P$ is a smooth map and*
$$d(g\circ f)_p = dg_{f(p)}\circ df_p$$
*for $p\in M$.*

**1.3.6 Exercise** Prove Proposition 1.3.5.

If $f\in C^\infty(M,N)$, $g\in C^\infty(N)$ and $v\in T_pM$, then it is a simple matter of unravelling the definitions to check that
$$df_p(v)(g) = v(g\circ f).$$
Now (1.3.4) together with this equation gives that
$$\begin{aligned} df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right) &= \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right)(y_i)\frac{\partial}{\partial y_i}\Big|_{f(p)}\\ &= \sum_{i=1}^n \frac{\partial(y_i\circ f)}{\partial x_j}\Big|_p\frac{\partial}{\partial y_i}\Big|_{f(p)}. \end{aligned}$$
The matrix
$$\left(\frac{\partial(y_i\circ f)}{\partial x_j}\Big|_p\right)$$
is called the *Jacobian matrix of $f$ at $p$* relative to the given coordinate systems. Observe that the chain rule (Proposition 1.3.5) is equivalent to saying that the Jacobian matrix of $g\circ f$ at a point is the product of the Jacobian matrices of $g$ and $f$ at the appropriate points.

Consider now the case in which $N=\mathbf{R}$ and $f\in C^\infty(M)$. Then $df_p : T_pM \to T_{f(p)}\mathbf{R}$, and upon the identification between $T_{f(p)}\mathbf{R}$ and $\mathbf{R}$, we easily see that $df_p(v) = v(f)$. Applying this to $f = x_i$, where $(U,\varphi = (x_1,\ldots,x_n))$ is a local chart around $p$, and using again (1.3.4) shows that

$$\{dx_1|_p,\ldots,dx_n|_p\} \tag{1.3.7}$$

is the basis of $T_pM^*$ dual of the basis (1.3.2), and hence
$$df_p = \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_i}\Big|_p\right)dx_i|_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dx_i|_p.$$

Finally, we discuss smooth curves on $M$. A *smooth curve* in $M$ is simply a smooth map $\gamma : (a,b) \to M$ where $(a,b)$ is an interval of $\mathbf{R}$. One can also consider smooth curves $\gamma$ in $M$ defined on a closed interval $[a,b]$. This simply means that $\gamma$ admits a smooth extension to an open interval $(a - \epsilon, b+\epsilon)$ for some $\epsilon > 0$.

If $\gamma : (a,b) \to M$ is a smooth curve, the *tangent vector* to $\gamma$ at $t \in (a,b)$ is

$$\dot\gamma(t) = d\gamma_t\left(\frac{\partial}{\partial r}\Big|_t\right) \in T_{\gamma(t)}M,$$

where $r$ is the canonical coordinate of $\mathbf{R}$. Note that an arbitrary vector $v \in T_pM$ can be considered to be the tangent vector at 0 to the curve $\gamma(t) = \varphi^{-1}(ta)$, where $(U,\varphi)$ is a local chart around $p$ with $\varphi(p) = 0$ and $d\varphi_p(v) = a$.

In the case in which $M = \mathbf{R}^n$, upon identifying $T_{\gamma(t)}\mathbf{R}^n$ and $\mathbf{R}^n$, it is easily seen that

$$\dot\gamma(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

### The inverse function theorem

It is now straightforward to state and prove the inverse function theorem for smooth manifolds.

**1.3.8 Theorem (Inverse function theorem)** *Let $f : M \to N$ be a smooth map between two smooth manifolds $M$, $N$, and let $p \in M$ and $q = f(p)$. If $df_p : T_pM \to T_qN$ is an isomorphism, then there exists an open neighborhood $W$ of $p$ such that $f(W)$ is an open neighborhood of $q$ and $f$ restricts to a diffeomorphism from $W$ onto $f(W)$.*

*Proof.* The proof is really a transposition of the inverse function theorem for smooth mappings between Euclidean spaces to manifolds using local charts. Note that $M$ and $N$ have the same dimension, say, $n$. Take local charts $(U,\varphi)$ of $M$ around $p$ and $(V,\psi)$ of $N$ around $q$ such that $f(U) \subset V$. Set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. By the inverse function theorem for smooth mappings of $\mathbf{R}^n$, there exists an open subset $\tilde W \subset \varphi(U)$ with $\varphi(p) \in \tilde W$ such that $\alpha(\tilde W)$ is an open neighborhood of $\psi(q)$ and $\alpha$ restricts to a diffeomorphism from $\tilde W$ onto $\alpha(\tilde W)$. It follows that $f = \psi^{-1} \circ \alpha \circ \varphi$ is a diffeomorphism from the open neighborhood $W = \varphi^{-1}(\tilde W)$ of $p$ onto the open neighborhood $\psi^{-1}(\alpha(\tilde W))$ of $q$. □

A smooth map $f : M \to N$ satisfying the conclusion of Theorem 1.3.8 at a point $p \in M$ is called a *local diffeomorphism* at $p$. It follows from the above and the chain rule that $f$ is a local diffeomorphism at $p$ if and only if $df_p : T_pM \to T_qN$ is an isomorphism. In this case, there exist local charts

$(U, \varphi)$ of $M$ around $p$ and $(V, \psi)$ of $N$ around $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ of $f$ is the identity, owing to Problem 1.2.5, after enlarging the atlas of $M$, if necessary.

**1.3.9 Exercise** Let $f : M \to N$ be a smooth bijective map that is a local diffeomorphism everywhere. Show that $f$ is a diffeomorphism.

## 1.4 Submanifolds of smooth manifolds

Similar to the situation of submanifolds of Euclidean spaces, some manifolds are contained in other manifolds in a natural way (compare Definition 1.1.1). Let $N$ be a smooth manifold of dimension $n + k$. A subset $M$ of $N$ is called an *embedded submanifold* of $N$ of dimension $n$ if, for every $p \in M$, there exists a local chart $(V, \psi)$ of $N$ such that $p \in V$ and $\psi(V \cap M) = \psi(V) \cap \mathbf{R}^n$, where we identify $\mathbf{R}^n$ with $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^n \times \mathbf{R}^k = \mathbf{R}^{n+k}$. We say that $(V, \psi)$ is a local chart of $N$ *adapted* to $M$. An embedded submanifold $M$ of $N$ is a smooth manifold in its own right, with respect to the relative topology, in a canonical way. In fact an atlas of $M$ is furnished by the restrictions to $M$ of those local charts of $N$ that are adapted to $M$. Namely, if $\{(V_\alpha, \psi_\alpha)\}$ is an atlas of $N$ consisting of adapted charts, then $\{(V_\alpha \cap M, \psi_\alpha|_{V_\alpha \cap M})\}$ becomes an atlas of $M$. Note that the compatibility condition for the local charts of $M$ follows automatically from the compatibility condition for $N$.

**1.4.1 Exercise** Let $N$ be a smooth manifold and let $M$ be an embedded submanifold of $N$. Prove that $T_pM$ is canonically isomorphic to a subspace of $T_pN$ for every $p \in M$.

### Immersions and embeddings

Another class of submanifolds can be introduced as follows. Let $f : M \to N$ be a smooth map between smooth manifolds. The map $f$ is called an *immersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is injective. If $f$ is an immersion everywhere it is simply called an *immersion*. Now call the pair $(M, f)$ an *immersed submanifold* or simply a *submanifold* of $N$ if $f : M \to N$ is an injective immersion.

Let $M$ be an embedded submanifold of $N$ and consider the inclusion $\iota : M \to N$. The existence of adapted local charts implies that $\iota$ can be locally represented around any point of $M$ by the standard inclusion $x \mapsto (x, 0)$, $\mathbf{R}^n \to \mathbf{R}^{n+k}$. Since this map is an immersion, also $\iota$ is an immersion. It follows that $(M, \iota)$ is an immersed submanifold of $N$. This shows that every embedded submanifold of a smooth manifold is an immersed submanifold, but the converse is not true.

**1.4.2 Example** Let $N$ be the 2-torus $T^2 = S^1 \times S^1$ viewed as an embedded submanifold of $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$ and consider the smooth map

$$F : \mathbf{R} \to \mathbf{R}^4, \qquad F(t) = (\cos at, \sin at, \cos bt, \sin bt),$$

where $a$, $b$ are non-zero real numbers. Note that the image of $F$ lies in $T^2$. Denote by $(r_1, r_2, r_3, r_4)$ the coordinates on $\mathbf{R}^4$. Choosing $r_i$, $r_j$ where $i \in \{1, 2\}$ and $j \in \{3, 4\}$ gives a system of coordinates defined on an open subset of $T^2$, and in this way we obtain atlas for $T^2$. It follows that the induced map $f : \mathbf{R} \to T^2$ is smooth. Since $N$ is an embedded submanifold of $\mathbf{R}^4$, we can consider $T_{f(t)}N$ to be a subspace of $\mathbf{R}^4$, and the tangent vector $f'(t) \in T_{f(t)}N$ is the usual derivative $F'(t)$. Since $f'(t)$ never vanishes, $f$ is an immersion. Note that if $b/a$ is an irrational number, then $f$ is an injective map, so $(\mathbf{R}, f)$ is an immersed submanifold which we claim is *not* an embedded submanifold of $T^2$. In fact, the assumption on $b/a$ implies that $M$ is a dense subset of $T^2$, but an embedded submanifold of another manifold is always locally closed.

We would like to further investigate the gap between immersed submanifolds and embedded submanifolds.

**1.4.3 Lemma (Local form of an immersion)** *Let $M$ and $N$ be smooth manifolds of dimensions $n$ and $n + k$, respectively, and suppose that $f : M \to N$ is an immersion at $p \in M$. Then there exist local charts of $M$ and $N$ such that the local expression of $f$ at $p$ is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$.*

*Proof.* Let $(U, \varphi)$ and $(V, \psi)$ be local charts of $M$ and $N$ around $p$ and $q = f(p)$, respectively, such that $f(U) \subset V$, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective, so, up to rearranging indices, we can assume that $d(\pi_1 \circ \alpha)_{\varphi(p)} = \pi_1 \circ d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\pi_1 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$ is the projection onto the first factor. By the inverse function theorem, by shrinking $U$, we can assume that $\pi_1 \circ \alpha$ is a diffeomorphism from $U_0 = \varphi(U)$ onto its image $V_0$; let $\beta : V_0 \to U_0$ be its smooth inverse. Now we can describe $\alpha(U_0)$ as being the graph of the smooth map $\gamma = \pi_2 \circ \alpha \circ \beta : V_0 \subset \mathbf{R}^n \to \mathbf{R}^k$, where $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ is the projection onto the second factor. By Exercise 1.1.3, $\alpha(U_0)$ is a submanifold of $\mathbf{R}^{n+k}$ and the map $\tau : V_0 \times \mathbf{R}^k \to V_0 \times \mathbf{R}^k$ given by $\tau(x, y) = (x, y - \gamma(x))$ is a diffeomorphism such that $\tau(\alpha(U_0)) = V_0 \times \{0\}$. Finally, we put $\tilde{\varphi} = \pi_1 \circ \alpha \circ \varphi$ and $\tilde{\psi} = \tau \circ \psi$, shrinking $V$ if necessary. Then $(U, \tilde{\varphi})$ and $(V, \tilde{\psi})$ are local charts, and for $x \in \tilde{\varphi}(U) = V_0$ we have that

$$\begin{aligned}\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}(x) &= \tau \circ \psi \circ f \circ \varphi^{-1} \circ \beta(x) = \tau \circ \alpha \circ \beta(x) \\ &= \tau(x, \gamma(x)) = (x, 0).\end{aligned}$$

□

**1.4.4 Proposition** *If $f : M \to N$ is an immersion at $p \in M$, then there exists an open neighborhood $U$ of $p$ in $M$ such that $f|_U$ is injective and $f(U)$ is an embedded submanifold of $N$.*

*Proof.* The local injectivity of $f$ at $p$ is an immediate consequence of the fact that some local expression of $f$ at $p$ is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$, hence, injective. Moreover, in the course of proof of Lemma 1.4.3, we have produced a local chart $(V, \tilde{\psi})$ of $N$ adapted to $f(U)$. $\square$

A smooth map $f : M \to N$ is called an *embedding* if it is an immersion and a homeomorphism from $M$ onto $f(M)$ with the induced topology.

**1.4.5 Proposition** *Let $N$ be a smooth manifold. A subset $P \subset N$ is an embedded submanifold of $N$ if and only if it is the image of an embedding.*

*Proof.* Let $f : M \to N$ be an embedding with $P = f(M)$. To prove that $P$ is an embedded submanifold of $N$, it suffices to check that it can be covered by open sets in the relative topology each of which is an embedded submanifold of $N$. Owing to Proposition 1.4.4, any point of $P$ lies in a set of the form $f(U)$, where $U$ is an open subset of $M$ and $f(U)$ is an embedded submanifold of $N$. Since $f$ is an open map into $P$ with the relative topology, $f(U)$ is open in the relative topology and we are done. Conversely, if $P$ is an embedded submanifold of $N$, it has the relative topology and thus the inclusion $\iota : P \to N$ is a homeomorphism onto its image. Moreover, we have seen above that $\iota$ is an immersion, whence it is an embedding. $\square$

Recall that a continuous map between locally compact, Hausdorff topological spaces is called *proper* if the inverse image of a compact subset of the target space is a compact subset of the domain. It is known that proper maps are closed. Also, it is clear that if the domain is compact, then every continuous map is automatically proper. An embedded submanifold $M$ of a smooth manifold $N$ is called *properly embedded* if the inclusion map is proper.

**1.4.6 Proposition** *If $f : M \to N$ is an injective immersion which is also a proper map, then the image $f(M)$ is a properly embedded submanifold of $N$.*

*Proof.* Let $P = f(M)$ have the relative topology. A proper map is closed. Since $f$ viewed as a map $M \to P$ is bijective and closed, it is an open map and thus a homeomorphism. Due to Proposition 1.4.5, $P$ is an embedded submanifold of $N$. The properness of the inclusion $P \to N$ clearly follows from that of $f$. $\square$

**1.4.7 Exercise** Give an example of an embedded submanifold of a smooth manifold which is not properly embedded.

**1.4.8 Exercise** Decide whether a closed embedded submanifold of a smooth manifold is necessarily properly embedded.

Exercise 1.1.4 dealt with a situation in which a smooth map $f : M \to N$ factors through an immersed submanifold $(P, g)$ of $N$ (namely, $f(M) \subset g(P)$) and the induced map $f_0 : M \to P$ (namely, $g \circ f_0 = f$) is discontinuous.

**1.4.9 Proposition** *Suppose that $f : M \to N$ is smooth and $(P, g)$ is an immersed submanifold of $N$ such that $f(M) \subset g(P)$. Consider the induced map $f_0 : M \to P$ that satisfies $g \circ f_0 = f$.*
*a. If $g$ is an embedding, then $f_0$ is continuous.*
*b. If $f_0$ is continuous, then it is smooth.*

*Proof.* (a) In this case $g$ is a homeomorphism onto $g(P)$ with the relative topology. If $V \subset P$ is open, then $g(V) = W \cap g(P)$ for some open subset $W \subset N$. By continuity of $f$, we have that $f_0^{-1}(V) = f_0^{-1}(g^{-1}(W)) = f^{-1}(W)$ is open in $M$, hence also $f_0$ is continuous.

(b) Let $p \in M$ and $q = f_0(p) \in P$. By Proposition 1.4.4, there exists a neighborhood $U$ of $q$ and a local chart $(V, \psi)$ of $N^n$ adapted to $g(U)$, with $g(U) \subset V$. In particular, there exists a projection $\pi$ from $\mathbf{R}^n$ onto a subspace obtained by setting some coordinates equal to 0 such that $\tau = \pi \circ \psi \circ g$ is a local chart of $P$ around $q$. Note that $f_0^{-1}(U)$ is a neighborhood of $p$ in $M$. Now

$$\tau \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ g \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ f|_{f_0^{-1}(U)},$$

and the latter is smooth. □

An immersed submanifold $(P, g)$ of $N$ with the property that $f_0 : M \to P$ is smooth for every smooth map $f : M \to N$ with $f(M) \subset g(P)$ will be called an *initial submanifold*.

**1.4.10 Exercise** Use Exercise 1.3.9 and Propositions 1.4.5 and 1.4.9 to deduce that an embedding $f : M \to N$ induces a diffeomorphism from $M$ onto a submanifold of $N$.

**1.4.11 Exercise** For an immersed submanifold $(M, f)$ of $N$, show that there is a natural structure of smooth manifold on $f(M)$ and that $(f(M), \iota)$ is an immersed submanifold of $N$, where $\iota : f(M) \to N$ denotes the inclusion.

### Submersions

A smooth map $f : M \to N$ is called a *submersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is surjective. If $f$ is a submersion everywhere it is simply called a *submersion*. A point $q \in N$ is called a *regular value* of $f$ if $f$ is a submersion at all points in $f^{-1}(q)$; otherwise $q$ is called a *singular value* of $f$.

**1.4.12 Lemma (Local form of a submersion)** *Let $M$ an $N$ be smooth manifolds of dimensions $n+k$ and $k$, respectively, and suppose that $f : M \to N$ is a submersion at $p \in M$. Then there exist local charts of $M$ and $N$ such that the local expression of $f$ at $p$ is the standard projection of $\mathbf{R}^{n+k}$ onto $\mathbf{R}^k$.*

*Proof.* Let $(U, \varphi)$ and $(V, \psi)$ be local charts of $M$ and $N$ around $p$ and $q = f(p)$, respectively, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^{n+k} \to \mathbf{R}^k$ is surjective, so, up to rearranging indices, we can assume that $d(\alpha \circ \iota_2)_{\varphi(p)} = d\alpha_{\varphi(p)} \circ \iota_2 : \mathbf{R}^k \to \mathbf{R}^k$ is an isomorphism, where $\iota_2 : \mathbf{R}^k \to \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k$ is the standard inclusion. Define $\tilde{\alpha} : \varphi(U) \subset \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k$ by $\tilde{\alpha}(x, y) = (x, \alpha(x, y))$. Since $d\alpha_{\varphi(p)} \circ \iota_2$ is an isomorphism, it is clear that $d\tilde{\alpha}_{\varphi(p)} : \mathbf{R}^n \oplus \mathbf{R}^k \to \mathbf{R}^n \oplus \mathbf{R}^k$ is an isomorphism. By the inverse function theorem, there exists an open neighborhood $U_0$ of $\varphi(p)$ contained in $\varphi(U)$ such that $\tilde{\alpha}$ is a diffeomorphism from $U_0$ onto its image $V_0$; let $\tilde{\beta} : V_0 \to U_0$ be its smooth inverse. We put $\tilde{\varphi} = \tilde{\alpha} \circ \varphi$. Then $(\varphi^{-1}(U_0), \tilde{\varphi})$ is a local chart of $M$ around $p$ and

$$\begin{aligned} \psi \circ f \circ \tilde{\varphi}^{-1}(x, y) &= \psi \circ f \circ \varphi^{-1} \circ \tilde{\beta}(x, y) = \alpha \circ \tilde{\beta}(x, y) \\ &= \pi_2 \circ \tilde{\alpha} \circ \tilde{\beta}(x, y) = y. \end{aligned}$$

□

**1.4.13 Proposition** *Let $f : M \to N$ be a smooth map, and let $q \in N$ be a regular value of $f$ such that $f^{-1}(q) \neq \varnothing$. Then $P = f^{-1}(q)$ is an embedded submanifold of $M$ of dimension $\dim M - \dim N$. Moreover, for $p \in P$ we have $T_pP = \ker df_p$.*

*Proof.* It is enough to construct local charts of $M$ that are adapted to $P$ and whose domains cover $P$. So suppose $\dim M = n + k$, $\dim N = k$, let $p \in P$ and consider local charts $(W := \varphi^{-1}(U_0), \tilde{\varphi})$ and $(V, \psi)$ as in Theorem 1.4.12 such that $p \in U$ and $q \in V$. We can assume that $\psi(q) = 0$. Now

$$\pi_2 \circ \tilde{\varphi}(W \cap P) = \alpha \circ \varphi(W \cap P) = \psi \circ f(W \cap P) = \{0\},$$

so $\tilde{\varphi}(W \cap P) = \tilde{\varphi}(W) \cap \mathbf{R}^n$ and thus $\varphi$ is an adapted chart around $p$. Finally, the local representation of $f$ at $p$ is the projection $\mathbf{R}^{n+k} \to \mathbf{R}^k$. This is a linear map with kernel $\mathbf{R}^n$. It follows that $\ker df_p = (d\tilde{\varphi}^{-1})_{\varphi(p)}(\mathbf{R}^n) = T_pP$. □

**1.4.14 Examples** (a) Let $A$ be a non-singular real symmetric matrix of order $n + 1$ and define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(p) = \langle Ap, p \rangle$ where $\langle , \rangle$ is the standard Euclidean inner product. Then $df_p : \mathbf{R}^{n+1} \to \mathbf{R}$ is given by $df_p(v) = 2\langle Ap, v \rangle$, so it is surjective if $p \neq 0$. It follows that $f$ is a submersion on $\mathbf{R}^{n+1} \setminus \{0\}$, and then $f^{-1}(r)$ for $r \in \mathbf{R}$ is an embedded submanifold of $\mathbf{R}^{n+1}$ of dimension $n$ if it is nonempty. In particular, by taking $A$ to be the identity matrix we get a manifold structure for $S^n$ which coincides with the one previously constructed.

(b) Denote by $Sym(n, \mathbf{R})$ the vector space of real symmetric matrices of order $n$, and define $f : M(n, \mathbf{R}) \to Sym(n, \mathbf{R})$ by $f(A) = AA^t$. This is map between vector spaces whose local representations components are quadratic polynomials. It follows that $f$ is smooth and that $df_A$ can be viewed as a map $M(n, \mathbf{R}) \to Sym(n, \mathbf{R})$ for all $A \in M(n, \mathbf{R})$. We claim that $I$ is a regular value of $f$. For the purpose of checking that, we first compute for $A \in f^{-1}(I)$ and $B \in M(n, \mathbf{R})$ that

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \frac{(A+hB)(A+hB)^t - I}{h} \\ &= \lim_{h\to 0} \frac{h(AB^t + BA^t) + h^2 BB^t}{h} \\ &= AB^t + BA^t. \end{aligned}$$

Now given $C \in Sym(n, \mathbf{R})$, we have $df_A(\frac{1}{2}CA) = C$, and this proves that $f$ is a submersion at $A$, as desired. Hence $f^{-1}(I) = \{ A \in ML(n, \mathbf{R}) \mid AA^t = I \}$ is an embedded submanifold of $M(n, \mathbf{R})$ of dimension

$$\dim M(n, \mathbf{R}) - \dim V = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Note that $f^{-1}(I)$ is a group with respect to the multiplication of matrices; it is called the *orthogonal group* of order $n$ and is usually denoted by $O(n)$. It is obvious that $O(n) \subset GL(n, \mathbf{R})$.

We close this section by mentioning a generalization of Proposition 1.4.13. Let $f : M \to N$ be a smooth map and let $Q$ be an embedded submanifold of $N$. We say that $f$ is *transverse* to $Q$, in symbols $f \pitchfork Q$, if

$$df_p(T_pM) + T_{f(p)}Q = T_{f(p)}N$$

for every $p \in f^{-1}(Q)$.

**1.4.15 Exercise** Let $f : M \to N$ be a smooth map and let $q \in N$. Prove that $f \pitchfork \{q\}$ if and only if $q$ is a regular value of $f$.

For an immersed submanifold $(M, f)$ of a smooth manifold $N$, its *codimension* is the number $\dim N - \dim M$.

**1.4.16 Proposition** *If $f : M \to N$ is a smooth map which is transverse to an embedded submanifold $Q$ of $N$ of codimension $k$ and $P = f^{-1}(Q)$ is non-empty, then $P$ is an embedded submanifold of $M$ of codimension $k$. Moreover $T_pP = (df_p)^{-1}(T_{f(p)}Q)$ for every $p \in P$.*

*Proof.* For the first assertion, it suffices to check that $P$ is an embedded submanifold of $M$ in a neighborhod of a point $p \in P$. Let $(V, \psi)$ be a local chart of $N$ adapted to $Q$ around $q := f(p)$. Then $\psi : V \to \mathbf{R}^{n+k}$ and

$\psi(V \cap Q) = \psi(V) \cap \mathbf{R}^n$, where $n = \dim Q$. Let $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ be the standard projection and put $g = \pi_2 \circ \psi$. Then $g : V \to \mathbf{R}^k$ is a submersion and $g^{-1}(0) = V \cap Q$. Moreover

$$\begin{aligned} d(g \circ f)_p(T_pM) &= dg_q \circ df_p(T_pM) \\ &= dg_q(T_qN) \\ &= \mathbf{R}^k \end{aligned}$$

where, in view of $\ker dg_q = T_qQ$, the second equality follows from the assumption $f \pitchfork Q$. Now $h := g \circ f : f^{-1}(V) \to \mathbf{R}^k$ is a submersion at $p$ and $h^{-1}(0) = f^{-1}(V \cap Q) = f^{-1}(V) \cap P$ and $f^{-1}(V)$ is an open neighborhood of $p$ in $M$, so we can apply Proposition 1.4.13. All the assertions follow. $\square$

As a most important special case, two embedded submanifolds $M$, $P$ of $N$ are called *transverse*, denoted $M \pitchfork P$, if the inclusion map $\iota : M \to N$ is transverse to $P$. It is easy to see that this is a symmetric relation.

**1.4.17 Corollary** *If $M$ and $P$ are transverse embedded submanifolds of $N$ then $M \cap P$ is an embedded submanifold of $N$ and*

$$\operatorname{codim}(M \cap P) = \operatorname{codim}(M) + \operatorname{codim}(P).$$

## 1.5 Partitions of unity

Many important constructions for smooth manifolds rely on the existence of smooth partitions of unity. This technique allows for a much greater flexibility of smooth manifolds as compared, for instance, with real analytic or complex manifolds.

### Bump functions

We start with the remark that the function

$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is smooth everywhere. Therefore the function

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

is smooth, flat and equal to $0$ on $(-\infty, 0]$, and flat and equal to $1$ on $[1, +\infty)$. Finally,

$$h(t) = g(t+2)g(2-t)$$

is smooth, flat and equal to 1 on $[-1,1]$ and its support lies in $(-2,2)$; $h$ is called a *bump function*. We can also make an $n$-dimensional version of a bump function by setting

$$k(x_1,\ldots,x_n)=h(x_1)\cdots h(x_n),$$

and we can rescale $k$ by precomposing with $x\mapsto r^{-1}x$ to have a smooth function on $\mathbf{R}^n$ which is flat and equal to 1 on a closed ball of radius $r$ and with support contained in an open ball of radius $2r$.

Bump functions are very useful. As one application, note that for a given smooth manifold $M$ so far we do not know whether the algebra $C^\infty(M)$ of smooth functions on $M$ contains functions other than the constants (of course, the components of local charts are smooth, but these are not *globally* defined on $M$). We claim that $C^\infty(M)$ is indeed in general huge. In fact, let $(U,\varphi)$ be a local chart of $M$ and take a bump function $k:\mathbf{R}^n\to\mathbf{R}$ whose support lies in $\varphi(U)$. Then

$$f(x):=\begin{cases} k\circ\varphi(x) & \text{if} \in U,\\ 0 & \text{if } x\in M\setminus U\end{cases}$$

is a smooth function on $M$: this is clear for a point $p\in U$; if $p\notin U$, then we can find a neighborhood $V$ of $p$ which does not meet the compact set $\varphi^{-1}(\mathrm{supp}(k))$, so $f|_V=0$ and thus $f$ is smooth at $p$.

**Partitions of unity**

Let $M$ be a smooth manifold. A *partition of unity* on $M$ is a collection $\{\rho_i\}_{i\in I}$ of smooth functions on $M$, where $I$ is an index set, such that:

(i) $\rho_i(p)\geq 0$ for all $p\in M$ and all $i\in I$;

(ii) the collection of supports $\{\mathrm{supp}(\rho)\}_{i\in I}$ is locally finite (i.e. every point of $M$ admits a neighborhood meeting $\mathrm{supp}(\rho_i)$ for only finitely many indices $i$);

(iii) $\sum_{i\in I}\rho_i(p)=1$ for all $p\in M$ (the sum is finite in view of (ii)).

Let $\{U_\alpha\}_{\alpha\in A}$ be a cover of $M$ by open sets. We say that a partition of unity $\{\rho_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in A}$ if for every $i\in I$ there is some $\alpha\in A$ such that $\mathrm{supp}(\rho_i)\subset U_\alpha$; and we say $\{\rho_i\}_{i\in I}$ is *strictly subordinate* to $\{U_\alpha\}_{\alpha\in A}$ if $I=A$ and $\mathrm{supp}(\rho_\alpha)\subset U_\alpha$ for every $\alpha\in A$.

Partitions of unity are used to piece together global objects out of local ones, and conversely to decompose global objects as locally finite sums of locally defined ones. For instance, suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of $M$ and $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity strictly subordinate to $\{U_\alpha\}$. If we are given $f_\alpha\in C^\infty(U_\alpha)$ for all $\alpha\in A$, then $f=\sum_{\alpha\in A}\rho_\alpha f_\alpha$ is a smooth function on $M$. Indeed for $p\in M$ and $\alpha\in A$, it is true that either $p\in U_\alpha$ and then $f_\alpha$ is defined at $p$, or $p\notin U_\alpha$ and then $\rho_\alpha(p)=0$. Moreover, since the sum is locally finite, $f$ is locally the sum of finitely many smooth functions and hence smooth. Conversely, if we start with $f \in C^\infty(M)$ then $f = \sum_{\alpha\in A} f_\alpha$ for smooth functions $f_\alpha$ with $\mathrm{supp}(f_\alpha) \subset U_\alpha$, namely, $f_\alpha := \rho_\alpha f$.

**1.5.1 Exercise** Let $C$ be closed in $M$ and let $U$ be open in $M$ with $C \subset U$. Prove that there exists a smooth function $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda|_C = 1$ and $\mathrm{supp}\,\lambda \subset U$.

If $M$ is compact, it is a lot easier to prove the existence of a partition of unity subordinate to any given open cover $\{U_\alpha\}$ of $M$. In fact for each $x \in U_\alpha$ we construct as above a bump function $\lambda_x$ which is flat and equal to $1$ on a neighborhood $V_x$ of $x$ and whose (compact) support lies in $U_\alpha$. Owing to compactness of $M$, we can extract a finite subcover of $\{V_x\}$ and thus we get non-negative smooth functions $\lambda_i := \lambda_{x_i}$ for $i = 1, \ldots, n$ such that $\lambda_i$ is $1$ on $V_{x_i}$. In particular, their sum is positive, so

$$\rho_i := \frac{\lambda_i}{\sum_{i=1}^n \lambda_i}$$

for $i = 1, \ldots, n$ yields the desired partition of unity.

**1.5.2 Theorem (Easy Whitney embedding theorem)** *Let $M$ be a compact smooth manifold. Then there exists an embedding of $M$ into $\mathbf{R}^m$ for $m$ suffciently big.*

*Proof.* Since $M$ is compact, there exists an open covering $\{V_i\}_{i=1}^a$ such that for each $i$, $\bar{V}_i \subset U_i$ where $(U_i, \varphi_i)$ is a local chart of $M$. For each $i$, we can find $\rho_i \in C^\infty(M)$ such that $0 \leq \rho_i \leq 1$, $\rho_i|_{\bar{V}_i} = 1$ and $\mathrm{supp}\,\rho_i \subset U_i$. Put

$$f_i(x) = \begin{cases} \rho_i(x)\varphi_i(x), & \text{if } x \in U_i, \\ 0, & \text{if } x \in M \setminus U_i. \end{cases}$$

Then $f_i : M \to \mathbf{R}^n$ is smooth, where $n = \dim M$. Define also smooth functions

$$g_i = (f_i, \rho_i) : M \to \mathbf{R}^{n+1} \quad \text{and} \quad g = (g_1, \ldots, g_a) : M \to \mathbf{R}^{a(n+1)}.$$

It is enough to check that $g$ is an injective immersion. In fact, on the open set $V_i$, we have that $g_i = (\varphi_i, 1)$ is an immersion, so $g$ is an immersion. Further, if $g(x) = g(y)$ for $x$, $y \in M$, then $\rho_i(x) = \rho_i(y)$ and $f_i(x) = f_i(y)$ for all $i$. Take an index $j$ such that $\rho_j(x) = \rho_j(y) \neq 0$. Then $x$, $y \in U_j$ and $\varphi_j(x) = \varphi_j(y)$. Due to the injectivity of $\varphi_j$, we must have $x = y$. Hence $g$ is injective. □

**1.5.3 Remark** In the noncompact case, one can still construct partitions of unity and modify the proof of Theorem 1.5.2 to prove that $M$ properly embedds into $\mathbf{R}^m$ for some $m$. Then a standard trick involving Sard's theorem and projections into lower dimensional subspaces of $\mathbf{R}^m$ allows to find the bound $m \leq 2n + 1$, where $n = \dim M$. A more difficult result, the *strong Whitney embedding theorem* asserts that in fact $m \leq 2n$.

In general, a reasonable substitute for compactness is paracompactness. A topological space is called *paracompact* if every open covering admits an open locally finite refinement. It turns out that every locally compact, second countable, Hausdorff space is paracompact. Hence manifolds are paracompact. Now the above argument can be extended to give the following theorem, for whose proof we refer the reader to [War83].

**1.5.4 Theorem (Existence of partitions of unity)** *Let $M$ be a smooth manifold and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $M$. Then there exists a countable partition of unity $\{\rho_i : i = 1, 2, 3, \ldots\}$ subordinate to $\{U_\alpha\}$ with $\mathrm{supp}(\rho_i)$ compact for each $i$. If one does not require compact supports, then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ strictly subordinate to $\{U_\alpha\}$ with at most countably many of the $\rho_\alpha$ not zero.*

## 1.6 Vector fields

Let $M$ be a smooth manifold. A *vector field* on $M$ is an assigment of a tangent vector $X(p)$ in $T_pM$ for all $p \in M$. Sometimes, we also write $X_p$ instead of $X(p)$. So a vector field is a map $X : M \to TM$ where $TM = \dot{\cup}_{p\in M} T_pM$ (disjoint union), and

$$\pi \circ X = \mathrm{id} \tag{1.6.1}$$

where $\pi : TM \to M$ is the natural projection $\pi(v) = p$ if $v \in T_pM$. In account of property (1.6.1), we say that $X$ is a *section* of $TM$.

We shall need to talk about continuity and differentiability of vector fields, so we next explain that $TM$ carries a canonical manifold structure induced from that of $M$.

### The tangent bundle

Let $M$ be a smooth manifold and consider the disjoint union

$$TM = \dot{\bigcup}_{p\in M} T_pM.$$

We can view the elements of $TM$ as equivalence classes of triples $(p, a, \varphi)$, where $p \in M$, $a \in \mathbf{R}^n$ and $(U, \varphi)$ is a local chart of $M$ such that $p \in U$, and

$$(p, a, \varphi) \sim (q, b, \psi) \quad \text{if and only if } p = q \text{ and } d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

There is a natural projection $\pi : TM \to M$ given by $\pi[p, a, \varphi] = p$, and then $\pi^{-1}(p) = T_pM$.

Suppose $\dim M = n$. Note that we have $n$ degrees of freedom for a point $p$ in $M$ and $n$ degrees of freedom for a vector $v \in T_pM$, so we expect

$TM$ to be $2n$-dimensional. We will use Proposition 1.2.10 to simultaneously introduce a topology and smooth structure on $TM$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for $M$ with countably many elements (recall that every second countable space is Lindelöf). For each $\alpha$, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a diffeomorphism and, for each $p \in U_\alpha$, $d(\varphi_\alpha)_p : T_pU_\alpha = T_pM \to \mathbf{R}^n$ is the isomorphism mapping $[p, a, \varphi]$ to $a$. Set

$$\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbf{R}^n, \qquad [p, a, \varphi] \to (\varphi_\alpha(p), a).$$

Then $\tilde{\varphi}_\alpha$ is a bijection and $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbf{R}^{2n}$. Moreover, the maps

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

are defined on open subsets of $\mathbf{R}^{2n}$ and are given by

$$(x, a) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(x)\,,\, d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)).$$

Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth diffeomorphism, we have that $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x$ is a linear isomorphism and $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)$ is also smooth on $x$. It follows that $\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}$ defines a topology and a smooth atlas for $M$ and we need only to check the Hausdorff condition. Namely, let $v$, $w \in TM$ with $v \neq w$. Note that $\pi$ is an open map. If $v$, $w \in TM$ and $\pi(v) \neq \pi(w)$, we can use the Hausdorff property of $M$ to separate $v$ and $w$ from each other with open sets of $TM$. On the other hand, if $v$, $w \in T_pM$, they lie in the domain of the same local chart of $TM$ and the result also follows.

Note that, in particular, we have shown that every system of local coordinates $(x_1, \ldots, x_n)$ on an open subset $U$ of $M$ induces a system of local coordinates $(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ on $TM|_U$.

If $f \in C^\infty(M, N)$, then we define the *differential of $f$* to be the map

$$df : TM \to TN$$

that restricts to $df_p : T_pM \to T_{f(p)}N$ for each $p \in M$. Using the above atlases for $TM$ and $TN$, we immediately see that $df \in C^\infty(TM, TN)$.

**1.6.2 Remark** The mapping that associates to each manifold $M$ its tangent bundle $TM$ and associates to each smooth map $f : M \to N$ its tangent map $df : TM \to TN$ can be thought of a functor $\mathbf{DIFF} \to \mathbf{VB}$ from the category of smooth manifolds to the category of smooth vector bundles. In fact, $d(\mathrm{id}_M) = \mathrm{id}_{TM}$, and $d(g \circ f) = dg \circ df$ for a sequence of smooth maps $M \xrightarrow{f} N \xrightarrow{g} P$.

### Smooth vector fields

A vector field $X$ on $M$ is called *smooth* (resp. *continuous*) if the map $X : M \to TM$ is smooth (resp. continuous).

More generally, let $f : M \to N$ be a smooth mapping. Then a (smooth, continuous) *vector field along* $f$ is a (smooth, continuous) map $X : M \to TN$ such that $X(p) \in T_{f(p)}N$ for $p \in M$. The most important case is that in which $f$ is a smooth curve $\gamma : [a, b] \to N$. A vector field along $\gamma$ is a map $X : [a, b] \to TN$ such that $X(t) \in T_{\gamma(t)}N$ for $t \in [a, b]$. A typical example is the tangent vector field $\dot{\gamma}$.

For practical purposes, we reformulate the notion of smoothness as follows. Let $X$ be a vector field on $M$. Given a smooth function $f \in C^\infty(U)$ where $U$ is an open subset of $M$, the directional derivative $X(f) : U \to \mathbf{R}$ is defined to be the function $p \in U \mapsto X_p(f)$. Further, if $(x_1, \ldots, x_n)$ is a coordinate system on $U$, we have already seen that $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of $T_pM$ for $p \in U$. It follows that there are functions $a_i : U \to \mathbf{R}$ such that

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}. \tag{1.6.3}$$

**1.6.4 Proposition** *Let $X$ be a vector field on $M$. Then the following assertions are equivalent:*

*a. $X$ is smooth.*

*b. For every coordinate system $(U, (x_1, \ldots, x_n))$ of $M$, the functions $a_i$ defined by (1.6.3) are smooth.*

*c. For every open set $V$ of $M$ and $f \in C^\infty(V)$, the function $X(f) \in C^\infty(V)$.*

*Proof.* Suppose $X$ is smooth and let $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ be a coordinate system on $U$. Then $X|_U$ is smooth and $a_i = dx_i \circ X|_U$ is also smooth.

Next, assume (b) and let $f \in C^\infty(V)$. Take a coordinate system

$$(U, (x_1, \ldots, x_n))$$

with $U \subset V$. Then, by using (b) and the fact that $\frac{\partial f}{\partial x_i}$ is smooth,

$$X(f)|_U = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} \in C^\infty(U).$$

Since $V$ can be covered by such $U$, this proves (c).

Finally, assume (c). For every coordinate system $(U, (x_1, \ldots, x_n))$ of $M$, we have a corresponding coordinate system $(\pi^{-1}(U), x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$ of $TM$. Then

$$(x_i \circ \pi) \circ X|_U = x_i \quad \text{and} \quad dx_i \circ X|_U = X(x_i)$$

are smooth. This proves that $X$ is smooth. $\square$

In particular, the proposition shows that the coordinate vector fields $\frac{\partial}{\partial x_i}$ associated to a local chart are smooth. Since $a_i = X(x_i)$ in (1.6.3), we have

**1.6.5 Scholium** *If $X$ is a smooth vector field on $M$ and $X(f) = 0$ for every smooth function, then $X = 0$.*

**1.6.6 Remark** Part (c) of Proposition 1.6.4 in fact says that every smooth vector field on $M$ defines a *derivation* of the algebra $C^\infty(M)$, namely, a differential operator that maps constants to zero and satisfies the Leibniz identity $X(fg) = X(f)g + fX(g)$.

### Flow of a vector field

We have now come to the integration of vector fields. Let $\varphi_t : M \to M$ be a diffeomorphism such that the curve $t \mapsto \varphi_t(p)$ is smooth for each $p$. Then $X_p := \frac{d}{dt}\big|_{t=0}\varphi_t(p)$ defines a vector field on $M$. Conversely, one can integrate smooth vector fields to obtain (local) diffeomorphisms. Actually, this is the extension of ODE theory to smooth manifolds that we discuss below.

An *integral curve* of $X$ is a smooth curve $\gamma : I \to M$, where $I$ is an open interval, such that

$$\dot\gamma(t) = X(\gamma(t))$$

for all $t \in I$. We write this equation in local coordinates. Suppose $X$ has the form (1.6.3), $\gamma_i = x_i \circ \gamma$ and $\tilde a_i = a_i \circ \varphi^{-1}$. Then $\gamma$ is an integral curve of $X$ in $\gamma^{-1}(U)$ if and only if

$$\frac{d\gamma_i}{dr}\bigg|_t = \tilde a_i(\gamma_1(t), \ldots, \gamma_n(t)) \tag{1.6.7}$$

for $i = 1, \ldots, n$ and $t \in \gamma^{-1}(U)$. Equation (1.6.7) is a system of first order ordinary differential equations for which existence and uniqueness theorems are known. These, translated into manifold terminology yield local existence and uniqueness of integral curves for smooth vector fields. Moreover, one can cover $M$ by domains of local charts and, using uniqueness, piece together the locally defined integral curves of $X$ to obtain, for any given point $p \in M$, a *maximal* integral curve $\gamma_p$ of $X$ through $p$ defined on a possibly infinite interval $(a(p), b(p))$.

Even more interesting is to reverse the rôles of $p$ and $t$ by setting

$$\varphi_t(p) := \gamma_p(t)$$

for all $p$ such that $t \in (a(p), b(p))$. The smooth dependence of solutions of ODE on the initial conditions implies that for every $p \in M$, there exists an open neighborhood $V$ of $p$ and $\epsilon > 0$ such that the map

$$(-\epsilon, \epsilon) \times V \to M, \quad (t, q) \mapsto \varphi_t(q) \tag{1.6.8}$$

is well defined and smooth. The same theorem also shows that, for fixed $t > 0$, the domain of $\varphi_t$ is an open subset $\mathcal{D}_t$ of $M$.

The uniqueness of solutions of ODE with given initial conditions implies that

$$\varphi_{s+t} = \varphi_s \circ \varphi_t \tag{1.6.9}$$

whenever both hand sides are defined. In fact, for each $t$, the curve $s \mapsto \varphi_{s+t}(p)$ is an integral curve of $X$ passing through the point $\varphi_t(p)$ at $s = 0$, so it must locally coincide with $\varphi_s(\varphi_t(p))$.

Obviously $\varphi_0$ is the identity, so $\varphi_t$ is a diffeomorphism $\mathcal{D}_t \to \mathcal{D}_{-t}$ with inverse $\varphi_{-t}$. The collection $\{\varphi_t\}$ is called the *flow* of $X$. Owing to property (1.6.9), the flow of $X$ is also called the *one-parameter local group* of locally defined diffeomorphisms generated by $X$, and $X$ is called the *infinitesimal generator* of $\{\varphi_t\}$. If $\varphi_t$ is defined for all $t \in \mathbf{R}$, the vector field $X$ is called *complete*. This is equivalent to requiring that the maximal integral curves of $X$ be defined on the entire $\mathbf{R}$, or yet, that the domain of each $\varphi_t$ be $M$. In this case we refer to $\{\varphi_t\}$ as the *one-parameter group* of diffeomorphisms of $M$ generated by $X$.

**1.6.10 Proposition** *Every smooth vector field $X$ defined on a compact smooth manifold $M$ is complete.*

*Proof.* If $M$ is compact, we can find a finite open covering $\{V_i\}$ of it and $\epsilon_i > 0$ such that $(-\epsilon_i, \epsilon_i) \times V_i \to M$, $(t, p) \mapsto \varphi_t(p)$ is well defined and smooth for all $i$, as in (1.6.8). Let $\epsilon = \min_i\{\epsilon_i\}$. Now this map is defined on $(-\epsilon, \epsilon) \times M \to M$. This means that any integral curve of $X$ starting at any point of $M$ is defined at least on the interval $(-\epsilon, \epsilon)$. The argument using the uniqueness of solutions of ODE as in (1.6.9) and piecing together integral curves of $X$ shows that any integral curve of $X$ is defined on $(-k\epsilon, k\epsilon)$ for all positive integer $k$, hence it is defined on $\mathbf{R}$. $\square$

**1.6.11 Examples** (a) Take $M = \mathbf{R}^2$ and $X = \frac{\partial}{\partial x_1}$. Then $X$ is complete and $\varphi_t(x_1, x_2) = (x_1 + t, x_2)$ for $(x_1, x_2) \in \mathbf{R}^2$. Note that if we replace $\mathbf{R}^2$ by the punctured plane $\mathbf{R}^2 \setminus \{(0, 0)\}$, the domains of $\varphi_t$ become proper subsets of $M$.

(b) Consider the smooth vector field on $\mathbf{R}^{2n}$ defined by

$$X(x_1, \ldots, x_{2n}) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \cdots - x_{2n}\frac{\partial}{\partial x_{2n-1}} + x_{2n-1}\frac{\partial}{\partial x_{2n}}.$$

The flow of $X$ is given the linear map

$$\varphi_t \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} R_t & & \\ & \ddots & \\ & & R_t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix}$$

where $R_t$ is the $2 \times 2$ block

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

It is clear that $X$ restricts to a smooth vector field $\bar{X}$ on $S^{2n-1}$. The flow of $\bar{X}$ is of course the restriction of $\varphi_t$ to $S^{2n-1}$. $X$ and $\bar{X}$ are complete vector fields.

(c) Take $M = \mathbf{R}$ and $X(x) = x^2 \frac{\partial}{\partial x}$. Solving the ODE we find $\varphi_t(x) = \frac{x}{1-tx}$. It follows that the domain of $\varphi_t$ is $(-\infty, \frac{1}{t})$ if $t > 0$ and $(\frac{1}{t}, +\infty)$ if $t < 0$.

**Lie bracket**

If $X$ is a smooth vector field on $M$ and $f : M \to \mathbf{R}$ is a smooth function, the directional derivative $X(f) : M \to \mathbf{R}$ is also smooth and so it makes sense to derivate it again as in $Y(X(f))$ where $Y$ is another smooth vector field on $M$. For instance, in a local chart $(U, \varphi = (x_1, \ldots, x_n))$, we have the first order partial derivative

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial f}{\partial x_i}\Big|_p$$

and the second order partial derivative

$$\left(\frac{\partial}{\partial x_j}\right)_p \left(\frac{\partial}{\partial x_i}(f)\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}\Big|_p$$

and it follows from Schwarz theorem on the commutativity of mixed partial derivatives of smooth functions on $\mathbf{R}^n$ that

$$\text{(1.6.12)} \qquad \frac{\partial^2 f}{\partial x_j \partial x_i}\Big|_p = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial r_j \partial r_i}\Big|_p = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial r_i \partial r_j}\Big|_p = \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p,$$

where $\mathrm{id} = (r_1, \ldots, r_n)$ denote the canonical coordinates on $\mathbf{R}^n$.

On the other hand, for general smooth vector fields $X$, $Y$ on $M$ the second derivative depends on the order of the vector fields and the failure of the commutativity is measured by the *commutator* or *Lie bracket*

$$\text{(1.6.13)} \qquad [X, Y](f) = X(Y(f)) - Y(X(f))$$

for every smooth function $f : M \to \mathbf{R}$. We say that $X$, $Y$ *commute* if $[X, Y] = 0$. It turns out that formula (1.6.13) defines a smooth vector field on $M$! Indeed, Scholium 1.6.5 says that such a vector field is unique, if it exists. In order to prove existence, consider a coordinate system $(U, (x_1, \ldots, x_n))$. Then we can write

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y|_U = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$$

for $a_i, b_j \in C^\infty(U)$. If $[X,Y]$ exists, we must have

$$(1.6.14) \qquad [X,Y]|_U = \sum_{i,j=1}^n \left( a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

because the coefficients of $[X,Y]|_U$ in the local frame $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ must be given by $[X,Y](x_j) = X(Y(x_j)) - Y(X(x_j))$. We can use formula (1.6.14) as the definition of a vector field on $U$; note that such a vector field is smooth and satisfies property (1.6.13) for functions in $C^\infty(U)$. We finally define $[X,Y]$ globally by covering $M$ with domains of local charts: on the overlap of two charts, the different definitions coming from the two charts must agree by the above uniqueness result; it follows that $[X,Y]$ is well defined.

**1.6.15 Examples** (a) Schwarz theorem (1.6.12) now means $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for coordinate vector fields associated to a local chart.

(b) Let $X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial z}$ be smooth vector fields on $\mathbf{R}^3$. Then $[X,Y] = Z$, $[Z,X] = [Z,Y] = 0$.

**1.6.16 Proposition** *Let $X$, $Y$ and $Z$ be smooth vector fields on $M$. Then*

*a.* $[Y,X] = -[X,Y]$.

*b. If $f, g \in C^\infty(M)$, then*

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

*c.* $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. *(Jacobi identity)*

**1.6.17 Exercise** Prove Proposition 1.6.16. (Hint: Use (1.6.13).)

Let $f : M \to N$ be a diffeomorphism. For every smooth vector field $X$ on $M$, the formula $df \circ X \circ f^{-1}$ defines a smooth vector field on $N$, called the *push-forward* of $X$ under $f$, which we denote by $f_*X$. If the flow of $X$ is $\{\varphi_t\}$, then the flow of $f_*X$ is $f \circ \varphi_t \circ f^{-1}$, as

$$\frac{d}{dt} f(\varphi_t(f^{-1}(p))) = df\left(\frac{d}{dt}\varphi_t(f^{-1}(p))\right) = df(X_{f^{-1}(p)}).$$

More generally, if $f : M \to N$ is a smooth map which needs not be a diffeomorphism, smooth vector fields $X$ on $M$ and $Y$ on $N$ are called $f$-*related* if $df \circ X = Y \circ f$.

$$\begin{tikzcd} TM \arrow[r, "df"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, dotted, "f_*X"'] \end{tikzcd}$$

**1.6.18 Proposition** *Let $f : M \to M'$ be smooth. Let $X, Y$ be smooth vector fields on $M$, and let $X', Y'$ be smooth vector fields on $M'$. If $X$ and $X'$ are $f$-related and $Y$ and $Y'$ are $f$-related, then also $[X, Y]$ and $[X', Y']$ are $f$-related.*

*Proof*. Let $h \in C^\infty(M')$ and $q \in M$. Note first that

$$\begin{aligned} X_q(h \circ f) &= d(h \circ f)(X_q) \\ &= dh(df(X_q)) \\ &= (df \circ X)_q(h) \\ &= X'_{f(q)}(h), \end{aligned}$$

namely,

$$X(h \circ f) = X'(h) \circ f. \tag{1.6.19}$$

Similarly, $Y(h \circ f) = Y'(h) \circ f$.

We now prove $df \circ [X, Y] = [X', Y'] \circ f$. Let $g \in C^\infty(M')$ and $p \in M$. Use (1.6.13) and the above identities:

$$\begin{aligned} df([X, Y]_p)(g) &= [X, Y]_p(g \circ f) \\ &= X_p(Y(g \circ f)) - Y_p(X(g \circ f)) \\ &= X_p(Y'(g) \circ f) - Y_p(X'(g) \circ f) \\ &= X'_{f(p)}(Y'(g)) - Y'_{f(p)}(X'(g)) \\ &= [X', Y']_{f(p)}(g), \end{aligned}$$

as we wished. □

What is the relation between flows and Lie brackets? In order to discuss that, let $X, Y$ be smooth vector fields on $M$. Denote the flow of $X$ by $\{\varphi_t\}$ and let $f$ be a smooth function on $M$. Then

$$\frac{d}{dt} f(\varphi_t) = X(f),$$

and

$$((\varphi_{-t})_* Y)(f \circ \varphi_t) = Y(f) \circ \varphi_t \tag{1.6.20}$$

as $(\varphi_{-t})_* Y$ and $Y$ are $\varphi_t$-related (cf. (1.6.19)).

**1.6.21 Exercise** Let $Z_t$ be a smooth curve in $T_pM$ and let $h_t(x) = H(t, x)$, where $H \in C^\infty(\mathbf{R} \times M)$. Prove that

$$\left.\frac{d}{dt}\right|_{t=0} Z_t(h_t) = \left(\left.\frac{d}{dt}\right|_{t=0} Z_t\right)(h_0) + Z_0\left(\left.\frac{d}{dt}\right|_{t=0} h_t\right).$$

(Hint: Here $\frac{d}{dt}|_{t=0} h_t(x)$ means $\frac{\partial H}{\partial t}(0, x)$. Consider $\Gamma \in C^\infty(\mathbf{R} \times \mathbf{R})$ such that $\Gamma(t, 0) = p$ and $\frac{\partial}{\partial s}|_{s=0}\Gamma(t, s) = Z_t$ for all $t \in \mathbf{R}$, and use the chain rule.)

Differentiate identity (1.6.20) at $t = 0$ to get

$$\frac{d}{dt}\Big|_{t=0} \left((\varphi_{-t})_* Y\right)(f) + Y(X(f)) = X(Y(f)).$$

Note that $t \mapsto ((\varphi_{-t})_* Y)_p$ is a smooth curve in $T_pM$. Its tangent vector at $t = 0$ is called the *Lie derivative* of $Y$ with respect to $X$ at $p$, denoted by $(L_X Y)_p$, and this defines the Lie derivative $L_X Y$ as a smooth vector field on $M$. The above calculation shows that

$$L_X Y = [X, Y]. \tag{1.6.22}$$

**1.6.23 Proposition** *$X$ and $Y$ commute if and only if their corresponding flows $\{\varphi_t\}$, $\{\psi_s\}$ commute.*

*Proof.* $[X, Y] = 0$ if and only if $0 = \frac{d}{dt}\big|_{t=0}(\varphi_{-t})_* Y$. Since $\{\varphi_t\}$ is a one-parameter group,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_0}(\varphi_{-t})_* Y &= \frac{d}{dh}\Big|_{h=0}(\varphi_{-(t_0+h)})_* Y \\ &= d(\varphi_{-t_0})\left(\frac{d}{dh}\Big|_{h=0}(\varphi_{-h})_* Y \circ \varphi_{t_0}\right), \end{aligned}$$

this is equivalent to $(\varphi_{-t})_* Y = Y$ for all $t$. However the flow of $(\varphi_{-t})_* Y$ is $\{\varphi_{-t}\psi_s\varphi_t\}$, so this means $\varphi_{-t}\psi_s\varphi_t = \psi_s$. $\square$

We know that, for a local chart $(U, \varphi)$, the set of coordinate vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is linearly independent at every point of $U$ and the $\frac{\partial}{\partial x_i}$ pairwise commute. It turns out these two conditions locally characterize coordinate vector fields. Namely, we call a set $\{X_1, \dots, X_k\}$ of smooth vector fields defined on an open set $V$ of $M$ a *local $k$-frame* if it is linearly independent at every point of $V$; if $k = \dim M$, we simply say *local frame*.

**1.6.24 Proposition** *Let $\{X_1, \dots, X_k\}$ be a local $k$-frame on $V$ such that $[X_i, X_j] = 0$ for all $i, j = 1, \dots, k$. Then for every $p \in V$ there exists an open neighborhood $U$ of $p$ in $V$ and a local chart $(U, \varphi)$ whose first $k$ coordinate vector fields are exactly the $X_i$.*

*Proof.* Complete $\{X_1, \dots, X_k\}$ to a local frame $\{X_1, \dots, X_n\}$ in smaller neighborhood $\tilde{V} \subset V$ of $p$. (One can do that by first completing

$$\{X_1(p), \dots, X_k(p)\}$$

to a basis

$$\{X_1(p), \dots, X_k(p), v_{k+1}, \dots, v_n\}$$

of $T_pM$ and then declaring $X_{k+1}, \dots, X_n$ to be the vector fields defined on the domain of a system of local coordinates $(W, y_1, \dots, y_n)$ around $p$, $W \subset$

$V$, with constant coefficients in $\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ that extend $v_{k+1}, \ldots, v_n$. By continuity, $\{X_1, \ldots, X_k\}$ will be a local frame in a neighborhood $\tilde{V} \subset W$ of $p$.) Let $\{\varphi_t^i\}$ be the flow of $X_i$ and put $F(t_1, \ldots, t_n) := \varphi_{t_1}^1 \circ \cdots \circ \varphi_{t_n}^n(p)$, smooth map defined on a neighborhood of 0 in $\mathbf{R}^n$. Then $dF_0(e_i) = X_i(p)$ for all $i$, so $F$ is a local diffeomorphism at 0 by the inverse function theorem. The local inverse $F^{-1}$ defines a local chart $(U, x_1, \ldots, x_n)$ around $p$. Finally, for $q = F(t_1, \ldots, t_n)$,

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big|_q &= dF_{F^{-1}(q)}(e_i) \\
&= \frac{d}{dh}\Big|_{t=0} \varphi^i_{t_i+h}\varphi^1_{t_1}\cdots\hat{\varphi^i_{t_i}}\cdots\varphi^n_{t_n}(p) \\
&= X_i\left(\varphi^i_{t_i}\varphi^1_{t_1}\cdots\hat{\varphi^i_{t_i}}\cdots\varphi^n_{t_n}(p)\right) \\
&= X_i\left(\varphi^1_{t_1}\cdots\varphi^n_{t_n}(p)\right) \\
&= X_i(q),
\end{aligned}$$

where we have used Proposition 1.6.23 twice. □

## 1.7 Distributions and foliations

We seek to generalize the theory of the previous section to higher dimensions, so let us rephrase it in the following terms. Let $X$ be a smooth vector field on $M$ *which is nowhere zero*. On one hand, the $\mathbf{R}$-span of $X_p$ defines a family $\mathcal{D}$ of one-dimensional subspaces $\mathcal{D}_p$ of $T_pM$ for each $p \in M$. On the other hand, the maximal integral curves of $X$ define a partition $\mathcal{F}$ of $M$ into regular parametrized curves, or 1-dimensional immersed submanifolds of $M$. The relation between $\mathcal{D}$ and $\mathcal{F}$ is that $T_pL = \mathcal{D}_p$ for every $L \in \mathcal{F}$ and every $p \in L$.

In view of the above, we give the following definition. Suppose $\dim M = n$. A *rank $k$ (smooth) distribution* $\mathcal{D}$ on $M$, $0 \leq k \leq n$, is an assignment of a $k$-dimensional subspace $\mathcal{D}_p$ of $T_pM$ to each $p \in M$, where any $p \in M$ admits an open neighborhood $U$ with the property that there exist smooth vector fields $X_1, \ldots, X_k$ on $U$ such that the span of $X_1(q), \ldots, X_k(q)$ coincides with $\mathcal{D}_q$ for all $q \in U$.

Before continuing, we recall a consequence of Proposition 1.6.24, namely, that the flow of a non-vanishing vector field can be locally "rectified" in the following sense.

**1.7.1 Proposition** *Let $X$ be a smooth vector field on $M$ such that $X_p \neq 0$ for some $p \in M$. Then there exists a system of local coordinates $(U, (x_1, \ldots, x_n))$ around $p$ such that $X|_U = \frac{\partial}{\partial x_1}$. Equivalently, the integral curves of $X$ in $U$ are of the form $x_2 = c_2, \ldots, x_n = c_n$ for some $c_2, \ldots, c_n \in \mathbf{R}$.*

Based on Proposition 1.7.1, we make the following definition. A *$k$-dimensional foliation* of $M$, $0 \leq k \leq n$, is a partition $\mathcal{F}$ of $M$ into piecewise smooth arc-connected subsets, where any $p \in M$ admits a coordinate neighborhood $(U, (x_1, \ldots, x_n))$ such that $U$ is connected and, for every $L \in \mathcal{F}$, the piecewise smooth arc-connected components of $L \cap U$ are coincide with the "slices"

$$x_{k+1} = c_{k+1}, \ \ldots, \ x_n = c_n$$

for some $c_{k+1}, \ldots, c_n \in \mathbf{R}$. The elements of $\mathcal{F}$ are called *leaves*. A coordinate system $(U, (x_1, \ldots, x_n))$ as above will be called *distinguished*. If $L \in \mathcal{F}$, the piecewise smooth arc-components of $L \cap U$ are called *plaques*.

**1.7.2 Examples** (i) The levels sets of a submersion $M \to N$ form a foliation of rank $\dim M - \dim N$, by the local form of a submersion, where the leaves are embedded submanifolds. Indeed, this is the local model of a general foliation, by definition.

(ii) Recall the skew-line in the torus in Example 1.4.2. The traces of the immersions

$$F_s : \mathbf{R} \to \mathbf{R}^4, \qquad F(t) = (\cos at, \sin at, \cos(bt + 2\pi s), \sin(bt + 2\pi s)),$$

where $a$, $b$ are non-zero real numbers, for $s \in [0, 1]$, form a foliation of rank 1 of $T^2$. If $b/a$ is an irrational number, the leaves are dense in $T^2$.

*Each leaf $L \in \mathcal{F}$ has a canonical structure of immersed submanifold of $M$ of dimension $k$.* In fact, we can use Proposition 1.2.10. For any distinguished chart $(U, \varphi)$, $\varphi|_P$ is a bijective map from a plaque (arc component) $P$ of $L \cap U$ onto an open subset of $\mathbf{R}^k$. In this way, if we start with a countable collection $\{(U_m, \varphi_m)\}_{m \in \mathbb{N}}$ of distinguished charts of $M$ whose domains cover $L$, we construct a collection $\{(P_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, where $P_\alpha$ is a plaque of $L \cap U_m$ for some $m$ and $\varphi_\alpha$ is the restriction of $\varphi_m$ to $P_\alpha$. It is clear that this collection satisfies conditions (a), (b) and (c) of Proposition 1.2.10, but it remains to be checked that the index set $A$ is countable. For that purpose, it suffices to see that $U_m \cap L$ has countably many arc components, for every $m$. Fix a plaque $P_0$ of $L$ in $\{U_m\}$. Since $L$ is arc connected, for any other plaque $P$ there exists a sequence $P_1, \ldots, P_\ell = P$ of plaques such that $P_{i-1} \cap P_i \neq \varnothing$ for all $i = 1, \ldots, \ell$. So any plaque of $L$ in $\{U_m\}$ can be reached by a finite path of plaques that originates at $P_0$. It suffices to show that the collection of such paths is countable. In order to do that, it is enough to prove that a given plaque $P'$ of $L$ in $\{U_m\}$ can meet only countably many other plaques of $L$ in $\{U_m\}$. For any $m$, $P' \cap (L \cap U_m) = P' \cap U_m$ is an open subset of the locally Euclidean space $P'$ and thus has countably many components, each such component being contained in a plaque of $L \cap U_m$. It follows that $P'$ can meet at most countably many components of $L \cap U_m$, as we wished.

In this way, we have a structure of smooth manifold of $L$ such that each plaque of $L$ is an open submanifold of $L$. The underlying topology in $L$ can be much finer than the induced topology. In any case, the Hausdorff condition follows because the inclusion map $L \to M$ is continuous and $M$ is Hausdorff. In addition (recall Proposition 1.4.9):

**1.7.3 Proposition** *Every leaf L of a foliation of N is an initial submanifold.*

*Proof.* Let $f : M \to N$ be a smooth map such that $f(M) \subset L$ and consider the induced map $f_0 : M \to L$ such that $\iota \circ f_0 = f$, where $\iota : L \to N$ is the inclusion. We need to show that $f_0$ is continuous. We will prove that $f_0^{-1}(U)$ is open in $M$ for any given open subset $U$ of $L$. We may assume $f_0^{-1}(U) \neq \varnothing$, so let $p \in f_0^{-1}(U)$ and $q = f_0(p) \in U$. It suffices to show that $p$ is an interior point of $f_0^{-1}(U)$. Let $(V, y_1, \ldots, y_n)$ be a distinguished chart of $N$ around $q$, so that the plaques of $L$ in $V$ are of the form

$$y_i = \text{constant} \qquad \text{for } i = k+1, \ldots, n \tag{1.7.4}$$

and the plaque containing $q$ is

$$y_{k+1} = \cdots = y_n = 0 \tag{1.7.5}$$

By shrinking $V$, we may assume that (1.7.5) is an open set $\tilde{U} \subset U$. Note that $f^{-1}(V)$ an open neighborhood of $p$ in $M$; let $W$ be its connected component containing $p$. Of course, $W$ is open. It is enough to show that $f_0(W) \subset \tilde{U}$, or what amounts to the same, $f(W)$ is contained in (1.7.5). Since $f(W)$ is connected, it is contained in a plaque of of $V \cap L$; since $f(W)$ meets $q$, it must be (1.7.5). $\square$

### The Frobenius theorem

Let $M$ be a smooth manifold. It is clear that every foliation of $M$ gives rise to a distribution simply by taking the tangent spaces to the leaves at each point; locally, for a distinguished chart $(U, (x_1, \ldots, x_n))$, the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ span the distribution on $U$. What about the converse? If we start with a distribution, can we produce an "integral" foliation? Well, in case $k = 1$, locally we can find a smooth vector field $X$ that spans the line distribution and we have seen how to construct a local foliation by integral curves of $X$; in fact, the global problem can also be solved by passing to a double covering of $M$. It turns out that in case $k = 1$ there are no obstructions to the integrability of distributions, and this is in line with the fact that there are no obstructions to the integrability of ordinary differential equations. On the other hand, the situation is different when we pass to distributions of rank $k > 1$, what amounts to consider certain kinds of partial differential equations.

Let $\mathcal{D}$ be a distribution on $M$. We say that $\mathcal{D}$ is *integrable* if there exists a foliation $\mathcal{F}$ such that $T_pL_p = \mathcal{D}_p$ for every $p \in M$, where $L_p \in \mathcal{F}$ denotes the leaf thorugh $p$. Such an $\mathcal{F}$ is called an *integral foliation* of $\mathcal{D}$.

**1.7.6 Proposition** *If $\mathcal{D}$ is an integrable foliation on $M$ then the integral foliation $\mathcal{F}$ is unique.*

*Proof*. Define an equivalence relation on $M$ by declaring two points equivalent if and only if they can be joined by a piecewise smooth curve whose smooth arcs are tangent to $\mathcal{D}$. For $p \in M$, denote by $L_p$ the leaf of $\mathcal{F}$ through $p$. Since $L_p$ is arc connected, it is a union of equivalence classes. Now the existence of distinguished charts implies that each such equivalence class is open in $L_p$, so $L_p$ coincides with the equivalence class of $p$. This already characterizes the leaves of $\mathcal{F}$ as subsets of $M$. Each leaf is an initial submanifold of $M$, so the structure of smooth manifold on the leaf is unique up to equivalence, as in Problem 19(d). □

More generally, an *integral manifold* of a distribution $\mathcal{D}$ on $M$ is a submanifold $(L, f)$ of $M$ such that $df_p(T_pL) = \mathcal{D}_{f(p)}$ for every $p \in L$. A *maximal integral manifold* of $\mathcal{D}$ is a connected integral manifold whose image in $M$ is not a proper subset of another connected integral manifold of $\mathcal{D}$, that is, there is no connected integral manifold $(L', f')$ such that $f(L)$ is a proper subset of $f'(L')$.

**1.7.7 Exercise** Let $L_1$, $L_2$ be two integral manifolds of a distribution $\mathcal{D}$ on $M$. Use adapted charts to show that either $L_1$ and $L_2$ are disjoint or $L_1 \cap L_2$ is open in both $L_1$ and $L_2$. Deduce that, if $\mathcal{D}$ is integrable, then the leaves of the integral foliation are the maximal integral manifolds of $\mathcal{D}$.

We say that a vector field $X$ on $M$ *lies* in $\mathcal{D}$ if $X(p) \in \mathcal{D}_p$ for all $p \in M$; in this case, we write $X \in \mathcal{D}$. We say that $\mathcal{D}$ is *involutive* if $X$, $Y \in \mathcal{D}$ implies $[X, Y] \in \mathcal{D}$, namely, if $\mathcal{D}$ is closed under Lie brackets. Involutivity is a necessary condition for a distribution to be integrable.

**1.7.8 Proposition** *Every integrable distribution is involutive.*

*Proof*. Let $\mathcal{D}$ be an integrable distribution on a smooth manifold $M$. Given smooth vector fields $X$, $Y \in \mathcal{D}$ and $p \in M$, we need to show that $[X, Y]_p \in \mathcal{D}_p$. By assumption, there exists a distinguished coordinate system $(U, (x_1, \ldots, x_n))$ around $p$ such that the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ span the distribution $\mathcal{D}$ on $U$. Now $X|_U$, $Y|_U$ are linear combinations of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ with $C^\infty(U)$-coefficients, and so is their bracket, as we wished. □

It so happens that involutivity is also a sufficient condition for a distribution to be integrable. This is the contents of the celebrated Frobenius theorem.

Despite being named after Frobenius, the theorem seems to be proved first by Clebsch and Deahna. The merit of Frobenius in his 1875 Crelle's paper was to apply the theorem to Pfaffian systems, or systems of partial differential equations that are usefully formulated, from the point of view of their underlying geometric and algebraic structure, in terms of a system of differential forms of degree one. The proof below is accredited to Lundell [Lun92] who found inspiration in Chern and Wolfson.

We first prove an elementary, general lemma.

**1.7.9 Lemma** *Let $\mathcal{D}$ be any rank $k$ distribution on a smooth manifold $M$. Then there exists a system of local coordinates $(U, x_1, \ldots, x_n)$ around any given point $p$ in $M$ such that $\mathcal{D}$ is spanned by the $k$ vector fields*

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=k+1}^{n} a_{ij} \frac{\partial}{\partial x_i} \qquad \textit{for } j = 1, \ldots, k$$

*at all points in $U$, where $a_{ij} \in C^\infty(U)$.*

*Proof.* Let $(V, x_1, \ldots, x_n)$ be any system of local coordinates around $p$. Let $Y_1, \ldots, Y_k$ be arbitrary smooth vector fields spanning $\mathcal{D}$ on an open set $\tilde{U} \subset V$. Then $Y_j = \sum_{i=1}^n b_{ij} \frac{\partial}{\partial x_i}$ for $j = 1, \ldots, k$ and $b_{ij} \in C^\infty(\tilde{U})$. Since $Y_1, \ldots, Y_k$ is linearly independent at every point of $\tilde{U}$, the matrix $B(q) = (b_{ij}(q))$ has rank $k$ for all $q \in \tilde{U}$. By relabeling the $x_i$, we may assume that the $1 \leq i, j \leq k$-block $B'$ is non-singular in an open neighborhood $U \subset \tilde{U}$ of $p$. Now the $1 \leq i, j \leq k$-block of $B(B')^{-1}$ is the identity, namely, $X_j = \sum_{i=1}^k \hat{b}_{ij} Y_i$ has the desired form, where $(B')^{-1} = (\hat{b}_{ij})$. $\square$

**1.7.10 Theorem** *Every involutive distribution is integrable.*

*Proof.* Let $\mathcal{D}$ be an involutive distribution on a smooth manifold $M$. We first prove the local integrability, namely, the existence around any given point $p \in M$ of a system of local coordinates $(V, y_1, \ldots, y_n)$ such that $\mathcal{D}_q$ is spanned by $\frac{\partial}{\partial y_1}|_1, \ldots, \frac{\partial}{\partial y_k}|_q$ for every $q \in V$. Indeed let $(U, x_1, \ldots, x_n)$ and $X_1, \ldots, X_k$ be as in Lemma 1.7.9. Note that

$$[X_i, X_j] \in \operatorname{span}\left\{ \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n} \right\},$$

so the involutivity of $\mathcal{D}$ implies that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, k$. The desired result follows from Proposition 1.6.24.

Finally, we construct the integral foliation. According to Proposition 1.7.6, the leaf $L_p$ through a given point $p \in M$ must be the set of points $q \in M$ that can be reached from $p$ by a piecewise smooth curve whose smooth arcs are tangent to $\mathcal{D}$. This defines a partition $\mathcal{F}$ of $M$ into piecewise smooth arc connected subsets. Given $q \in L_p$, let $(V, y_1, \ldots, y_n)$ be a system of local coordinates around $q$ such that $\mathcal{D}$ is spanned by $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_k}$ at all points in $V$. It is clear that the arc connected components of $L_p \cap V$ are

$$y_{k+1} = \text{constant}, \ \ldots, \ y_n = \text{constant}.$$

This proves that $\mathcal{F}$ is a foliation. □

## 1.8 Problems

§ 1.2

**1** $a$. Use stereographic projection $\varphi_N : U_N = S^2 \setminus \{(0,0,1)\} \to \mathbf{R}^2$ to define a local chart on $S^2$ and write a formula for $\varphi_N$ in terms of the coordinates of $\mathbf{R}^3$. Do the same for $\varphi_S : U_S = S^2 \setminus \{(0,0,-1)\} \to \mathbf{R}^2$.
$b$. Show that $\{(U_N,\varphi_N),(U_S,\varphi_S)\}$ is a smooth atlas for $S^2$. Compare the smooth structure defined by this atlas with that defined in example 1.2.7 (viewing $S^2$ as a union of graphs of smooth maps).

**2** Let $M$ be the set of all (affine) lines in $\mathbf{R}^2$. Construct a natural structure of smooth manifold in $M$. What is the dimension of $M$? (Hint: Parametrize lines in terms of their equations.)

**3** Let $M$, $N$, $P$ be smooth manifolds and denote by $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ the canonical projections. Define maps $\iota_1 : M \to M \times N$, $\iota_2 : N \to M \times N$, where $\iota_1(x) = (x,q)$, $\iota_2(y) = (p,y)$ and $p \in M$, $q \in N$.
$a$. Show that $\pi_1$, $\pi_2$, $\iota_1$, $\iota_2$ are smooth maps.
$b$. Show that $f : P \to M \times N$ is smooth if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are smooth.

**4** Let $f : M \to N$ be a map. Prove that $f \in C^\infty(M,N)$ if and only if $g \circ f \in C^\infty(M)$ for all $g \in C^\infty(N)$.

**5** Let $\pi : \tilde{M} \to M$ be a topological covering of a smooth manifold $M$. Check that $\tilde{M}$ is necessarily Hausdorff, second-countable (here you need to know that the fundamental group $\pi(M)$ is at most countable) and locally Euclidean. Prove also that there exists a unique smooth structure on $\tilde{M}$ which makes $\pi$ smooth and a local diffeomorphism (compare Appendix A).

§ 1.4

**6** $a$. Prove that the composition and the product of immersions are immersions.
$b$. In case $\dim M = \dim N$, check that the immersions $M \to N$ coincide with the local diffeomorphisms.

**7** Prove that every submersion is an open map.

**8** $a$. Prove that if $M$ is compact and $N$ is connected then every submersion $M \to N$ is surjective.

*b.* Show that there are no submersions of compact manifolds into Euclidean spaces.

**9** Show that every smooth real function on a compact manifold has at least two critical points.

**10** Let $M$ be a compact manifold of dimension $n$ and let $f : M \to \mathbf{R}^n$ be smooth. Prove that $f$ has at least one critical point.

**11** Let $p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a polynomial with complex coefficients and consider the associated polynomial map $\mathbf{C} \to \mathbf{C}$. Show that this map is a submersion out of finitely many points.

**12** (*Generalized inverse function theorem.*) Let $f : M \to N$ be a smooth map which is injective on a compact submanifold $P$ of $M$. Assume that $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for every $p \in P$.

*a.* Prove that $f(P)$ is a submanifold of $N$ and that $f$ restricts to a diffeomorphism $P \to f(P)$.

*b.* Prove that indeed $f$ maps some open neighborhood of $P$ in $M$ diffeomorphically onto an open neighborhood of $f(P)$ in $N$. (Hint: It suffices to show that $f$ is injective on some neighborhood of $P$; if this is not the case, there exist sequences $\{p_i\}$, $\{q_i\}$ in $M$ both converging to a point $p \in P$, with $p_i \neq q_i$ but $f(p_i) = f(q_i)$ for all $i$, and this contradicts the non-singularity of $df_p$.)

**13** Let $p$ be a homogeneous polynomial of degree $m$ in $n$ variables $t_1, \ldots, t_n$. Show that $p^{-1}(a)$ is a submanifold of codimension one of $\mathbf{R}^n$ if $a \neq 0$. Show that the submanifolds obtained with $a > 0$ are all diffeomorphic, as well as those with $a < 0$. (Hint: Use Euler's identity

$$\sum_{i=1}^{n} t_i \frac{\partial p}{\partial t_i} = mp.)$$

**14** The $n \times n$ real matrices with determinar 1 form a group denoted $SL(n, \mathbf{R})$. Prove that $SL(n, \mathbf{R})$ is a submanifold of $GL(n, \mathbf{R})$. (Hint: Use Problem 13.)

**15** Consider the submanifolds $GL(n, \mathbf{R})$, $O(n)$ and $SL(n, \mathbf{R})$ of the vector space $M(n, \mathbf{R})$ (see Examples 1.2.7(ix) and 1.4.14(b), and Problem 14, respectively).

*a.* Check that the tangent space of $GL(n, \mathbf{R})$ at the identity is canonically isomorphic to $M(n, \mathbf{R})$.

*b.* Check that the tangent space of $SL(n, \mathbf{R})$ at the identity is canonically isomorphic to the subspace of $M(n, \mathbf{R})$ consisting of matrices of trace zero.

*c.* Check that the tangent space of $O(n)$ at the identity is canonically isomorphic to the subspace of $M(n, \mathbf{R})$ consisting of the skew-symmetric matrices.

**16** Denote by $M(m \times n, \mathbf{R})$ the vector space of real $m \times n$ matrices.

*a.* Show that the subset of $M(m \times n, \mathbf{R})$ consisting of matrices of rank at least $k$ ($0 \le k \le \min\{m, n\}$) is a smooth manifold.

*b.* Show that the subset of $M(m \times n, \mathbf{R})$ consisting of matrices of rank equal to $k$ ($0 \le k \le \min\{m, n\}$) is a smooth manifold. What is its dimension? (Hint: We may work in a neighborhood of a matrix

$$g = \begin{array}{c} \\ k \\ m-k \end{array} \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{array}$$

where $A$ is nonsingular and right multiply by

$$\left(\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right)$$

to check that $g$ has rank $k$ if and only if $D - CA^{-1}B = 0$.)

**17** Let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of smooth maps between smooth manifolds. Assume that $g \pitchfork Q$ for a submanifold $Q$ of $P$. Prove that $f \pitchfork g^{-1}(Q)$ if and only if $g \circ f \pitchfork Q$.

**18** Let $G \subset \mathbf{R}^2$ be the graph of $g : \mathbf{R} \to \mathbf{R}$, $g(x) = |x|^{1/3}$. Show that $G$ admits a smooth structure so that the inclusion $G \to \mathbf{R}^2$ is smooth. Is it an immersion? (Hint: consider the map $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(t) = \begin{cases} te^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ te^{1/t} & \text{if } t < 0.) \end{cases}$$

**19** Define submanifolds $(M_1, f_1)$, $(M_2, f_2)$ of $N$ to be *equivalent* if there exists a diffeomorphism $g : M_1 \to M_2$ such that $f_2 \circ g = f_1$.

*a.* Show that this is indeed an equivalence relation.

*b.* Show that each equivalence class of submanifolds of $N$ contains a unique representative of the form $(M, \iota)$, where $M$ is a subset of $N$ with a manifold structure such that $\iota : M \to N$ is a smooth immersion.

*c.* Let $N$ be a smooth manifold, and let $M$ be a subset of $N$ equipped with a given topology. Prove that there exists at most one smooth structure on $M$, up to equivalence, which makes $(M, \iota)$ an immersed submanifold of $N$, where $\iota : M \to N$ is the inclusion. (Hint: Use Proposition 1.4.9.)

*d*. Let $N$ be a smooth manifold, and let $M$ be a subset of $N$. Prove that there exists at most one structure of smooth manifold on $M$, up to equivalence, which makes $(M, \iota)$ an initial submanifold of $N$, where $\iota : M \to N$ is the inclusion. (Hint: Use Proposition 1.4.9.)

**20** Let $N$ be a smooth manifold of dimension $n + k$. For a point $q \in N$ and a subset $A \subset N$, denote by $C_q(A)$ the set of all points of $A$ that can be joined to $q$ by a smooth curve in $M$ whose image lies in $A$.

*a*. Prove that if $(P, g)$ is an initial submanifold of dimension $n$ of $N$ then for every $p \in P$ there exists a local chart $(V, \psi)$ of $N$ around $g(p)$ such that

$$\psi(C_{g(p)}(V \cap g(P))) = \psi(V) \cap (\mathbf{R}^n \times \{0\}).$$

(Hint: Use Proposition 1.4.5.)

*b*. Conversely, assume $P$ is a subset of $N$ with the property that around any point $p \in P$ there exists a local chart $(V, \psi)$ of $N$ around $p$ such that

$$\psi(C_p(V \cap P)) = \psi(V) \cap (\mathbf{R}^n \times \{0\}).$$

Prove that there exists a topology on $P$ that makes each connected component of $P$ into an initial submanifold of dimension $n$ of $N$ with respect to the inclusion. (Hint: Apply Proposition 1.2.10 to the restrictions $\psi|_{C_p(V \cap P)}$. Proving second-countability requires the following facts: for locally Euclidean Hausdorff spaces, paracompactness is equivalent to the property that each connected component is second-countable; every metric space is paracompact; the topology on $P$ is metrizable since it is compatible with the Riemannian distance for the Riemannian metric induced from a given Riemannian metric on $N$; Riemannian metrics can be constructed on $N$ using partitions of unity.)

**21** Show that the product of any number of spheres can be embedded in some Euclidean space with codimension one.

§ 1.5

**22** Let $M$ be a closed submanifold of $N$. Prove that the restriction map $C^\infty(N) \to C^\infty(M)$ is well defined and surjective. Show that the result ceases to be true if: (i) $M$ is not closed; or (ii) $M \subset N$ is closed but merely assumed to be an immersed submanifold.

**23** Let $M$ be a smooth manifold of dimension $n$. Given $p \in M$, construct a local chart $(U, \varphi)$ of $M$ around $p$ such that $\varphi$ is the restriction of a smooth map $M \to \mathbf{R}^n$.

**24** Prove that on any smooth manifold $M$ there exists a proper smooth map $f : M \to \mathbf{R}$. (Hint: Use $\sigma$-compactness of manifolds and partitions of unity.)

§ 1.6

**25** Determine the vector field on $\mathbf{R}^2$ with flow $\varphi_t(x, y) = (xe^{2t}, ye^{-3t})$.

**26** Determine the flow of the vector field $X$ on $\mathbf{R}^2$ when:
*a.* $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.
*b.* $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

**27** Given the following vector fields in $\mathbf{R}^3$,

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

compute their Lie brackets.

**28** Show that the restriction of the vector field defined on $\mathbf{R}^{2n}$

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \cdots - x_{2n}\frac{\partial}{\partial x_{2n-1}} + x_{2n-1}\frac{\partial}{\partial x_{2n}}$$

to the unit sphere $S^{2n-1}$ defines a nowhere vanishing smooth vector field.

**29** Let $X$ and $Y$ be smooth vector fields on $M$ and $N$ with flows $\{\varphi_t\}$ and $\{\psi_t\}$, respectively, and let $f : M \to N$ be smooth. Show that $X$ and $Y$ are $f$-related if and only if $f \circ \varphi_t = \psi_t \circ f$ for all $t$.

**30** Let $M$ be a properly embedded submanifold of $N$. Prove that every smooth vector field on $M$ can be smoothly extended to a vector field on $N$.

**31** Construct a natural diffeomorphism $TS^1 \approx S^1 \times \mathbf{R}$ which restricts to a linear isomorphism $T_pS^1 \to \{p\} \times \mathbf{R}$ for every $p \in S^1$ (we say that such a diffeomorphism maps fibers to fibers and is linear on the fibers).

**32** Construct a natural diffeomorphism $T(M \times N) \approx TM \times TN$ that maps fibers to fibers and is linear on the fibers.

**33** Construct a natural diffeomorphism $T\mathbf{R}^n \approx \mathbf{R}^n \times \mathbf{R}^n$ that maps fibers to fibers and is linear on the fibers.

**34** Show that $TS^n \times \mathbf{R}$ is diffeomorphic to $S^n \times \mathbf{R}^{n+1}$. (Hint: There are natural isomorphisms $T_pS^n \oplus \mathbf{R} \cong \mathbf{R}^{n+1}$.)

**35** A smooth manifold $M$ of dimension $n$ is called *parallelizable* if $TM \approx M \times \mathbf{R}^n$ by a diffeomorphism that maps fibers to fibers and is linear on the fibers. Prove that $M$ is parallelizable if and only if there exists a globally defined $n$-frame $\{X_1, \ldots, X_n\}$ on $M$.

§ 1.7

**36** Is there a non-constant smooth function $f$ defined on an open subset of $\mathbf{R}^3$ such that

$$\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z} = 0?$$

(Hint: Consider a regular level set of $f$.)

**37** Consider the first order system of partial differential equations

$$\frac{\partial z}{\partial x} = \alpha(x, y, z), \quad \frac{\partial z}{\partial y} = \beta(x, y, z)$$

where $\alpha$, $\beta$ are smooth functions defined on an open subset of $\mathbf{R}^3$.

*a.* Show that if $f$ is a solution, then the smooth vector fields $X = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial z}$ e $Y = \frac{\partial}{\partial y} + \beta\frac{\partial}{\partial z}$ span the tangent space to the graph of $f$ at all points.

*b.* Prove that the system admits local solutions if and only if

$$\frac{\partial \beta}{\partial x} + \alpha\frac{\partial \beta}{\partial z} = \frac{\partial \alpha}{\partial y} + \beta\frac{\partial \alpha}{\partial z}.$$

**38** Prove that there exists a smooth function $f$ defined on a neighborhood of $(0, 0)$ in $\mathbf{R}^2$ such that $f(0, 0) = 0$ and $\frac{\partial f}{\partial x} = ye^{-(x+y)} - f$, $\frac{\partial f}{\partial y} = xe^{-(x+y)} - f$.

CHAPTER 2

# Tensor fields and differential forms

## 2.1 Multilinear algebra

Let $V$ be a real vector space. In this section, we construct the tensor algebra $T(V)$ and the exterior algebra $\Lambda(V)$ over $V$. Elements of $T(V)$ are called tensors on $V$. Later we will apply these constructions to the tangent space $T_pM$ of a manifold $M$ and let $p$ vary in $M$, similarly to the definition of the tangent bundle.

### Tensor algebra

All vector spaces are real and finite-dimensional. Let $V$ and $W$ be vector spaces. It is less important what the tensor product of $V$ and $W$ *is* than what it *does*. Namely, a tensor product of $V$ and $W$ is a vector space $V \otimes W$ together with a bilinear map $h : V \times W \to V \otimes W$ such that the following *universal property* holds: for every vector space $U$ and every bilinear map $B : V \times W \to U$, there exists a unique linear map $\tilde{B} : V \otimes W \to U$ such that $\tilde{B} \circ h = B$.

$$\begin{tikzcd} V \otimes W \arrow[dr, dotted, "\tilde{B}"] & \\ V \times W \arrow[u, "h"] \arrow[r, "B"'] & U \end{tikzcd}$$

There are different ways to construct $V \otimes W$. It does not actually matter which one we choose, in view of the following exercise.

**2.1.1 Exercise** Prove that the tensor product of $V$ and $W$ is uniquely defined by the universal property. In other words, if $(V \otimes_1 W, h_1)$, $(V \otimes_2 W, h_2)$ are two tensor products, then there exists an isomorphism $\ell : V \otimes_1 W \to V \otimes_2 W$ such that $\ell \circ h_1 = h_2$.

We proceed as follows. Start with the canonical isomorphism $V^{**} \cong V$ between $V$ and its bidual. It says that we can view an element $v$ in $V$ as the linear map on $V^*$ given by $f \mapsto f(v)$. Well, we can extend this idea and consider the space $\mathrm{Bil}(V, W)$ of bilinear forms on $V \times W$. Then there is a natural map $h : V \times W \to \mathrm{Bil}(V, W)^*$ given by $h(v, w)(b) = b(v, w)$ for $b \in \mathrm{Bil}(V, W)$. We claim that $(\mathrm{Bil}(V, W)^*, h)$ satisfies the universal property: given a bilinear map $B : V \times W \to U$, there is an associated map $U^* \to \mathrm{Bil}(V, W)$, $f \mapsto f \circ B$; let $\tilde{B} : \mathrm{Bil}(V, W)^* \to U^{**} = U$ be its transpose.

**2.1.2 Exercise** Check that $\tilde{B} \circ h = B$.

**2.1.3 Exercise** Let $\{e_i\}$, $\{f_j\}$ be bases of $V$, $W$, respectively. Define $b_{ij} \in \mathrm{Bil}(V, W)$ to be the bilinear form whose value on $(e_k, f_\ell)$ is $1$ if $(k, \ell) = (i, j)$ and $0$ otherwise. Prove that $\{b_{ij}\}$ is a basis of $\mathrm{Bil}(V, W)$. Prove also that $\{h(e_i, f_j)\}$ is the dual basis of $\mathrm{Bil}(V, W)^*$. Deduce that the image of $h$ spans $\mathrm{Bil}(V, W)^*$ and hence $\tilde{B}$ as in Exercise 2.1.2 is uniquely defined.

Now that $V \otimes W$ is constructed, we can forget about its definition and keep in mind its properties only (in the same way as when we work with real numbers and we do not need to know that they are equivalence classes of Cauchy sequences), namely, the universal property and those listed in the sequel. Henceforth, we write $v \otimes w = h(v, w)$ for $v \in V$ and $w \in W$.

**2.1.4 Proposition** *Let $V$ and $W$ be vector spaces. Then:*

*a.* $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$;
*b.* $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$;
*c.* $av \otimes w = v \otimes aw = a(v \otimes w)$;

*for all $v$, $v_1$, $v_2 \in V$; $w$, $w_1$, $w_2 \in W$; $a \in \mathbf{R}$.*

**2.1.5 Proposition** *Let $U$, $V$ and $W$ be vector spaces. Then there are canonical isomorphisms:*

*a.* $V \otimes W \cong W \otimes V$;
*b.* $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$;
*c.* $V^* \otimes W \cong \mathrm{Hom}(V, W)$; *in particular,* $\dim V \otimes W = (\dim V)(\dim W)$.

**2.1.6 Exercise** Prove Propositions 2.1.4 and 2.1.5.

**2.1.7 Exercise** Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be bases for $V$ and $W$, respectively. Prove that $\{e_i \otimes f_j \ : \ i = 1, \ldots, m$ and $j = 1, \ldots, n\}$ is a basis for $V \otimes W$.

**2.1.8 Exercise** Let $A = (a_{ij})$ be a real $m \times n$ matrix, viewed as an element of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$. Use the canonical inner product in $\mathbf{R}^n$ to identify $(\mathbf{R}^n)^* \cong \mathbf{R}^n$. What is the element of $\mathbf{R}^n \otimes \mathbf{R}^m$ that corresponds to $A$?

Taking $V = W$ and using Proposition 2.1.5(b), we can now inductively form the tensor $n$th power $\otimes^n V = \otimes^{n-1} V \otimes V$ for $n \geq 1$, where we adopt the convention that $\otimes^0 V = \mathbf{R}$. The *tensor algebra* $T(V)$ over $V$ is the direct sum

$$T(V) = \bigoplus_{r,s\geq 0} V^{r,s}$$

where

$$V^{r,s} = (\otimes^r V) \otimes (\otimes^s V^*)$$

is called the *tensor space of type* $(r,s)$. The elements of $T(V)$ are called *tensors*, and those of $V^{r,s}$ are called *homogeneous of type* $(r,s)$. The multiplication $\otimes$, read "tensor", is the $\mathbf{R}$-linear extension of

$$\begin{aligned}(u_1 \otimes \cdots \otimes u_{r_1} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^*) \otimes (v_1 \otimes \cdots \otimes v_{r_2} \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*)\\ = \quad u_1 \otimes \cdots \otimes u_{r_1} \otimes v_1 \otimes \cdots \otimes v_{r_2} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^* \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*.\end{aligned}$$

$T(V)$ is a non-commutative, associative *graded* algebra, in the sense that tensor multiplication is compatible with the natural grading:

$$V^{r_1,s_1} \otimes V^{r_2,s_2} \subset V^{r_1+r_2,s_1+s_2}.$$

Note that $V^{0,0} = \mathbf{R}$, $V^{1,0} = V$, $V^{0,1} = V^*$, so real numbers, vectors and linear forms are examples of tensors.

### Exterior algebra

Even more important to us will be a certain quotient of the subalgebra $T^+(V) = \bigoplus_{k\geq 0} V^{k,0}$ of $T(V)$. Let $\mathfrak{I}$ be the two-sided ideal of $T^+(V)$ generated by the set of elements of the form

$$v \otimes v \tag{2.1.9}$$

for $v \in V$.

**2.1.10 Exercise** Prove that another set of generators for $\mathfrak{I}$ is given by the elements of the form $u \otimes v + v \otimes u$ for $u, v \in V$.

The *exterior algebra* over $V$ is the quotient

$$\Lambda(V) = T^+(V)/\mathfrak{I}.$$

The induced multiplication is denoted by $\wedge$, and read "wedge" or "exterior product". In particular, the class of $v_1 \otimes \cdots \otimes v_k$ modulo $\mathfrak{I}$ is denoted $v_1 \wedge \cdots \wedge v_k$. This is also a graded algebra, where the space of elements of degree $k$ is

$$\Lambda^k(V) = V^{k,0}/\mathfrak{I} \cap V^{k,0}.$$

Since $\mathfrak{I}$ is generated by elements of degree $2$, we immediately get

$$\Lambda^0(V) = \mathbf{R} \quad \text{and} \quad \Lambda^1(V) = V.$$

$\Lambda(V)$ is not commutative, but we have:

**2.1.11 Proposition** *$\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$ for $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^\ell(V)$.*

*Proof.* Since $v \otimes v \in \mathfrak{I}$ for all $v \in V$, we have $v \wedge v = 0$. Since $\mathbf{R}$ is not a field of characteristic two, this relation is equivalent to $v_1 \wedge v_2 = -v_2 \wedge v_1$ for all $v_1, v_2 \in V$.

By linearity, we may assume that $\alpha = u_1 \wedge \cdots \wedge u_k$, $\beta = v_1 \wedge \cdots \wedge v_\ell$. Now

$$\begin{aligned}
\alpha \wedge \beta &= u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_\ell \\
&= -u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge u_k \wedge v_2 \cdots \wedge v_\ell \\
&= u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge v_2 \wedge u_k \wedge v_3 \cdots \wedge v_\ell \\
&= \cdots \\
&= (-1)^\ell u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge \cdots \wedge v_\ell \wedge u_k \\
&= (-1)^{2\ell} u_1 \wedge \cdots \wedge u_{k-2} \wedge v_1 \wedge \cdots \wedge v_\ell \wedge u_{k-1} \wedge u_k \\
&= \cdots \\
&= (-1)^{k\ell}\beta \wedge \alpha,
\end{aligned}$$

as we wished. □

**2.1.12 Lemma** *If $\dim V = n$, then $\dim \Lambda^n(V) = 1$ and $\Lambda^k(V) = 0$ for $k > n$.*

*Proof.* Let $\{e_1, \ldots, e_n\}$ be a basis of $V$. Since

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k} \;:\; i_1, \ldots, i_k \in \{1, \ldots, n\}\} \tag{2.1.13}$$

is a basis of $V^{k,0}$ (see Exercise 2.1.7), the image of this set under the projection $V^{k,0} \to \Lambda^k(V)$ is a set of generators of $\Lambda^k(V)$. Taking into account Proposition 2.1.11 yields $\Lambda^k(V) = 0$ for $k > n$ and that $\Lambda^n(V)$ is generated by $e_1 \wedge \cdots \wedge e_n$, so we need only show that this element is not zero.

Suppose, on the contrary, that $e_1 \otimes \cdots \otimes e_n \in \mathfrak{I}$. Then $e_1 \otimes \cdots \otimes e_n$ is a linear combination of elements of the form $\alpha \otimes v \otimes v \otimes \beta$ where $v \in V$, $\alpha \in V^{k,0}$, $\beta \in V^{\ell,0}$ and $k + \ell + 2 = n$. Writing $\alpha$ (resp. $\beta$) in terms of the basis (2.1.13), we may assume that the only appearing base elements are of the form $e_1 \otimes \cdots \otimes e_k$ (resp. $e_{n-\ell+1} \otimes \cdots \otimes e_n$). It follows that we can write

$$e_1 \otimes \cdots \otimes e_n = \sum_{k=0}^{n-2} c_k\, e_1 \otimes \cdots \otimes e_k \otimes v_k \otimes v_k \otimes e_{k+3} \otimes \cdots \otimes e_n \tag{2.1.14}$$

where $c_k \in \mathbf{R}$ and $v_k \in V$ for all $k$. Finally, write $v_k = \sum_{i=1}^n a_{ik} e_i$ for $a_k \in \mathbf{R}$. For $m = 0, \ldots, n-2$, the coefficient of

$$e_1 \otimes \cdots \otimes e_m \otimes e_{m+2} \otimes e_{m+1} \otimes e_{m+3} \otimes \cdots \otimes e_n$$

on the right hand side of (2.1.14) is

$$c_m\, a_{m+2,m} a_{m+1,m},$$

thus zero. However, the coefficient of $e_1 \otimes \cdots \otimes e_n$ on the right hand side is

$$\sum_{k=0}^{n-2} c_k\, a_{k+1,k} a_{k+2,k},$$

hence also zero, a contradiction. $\square$

**2.1.15 Proposition** *If $\{e_1, \ldots, e_n\}$ be a basis of $V$, then*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \;:\; i_1 < \cdots < i_k\}$$

*is a basis of $\Lambda^k(V)$ for all $0 \leq k \leq n$; in particular,* $\dim \Lambda^k(V) = \binom{n}{k}$.

*Proof.* Fix $k \in \{0, \ldots, n\}$. The above set is clearly a set of generators of $\Lambda^k(V)$ and we need only show linear independence. Suppose

$$\sum a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} = 0,$$

which we write as

$$\sum a_I e_I = 0$$

where the $I$ denotes increasing $k$-multi-indices, and $e_\varnothing = 1$. Multiply through this equation by $e_J$, where $J$ is an increasing $n-k$-multi-index, and note that $e_I \wedge e_J = 0$ unless $I$ is the multi-index $J^c$ complementary to $J$, in which case $e_{J^c} \wedge e_J = \pm e_1 \wedge \cdots \wedge e_n$. Since $e_1 \wedge \cdots \wedge e_n \neq 0$ by Lemma 2.1.12, this shows that $a_I = 0$ for all $I$. $\square$

## 2.2 Tensor bundles

### Cotangent bundle

In the same way as the fibers of the tangent bundle of $M$ are the tangent spaces $T_pM$ for $p \in M$, the fibers of the cotangent bundle of $M$ will be the dual spaces $T_pM^*$. Indeed, form the disjoint union

$$T^*M = \dot{\bigcup}_{p \in M} T_pM^*.$$

There is a natural projection $\pi^* : T^*M \to M$ given by $\pi(\tau) = p$ if $\tau \in T_pM^*$. Recall that every local chart $(U, \varphi)$ of $M$ induces a local chart $\tilde{\varphi} : \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^{2n}$ of $TM$, where $\tilde{\varphi}(v) = (\varphi(\pi(v)), d\varphi(v))$, and thus a map $\tilde{\varphi}^* : (\pi^*)^{-1}(U) \to \mathbf{R}^n \times (\mathbf{R}^n)^* = \mathbf{R}^{2n}$, $\tilde{\varphi}^*(\tau) = (\varphi(\pi^*(\tau)), ((d\varphi)^*)^{-1}(\tau))$, where $(d\varphi)^*$ denotes the transpose map of $d\varphi$ and we have identified $\mathbf{R}^n = \mathbf{R}^{n*}$ using the canonical Euclidean inner product. The collection

$$\{((\pi^*)^{-1}(U), \tilde{\varphi}^*) \mid (U, \varphi) \in \mathcal{A}\}, \tag{2.2.1}$$

for an atlas $\mathcal{A}$ of $M$, satisfies the conditions of Proposition 1.2.10 and defines a Hausdorff, second-countable topology and a smooth structure on $T^*M$ such that $\pi^* : TM \to M$ is smooth.

A section of $T^*M$ is a map $\omega : M \to T^*M$ such that $\pi^* \circ \omega = \mathrm{id}_M$. A smooth section of $T^*M$ is also called a *differential form of degree* 1 or *differential* 1*-form*. For instance, if $f : M \to \mathbf{R}$ is a smooth function then $df_p : T_pM \to \mathbf{R}$ is an element of $T_pM^*$ for all $p \in M$ and hence defines a differential 1-form $df$ on $M$.

If $(U, x_1, \ldots, x_n)$ is a system of local coordinates on $M$, the differentials $dx_1, \ldots, dx_n$ yield local smooth sections of $T^*M$ that form the dual basis to $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ at each point (recall (1.3.7)). Therefore any section $\omega$ of $T^*M$ can be locally written as $\omega|_U = \sum_{i=1}^n a_i dx_i$, and one proves similarly to Proposition 1.6.4 that $\omega$ is smooth if and only if the $a_i$ are smooth functions on $U$, for every coordinate system $(U, x_1, \ldots, x_n)$.

**2.2.2 Exercise** Prove that the differential of a smooth function on $M$ indeed gives a a smooth section of $T^*M$ by using the atlas (2.2.1).

**Tensor bundles**

We now generalize the construction of the tangent and cotangent bundles using the notion of tensor algebra. Let $M$ be a smooth manifold. Set:

$$\begin{array}{lcll} T^{r,s}(M) & = & \bigcup_{p\in M}(T_pM)^{r,s} & \textit{tensor bundle of type } (r,s) \textit{ over } M; \\ \Lambda^k(M) & = & \bigcup_{p\in M}\Lambda^k(T_pM^*) & \textit{exterior } k\textit{-bundle over } M; \\ \Lambda(M) & = & \bigcup_{p\in M}\Lambda(T_pM^*) & \textit{exterior algebra bundle over } M. \end{array}$$

Then $T^{r,s}(M)$, $\Lambda^k(M)$ and $\Lambda(M)$ admit natural structures of smooth manifolds such that the projections onto $M$ are smooth. If $(U, x_1, \ldots, x_n)$ is a coordinate system on $M$, then the bases $\{\frac{\partial}{\partial x_i}|_p\}_{i=1}^n$ of $T_pM$ and $\{dx_i|_p\}_{i=1}^n$ of $T_pM^*$, for $p \in U$, define bases of $(T_pM)^{r,s}$, $\Lambda^k(T_pM^*)$ and $\Lambda(T_pM)$. For instance, a section $\omega$ of $\Lambda^k(M)$ can be locally written as

$$\omega|_U = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{2.2.3}$$

where the $a_{i_1,\ldots,i_k}$ are functions on $U$.

**2.2.4 Exercise** Check that $T^{1,0}(M) = TM$, $T^{0,1}(M) = T^*M = \Lambda^1(M)$ and $\Lambda^0(M) = M \times \mathbf{R}$.

The smooth sections of $T^{r,s}(M)$, $\Lambda^k(M)$, $\Lambda^*(M)$ are respectively called *tensor fields of type* $(r,s)$, *differential* $k$*-forms*, *differential forms* on $M$. For instance, a section $\omega$ of $\Lambda^k(M)$ is a differential $k$-form if and only if the functions $a_i$ in all its local representations (2.2.3) are smooth.

We will denote the space of differential $k$-forms on $M$ by $\Omega^k(M)$ and the space of all differential forms on $M$ by $\Omega^*(M)$. Note that $\Omega^*(M)$ is a graded algebra over $\mathbf{R}$ with wedge multiplication and a module over the ring $C^\infty(M)$.

It follows from Problems 4 and 7(a) that a differential $k$-form $\omega$ on $M$ is an object that, at each point $p \in M$, yields a map $\omega_p$ that can be evaluated on $k$ tangent vectors $v_1, \ldots, v_k$ at $p$ to yield a real number, with some smoothness assumption. The meaning of the next proposition is that we can *equivalently* think of $\omega$ as being an object that, evaluated at $k$ vector fields $X_1, \ldots, X_k$ yields the smooth function

$$\omega(X_1, \ldots, X_k) : p \mapsto \omega_p(X_1(p), \ldots, X_k(p)).$$

We first prove a lemma.

Hereafter, it shall be convenient to denote the $C^\infty(M)$-module of smooth vector fields on $M$ by $\mathfrak{X}(M)$.

**2.2.5 Lemma** *Let*

$$\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k\ factors} \to C^\infty(M)$$

*be a $C^\infty(M)$-multilinear map. Then the value of $\omega(X_1, \ldots, X_k)$ at any given point $p$ depends only on the values of $X_1, \ldots, X_k$ at $p$.*

*Proof.* For simplicity of notation, let us do the proof for $k = 1$; the case $k > 1$ is similar. We first show that if $X|_U = X'|_U$ for some open subset $U$ of $M$, then $\omega(X)|_U = \omega(X')|_U$. Indeed let $p \in U$ be arbitrary, take an open neighborhood $V$ of $p$ such that $\bar{V} \subset U$ and a smooth function $\lambda \in C^\infty(M)$ with $\lambda|_{\bar{V}} = 1$ and $\operatorname{supp} \lambda \subset U$ (Exercise 1.5.1). Then

$$\begin{aligned}
\omega(X)(p) &= \lambda(p)\omega(X)(p) \\
&= (\lambda(\omega(X)))(p) \\
&= \omega(\lambda X)(p) \\
&= \omega(\lambda X')(p) \\
&= \lambda(\omega(X')))(p) \\
&= \lambda(p)\omega(X')(p) \\
&= \omega(X')(p),
\end{aligned}$$

where in the third and fifth equalities we have used $C^\infty(M)$-linearity of $\omega$, and in the fourth equality we have used that $\lambda X = \lambda X'$ as vector fields on $M$.

Finally, we prove that $\omega(X)(p)$ depends only on $X(p)$. By linearity, it suffices to prove that $X(p) = 0$ implies $\omega(X)(p) = 0$. Let $(W, x_1, \ldots, x_n)$

be a coordinate system around $p$ and write $X|_W = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for $a_i \in C^\infty(W)$. By assumption, $a_i(p) = 0$ for all $i$. Let $\lambda$ be a smooth function on $M$ with support contained in $W$ and such that it is equal to $1$ on an open neighborhood $U$ of $p$ with $\bar{U} \subset W$. Define also

$$\tilde{X}_i = \begin{cases} \lambda \frac{\partial}{\partial x_i} & \text{on } W \\ 0 & \text{on } M \setminus \bar{U} \end{cases} \quad \text{and} \quad \tilde{a}_i = \begin{cases} \lambda a_i & \text{on } W \\ 0 & \text{on } M \setminus \bar{U}. \end{cases}$$

Then $\tilde{X} := \sum_{i=1}^n \tilde{a}_i \tilde{X}_i$ is a globally defined smooth vector field on $M$ such that $\tilde{X}|_U = X|_U$ and we can apply the result in the previous paragraph to write

$$\begin{aligned} \omega(X)(p) &= \omega(\tilde{X})(p) \\ &= \left( \sum_{i=1}^n \tilde{a}_i \omega(\tilde{X}_i) \right)(p) \\ &= \sum_{i=1}^n \tilde{a}_i(p) \omega(\tilde{X}_i)(p) \\ &= 0 \end{aligned}$$

because $\tilde{a}_i(p) = a_i(p) = 0$ for all $i$. □

**2.2.6 Proposition** *$\Omega^*(M)$ is canonically isomorphic as a $C^\infty(M)$-module to the $C^\infty(M)$-module of alternating $C^\infty(M)$-multilinear maps*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ factors}} \to C^\infty(M) \tag{2.2.7}$$

*Proof.* Let $\omega \in \Omega^k(M)$. Then $\omega_p \in \Lambda^k(T_pM^*) \cong \Lambda^k(T_pM)^* \cong A_k(T_pM)$ for every $p \in M$, owing to Problems 4 and 7(a), namely, $\omega_p$ can be considered to be an alternating $k$-multilinear form on $T_pM$. Therefore, for vector fields $X_1, \ldots, X_k$ on $M$,

$$\tilde{\omega}(X_1, \ldots, X_k)(p) := \omega_p(X_1(p), \ldots, X_k(p))$$

defines a smooth function on $M$, $\tilde{\omega}(X_1, \ldots, X_k)$ is $C^\infty(M)$-linear in each argument $X_i$, thus $\tilde{\omega}$ is an alternating $C^\infty(M)$-multilinear map as in (2.2.7).

Conversely, let $\tilde{\omega}$ be a $C^\infty(M)$-multilinear map as in (2.2.7). Due to Lemma 2.2.5, we have $\tilde{\omega}_p \in A_k(T_pM) \cong \Lambda^k(T_pM^*)$, namely, $\tilde{\omega}$ defines a section $\omega$ of $\Lambda^k(M)$: given $v_1, \ldots, v_k \in T_pM$, choose $X_1, \ldots, X_k \in \mathfrak{X}(M)$ such that $X_i(p) = v_i$ for all $i$ and put

$$\omega_p(v_1, \ldots, v_k) := \tilde{\omega}(X_1, \ldots, X_k)(p).$$

The smoothness of the section $\omega$ follows from the fact that, in a coordinate system $(U, x_1, \ldots, x_n)$, we can write $\omega|_U = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

where $a_{i_1\cdots i_k}(q) = \omega_q(\frac{\partial}{\partial x_{i_1}}\big|_q, \ldots, \frac{\partial}{\partial x_{i_k}}\big|_q) = \tilde{\omega}(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}})(q)$ for all $q \in U$, and thus $a_{i_1\cdots i_k} \in C^\infty(U)$. It follows that $\omega$ is a differential $k$-form on $M$. □

Henceforth we will not distinguish between differential $k$-forms and alternating multilinear maps (2.2.7). Similarly to Proposition 2.2.6:

**2.2.8 Proposition** *The $C^\infty(M)$-module of tensor fields of type $(r,s)$ on $M$ is canonically isomorphic to the $C^\infty(M)$-module of $C^\infty(M)$-multilinear maps*

$$\underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{r \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text{ factors}} \to C^\infty(M).$$

## 2.3 The exterior derivative

Recall that $\Lambda^0(M) = M \times \mathbf{R}$, so a smooth section of this bundle is a map $M \to M \times \mathbf{R}$ of the form $p \mapsto (p, f(p))$ where $f \in C^\infty(M)$. This shows that $\Omega^0(M) \cong C^\infty(M)$. Furthermore, we have seen that the differential of $f \in C^\infty(M)$ can be viewed as a differential 1-form $df \in \Omega^1(M)$, so we have an operator $C^\infty(M) \to \Omega^1(M)$, $f \mapsto df$. In this section, we extend this operator to an operator $d : \Omega^*(M) \to \Omega^*(M)$, called *exterior derivative*, mapping $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all $k \geq 0$. It so happens that $d$ plays an *extremely* important rôle in the theory of smooth manifolds.

**2.3.1 Theorem** *There exists a unique $\mathbf{R}$-linear operator $d : \Omega^*(M) \to \Omega^*(M)$ with the following properties:*

*a.* $d\left(\Omega^k(M)\right) \subset \Omega^{k+1}(M)$ *for all* $k \geq 0$ *($d$ has* degree $+1$*);*
*b.* $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\, \omega \wedge d\eta$ *for every* $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ *($d$ is an* anti-derivation*);*
*c.* $d^2 = 0$*;*
*d.* $df$ *is the differential of* $f$ *for every* $f \in C^\infty(M) \cong \Omega^0(M)$.

*Proof.* We start with uniqueness, so let $d$ be as in the statement. The first case is when $M$ is a coordinate neighborhood $(U, x_1, \ldots, x_n)$. Then any $\omega \in \Omega^k(U)$ can be written as $\omega = \sum_I a_I dx_I$, where $I$ runs over increasing multi-indices $(i_1, \ldots, i_k)$ and $a_I \in C^\infty(U)$, and we get

$$\begin{aligned} d\omega &= \sum_I d(a_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \quad \text{(by } \mathbf{R}\text{-linearity)} \\ &= \sum_I d(a_I) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\qquad + \sum_{r=1}^{k} (-1)^{r-1}\, a_I\, dx_{i_1} \wedge \cdots \wedge d(dx_{i_r}) \wedge \cdots \wedge dx_{i_k} \quad \text{(by (b))} \\ &= \sum_I \sum_{r=1}^{n} \frac{\partial a_I}{\partial x_r} dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{(by (c) and (d).)} \end{aligned} \tag{2.3.2}$$

Next we go to the case of a general manifold $M$ and show that $d$ is a *local* operator, in the sense that $(d\omega)|_U = 0$ whenever $\omega|_U = 0$ and $U$ is an open subset of $M$. So assume $\omega|_U = 0$, take an arbitrary point $p \in U$, and choose $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda$ is flat equal to $1$ on $M \setminus U$ and has support disjoint from $\bar{V}$, where $V$ is a neighborhood of $p$ with $\bar{V} \subset U$. Then $\omega = \lambda\omega$ on the entire $M$ so that, using (b) we get

$$(d\omega)_p = d(\lambda\omega)_p = d\lambda_p \wedge \underbrace{\omega_p}_{=0} + \underbrace{\lambda(p)}_{=0} d\omega_p = 0,$$

as wished.

To continue, we verify that $d$ induces an operator $d_U$ on $\Omega^*(U)$ satisfying (a)-(d) for every open subset $U$ of $M$. So given $\omega \in \Omega^k(U)$ and $p \in U$, construct $\tilde{\omega} \in \Omega^k(M)$ which coincides with $\omega$ on a neighborhood $V$ of $p$ with $\bar{V} \subset U$, as usual by means of a bump function, and define $(d_U\omega)_p := (d\tilde{\omega})_p$. The definition is independent of the chosen extension, as $d$ is a local operator. It is easy to check that $d_U$ indeed satisfies (a)-(d); for instance, for (b), note that $\tilde{\omega} \wedge \tilde{\eta}$ is an extension of $\omega \wedge \eta$ and hence $d_U(\omega \wedge \eta)_p = (d(\tilde{\omega} \wedge \tilde{\eta}))_p = (d\tilde{\omega})_p \wedge \tilde{\eta}_p + (-1)^{\deg\tilde{\omega}}\tilde{\omega}_p \wedge (d\tilde{\eta})_p = (d_U\omega)_p \wedge \eta_p + (-1)^{\deg\omega}\omega_p \wedge (d_U\eta)_p$. Note also that the collection $\{d_U\}$ is *natural with respect to restrictions*, in the sense that if $U \subset V$ are open subsets of $M$ then $d_V|_U = d_U$.

Finally, for $\omega \in \Omega^*(M)$ and a coordinate neighborhood $(U, x_1, \ldots, x_n)$, on one hand $d_U(\omega|_U)$ is uniquely defined by formula (2.3.2). On the other hand, $\omega$ itself is an extension of $\omega|_U$, and hence $(d\omega)_p = (d_U(\omega|_U))_p$ for every $p \in U$. This proves that $d\omega$ is uniquely defined.

To prove existence, we first use formula (2.3.2) to define an $\mathbf{R}$-linear operator $d_U$ on $\Omega^k(U)$ for every coordinate neighborhood $U$ of $M$. It is clear that $d_U$ satisfies (a) and (d); let us prove that it also satisfies (b) and (c). So let $\omega = \sum_I a_I dx_I \in \Omega^k(U)$. Then $d_U\omega = \sum_I da_I \wedge dx_I$ and

$$\begin{aligned}
d_U^2\omega &= \sum_{I,r} d_U\left(\frac{\partial a_I}{\partial x_r} dx_r \wedge dx_I\right) \\
&= \sum_{I,r,s} \frac{\partial^2 a_I}{\partial x_s \partial x_r} dx_s \wedge dx_r \wedge dx_I \\
&= 0,
\end{aligned}$$

since $\dfrac{\partial^2 a_I}{\partial x_s \partial x_r}$ is symmetric and $dx_s \wedge dx_r$ is skew-symmetric in $r$, $s$. Let also $\eta = \sum_J b_J dx_J$. Then $\omega \wedge \eta = \sum_{I,J} a_I b_J dx_I \wedge dx_J$ and

$$\begin{aligned}
d_U(\omega \wedge \eta) &= \sum_{I,J} d_U(a_I b_J dx_I \wedge dx_J) \\
&= \sum_{I,J,r} \frac{\partial a_I}{\partial x_r} b_J dx_r \wedge dx_I \wedge dx_J + \sum_{I,J,s} a_I \frac{\partial b_J}{\partial x_s} dx_s \wedge dx_I \wedge dx_J \\
&= \left( \sum_{I,r} \frac{\partial a_I}{\partial x_r} dx_r \wedge dx_I \right) \wedge \left( \sum_J b_J dx_J \right) \\
&\qquad + (-1)^{|I|} \left( \sum_I a_I dx_I \right) \wedge \left( \sum_{J,s} \frac{\partial b_J}{\partial x_s} dx_s \wedge dx_J \right) \\
&= d_U \omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d_U \eta,
\end{aligned}$$

where we have used Proposition 2.1.11 in the third equality to write $dx_s \wedge dx_I = (-1)^{|I|} dx_I \wedge dx_s$.

We finish by noting that the operators $d_U$ for each coordinate system $U$ of $M$ can be pieced together to define a global operator $d$. Indeed for two coordinate systems $U$ and $V$, the operators $d_U$ and $d_V$ induce two operators on $\Omega^*(U \cap V)$ satisfying (a)-(d) by the remarks above which must coincide by the uniqueness part. Note also that the resulting $d$ satisfies (a)-(d) since it locally coincides with some $d_U$. □

**2.3.3 Remark** We have constructed the exterior derivative $d$ as an operator between sections of vector bundles which, locally, is such that the local coordinates of $d\omega$ are linear combinations of partial derivatives of the local coordinates of $\omega$ (cf. 2.3.2). For this reason, $d$ is called a *differential operator*.

### Pull-back

A nice feature of differential forms is that they can always be pulled-back under a smooth map. In contrast, the push-forward of a vector field under a smooth map need not exist if the map is not a diffeomorphism.

Let $f : M \to N$ be a smooth map. The differential $df_p : T_pM \to T_{f(p)}$ at a point $p$ in $M$ has a transpose map $(df_p)^* : T_{f(p)}N^* \to T_pM^*$ and there is an induced algebra homomorphism $\delta f_p := \Lambda((df_p)^*) : \Lambda(T_{f(p)}N^*) \to \Lambda(T_pM^*)$ (cf. Problem 6). For varying $p \in M$, this yields map $\delta f : \Lambda^*(N) \to \Lambda^*(M)$. Recall that a differential form $\omega$ on $N$ is a section of $\Lambda^*(N)$. The *pull-back of $\omega$ under $f$* is the section of $\Lambda^*(M)$ given by $f^*\omega = \delta f \circ \omega \circ f$, so that the following diagram is commutative:

$$\begin{tikzcd}
\Lambda^*(M) & \Lambda^*(N) \arrow[l, "\delta f"'] \\
M \arrow[u, "f^*\omega"] \arrow[r, "f"'] & N \arrow[u, dotted, "\omega"']
\end{tikzcd}$$

(We prove below that $f^*\omega$ is smooth, so that it is in fact a differential form on $M$. This fact would also follow from the formula $f^*\omega = \delta f \circ \omega \circ f$ if we checked that $\delta f$ is a smooth.) In more detail, we have

$$(f^*\omega)_p = \delta f(\omega_{f(p)})$$

for all $p \in M$. In particular, if $\omega$ is a $k$-form, then $(f^*\omega)_p \in \Lambda^k(T_pM^*) = \Lambda^k(T_pM)^* = A_k(T_pM)$ and

$$(f^*\omega)_p(v_1,\ldots,v_k) = \omega_{f(p)}(df_p(v_1),\ldots,df_p(v_k)) \tag{2.3.4}$$

for all $v_1,\ldots,v_k \in T_pM$.

**2.3.5 Exercise** Let $f : M \to N$ be a smooth map.

a. In the case of 0-forms, that is smooth functions, check that $f^*(g) = g \circ f$ for all $g \in \Omega^0(N) = C^\infty(N)$.
b. In the case $\omega = dg \in \Omega^1(N)$ for some $g \in C^\infty(N)$, check that $f^*(dg) = d(g \circ f)$.

**2.3.6 Proposition** *Let $f : M \to N$ be a smooth map. Then:*

a. *$f^* : \Omega^*(N) \to \Omega^*(M)$ is a homomorphism of algebras;*
b. *$d \circ f^* = f^* \circ d$;*
c. *$(f^*\omega)(X_1,\ldots,X_k)(p) = \omega_{f(p)}(df(X_1(p)),\ldots,df(X_k(p)))$ for all $\omega \in \Omega^*(N)$ and all $X_1,\ldots,X_k \in \mathfrak{X}(M)$.*

*Proof.* Result (c) follows from (2.3.4). The fact that $f^*$ is compatible with the wedge product is a consequence of Problem 6(b) applied to local expressions of the form (2.2.3). For (a), it only remains to prove that $f^*\omega$ is actually a *smooth* section of $\Lambda^*(M)$ for a differential form $\omega \in \Omega^*(M)$. So let $p \in M$, choose a coordinate system $(V, y_1,\ldots,y_n)$ of $N$ around $f(p)$ and a neighborhood $U$ of $p$ in $M$ with $f(U) \subset V$. Since $f^*$ is linear, we may assume that $\omega$ is a $k$-form. As $\omega$ is smooth, we can write

$$\omega|_V = \sum_I a_I dy_{i_1} \wedge \cdots \wedge dy_{i_k}.$$

It follows from Exercise 2.3.5 that

$$f^*\omega|_U = \sum_I (a_I \circ f)\, d(y_{i_1} \circ f) \wedge \cdots \wedge d(y_{i_k} \circ f), \tag{2.3.7}$$

which indeed is a smooth form on $U$. Finally, (b) is proved using (2.3.7):

$$\begin{aligned}
d(f^*\omega)_p &= d\left(\sum_I (a_I\circ f)\, d(y_{i_1}\circ f)\wedge\cdots\wedge d(y_{i_k}\circ f)\right)\Big|_p \\
&= \sum_I \left(d(a_I\circ f)\wedge d(y_{i_1}\circ f)\wedge\cdots\wedge d(y_{i_k}\circ f)\right)|_p \\
&= f^*\left(\sum_I da_I\wedge dy_{i_1}\wedge\cdots\wedge dy_{i_k}\right)\Big|_p \\
&= f^*(d\omega)_p,
\end{aligned}$$

as desired. □

## 2.4 The Lie derivative of tensors

In section 1.6, we defined the Lie derivative of a smooth vector field $Y$ on $M$ with respect to another smooth vector field $X$ by using the flow $\{\varphi_t\}$ of $X$ to identify different tangent spaces of $M$ along an integral curve of $X$. The same idea can be used to define the Lie derivative of a differential form $\omega$ or tensor field $S$ with respect to $X$. The main point is to understand the action of $\{\varphi_t\}$ on the space of differential forms or tensor fields.

So let $\{\varphi_t\}$ denote the flow of a vector field $X$ on $M$, and let $\omega$ be a differential form on $M$. Then the pull-back $\varphi_t^*\omega$ is a differential form and $t\mapsto(\varphi_t^*\omega)_p$ is a smooth curve in $\Lambda(T_pM^*)$, for all $p\in M$. The *Lie derivative* of $\omega$ with respect to $X$ is the section $L_X\omega$ of $\Lambda(M)$ given by

(2.4.1) $$(L_X\omega)_p = \frac{d}{dt}\Big|_{t=0}(\varphi_t^*\omega)_p.$$

We prove below that $L_X\omega$ is smooth, so it indeed yields a differential form on $M$. In view of (2.3.4), it is clear that the Lie derivative preserves the degree of a differential form.

We extend the definition of Lie derivative to an arbitrary tensor field $S$ of type $(r,s)$ as follows. Suppose

$$S_{\varphi_t(p)} = v_1\otimes\cdots\otimes v_r\otimes v_1^*\otimes\cdots\otimes v_s^*.$$

Then we define $(\varphi_t^*S)_p\in(T_pM)^{r,s}$ to be

$$d\varphi_{-t}(v_1)\otimes\cdots\otimes d\varphi_{-t}(v_r)\otimes\delta\varphi_t(v_1^*)\otimes\cdots\otimes\delta\varphi_t(v_s^*)$$

and put

(2.4.2) $$(L_XS)_p = \frac{d}{dt}\Big|_{t=0}(\varphi_t^*S)_p.$$

One can view definition 2.4.1 as the operator in the quotient obtained from definition 2.4.2 in the sense that the exterior algebra is a subquotient of the tensor algebra.

Before stating properties of the Lie derivative, it is convenient to introduce two more operators. For $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^{k+1}(M)$ with $k \geq 0$, the *interior multiplication* $\iota_X\omega \in \Omega^k(M)$ is the $k$-differential form given by

$$\iota_X\omega(X_1,\dots,X_k) = \omega(X,X_1,\dots,X_k)$$

for $X_1,\dots,X_k \in \mathfrak{X}(M)$, and $\iota_X$ is zero on $0$-forms.

**2.4.3 Exercise** Prove that $\iota_X\omega$ is indeed a *smooth* section of $\Lambda^{k-1}(M)$ for $\omega \in \Omega^k(M)$. Prove also that $\iota_X$ is an *anti-derivation* in the sense that

$$\iota_X(\omega\wedge\eta) = \iota_X\omega\wedge\eta + (-1)^k\omega\wedge\iota_X\eta$$

for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. (Hint: For the last assertion, it suffices to check the identity at one point.)

Let $V$ be a vector space. The *contraction* $c_{i,j} : V^{r,s} \to V^{r-1,s-1}$ is the linear map that operates on basis vectors as

$$\begin{aligned} &v_1\otimes\cdots\otimes v_r\otimes v_1^*\otimes\cdots\otimes v_s^* \\ &\qquad\mapsto v_j^*(v_i)\, v_1\otimes\cdots\otimes\hat{v}_i\otimes\cdots\otimes v_r\otimes v_1^*\otimes\cdots\hat{v_j^*}\otimes\cdots\otimes v_s^*. \end{aligned}$$

It is easy to see that $c_{i,j}$ extends to a map $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$.

**2.4.4 Exercise** Let $V$ be a vector space. Recall the canonical isomorphism $V^{1,1} \cong \mathrm{Hom}(V,V) = \mathrm{End}(V)$ (Proposition 2.1.5). Check that $c_{1,1} : V^{1,1} \to V^{0,0}$ is the trace map $\mathrm{tr} : \mathrm{End}(V) \to \mathbf{R}$.

**2.4.5 Proposition** *Let $X$ be a smooth vector field on $M$. Then:*

*a.* $L_Xf = X(f)$ *for all* $f \in C^\infty(M)$.

*b.* $L_XY = [X,Y]$ *for all* $X \in \mathfrak{X}(M)$.

*c.* $L_X$ *is a type-preserving* $\mathbf{R}$*-linear operator on the space* $\mathcal{T}(M)$ *of tensor fields on* $M$.

*d.* $L_X : \mathcal{T}(M) \to \mathcal{T}(M)$ *is a derivation, in the sense that*

$$L_X(S\otimes S') = (L_XS)\otimes S' + S\otimes(L_XS')$$

*e.* $L_X : \mathcal{T}(M) \to \mathcal{T}(M)$ *commutes with contractions:*

$$L_X(c(S)) = c(L_XS)$$

*for any contraction* $c : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$.

*f.* $L_X$ *is a degree-preserving* $\mathbf{R}$*-linear operator on the space of differential forms* $\Omega(M)$ *which is a derivation and commutes with exterior differentiation.*

*g.* $L_X = \iota_X \circ d + d \circ \iota_X$ *on* $\Omega(M)$ *(*Cartan's magical formula*)*
*h. For* $\omega \in \Omega^k(M)$ *and* $X_0, \ldots, X_k \in \mathfrak{X}(M)$*, we have:*

$$L_{X_0}\omega(X_1, \ldots, X_k) = X_0(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, [X_0, X_i], X_{i+1}, \ldots, X_k).$$

*i. Same assumption as in (h), we have:*

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

*Proof.* (a) follows from differentiation of $(\varphi_t^* f)_p = f(\varphi_t(p))$ at $t = 0$. (b) was proved in section 1.6. The type-preserving part of (c) is clear from the definition. For (d), differentiate the obvious formula $\varphi_t^*(S \otimes S')|_p = (\varphi_t^* S)_p \otimes (\varphi_t^* S')_p$ at $t = 0$; the derivation property follows using the fact that tensor multiplication is $\mathbf{R}$-bilinear. Smoothness of $L_X S$ as a section of $T^{r,s}(M)$ is proved noting that $L_X$ is a local operator and expressing $L_X S$ in a system of local coordinates, see below for the analogous argument in the case of differential forms. This covers (c) and (d).

(e) follows from the easily checked fact that $\varphi_t^*$ commutes with contractions. As a consequence, which we will use below, if $\omega \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$ then $\omega(Y) = c(Y \otimes \omega)$ so

$$\begin{aligned} X(\omega(Y)) &= L_X(c(Y \otimes \omega)) \qquad \text{(using (a))} \\ &= c(L_X(Y \otimes \omega)) \\ &= c(L_X Y \otimes \omega + Y \otimes L_X \omega) \qquad \text{(using (d))} \\ &= \omega([X, Y]) + L_X \omega(Y) \qquad \text{(using (b));} \end{aligned}$$

in other words,

$$L_X \omega(Y) = X(\omega(Y)) - \omega([X, Y]). \tag{2.4.6}$$

For (f), we first remark that $L_X$ is a derivation as a map from $\Omega(M)$ to non-necessarily smooth sections of $\Lambda(M)$: this is a pointwise check, and follows from the fact that $(\varphi_t)^*$ defines an automorphism of the algebra $\Omega(M)$. Next, check that $L_X$ commutes with $d$ on functions using (2.4.6):

$$\begin{aligned} L_X(df)(Y) &= X(df(Y)) - df([X, Y]) \\ &= X(Y(f)) - [X, Y](f) \\ &= Y(X(f)) \\ &= d(X(f))(Y) \\ &= d(L_X f)(Y) \end{aligned}$$

for all $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$. To continue, note that $L_X$ is a local operator: formula (2.4.1) shows that $L_X\omega|_U$ depends only on $\omega|_U$, for any open subset $U$ of $M$, and the same applies for (2.4.1). Finally, to see that $L_X\omega$ is smooth for any $\omega \in \Omega(M)$, we may assume that $\omega$ has degree $k$ and work in a coordinate system $(U, x_1, \ldots, x_n)$, where $\omega$ has a local representation as in (2.2.3). Using the above collected facts:

$$\begin{aligned} L_X\omega|_U = &\sum_{i_1<\cdots<i_k} X(a_{i_1\cdots i_k})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &+\sum_{j=1}^{k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge d(X(x_{i_j})) \wedge \cdots \wedge dx_{i_k} \end{aligned}$$

as wished. This formula can also be used to show that $L_X$ commutes with $d$ in general.

To prove (g), let $P_X = d\circ\iota_X + \iota_X\circ d$. Then $P_X$ and $L_X$ are local operators, derivations of $\Omega(M)$, that coincide on functions and commmute with $d$. Since any differential form is locally a sum of wedge products of functions and differentials of functions, it follows that $L_X = P_X$.

The case $k = 1$ in (h) is formula (2.4.6). The proof for $k > 1$ is completely analogous.

Finally, (i) is proved by induction on $k$. The initial case $k = 0$ is immediate. Assuming (i) holds for $k - 1$, one proves it for $k$ by starting with (h) and using (g) and the induction hypothesis. □

**2.4.7 Exercise** Carry out the calculations to prove (h) and (i) in Proposition 2.4.5.

## 2.5 Vector bundles

The tangent, cotangent and and all tensor bundles we have constructed so far are smooth manifolds of a special kind in that they have a fibered structure over another manifold. For instance, $TM$ fibers over $M$ so that the fiber over any point $p$ in $M$ is the tangent space $T_pM$. Moreover, there is some control on how the fibers vary with the point. In case of $TM$, this is reflected on the way a chart $(\pi^{-1}(U), \tilde\varphi)$ is constructed from a given chart $(U, \varphi)$ of $M$. Recall that $\tilde\varphi : \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n$ where $\tilde\varphi(v) = (\varphi(\pi(v)), d\varphi(v))$. So $\tilde\varphi$ induces a diffeomorphism $\cup_{p\in U} T_pM \to \varphi(U) \times \mathbf{R}^n$ so that each fiber $T_pM$ is mapped linearly and isomorphically onto $\{\varphi(p)\} \times \mathbf{R}^n$. We could also compose this map with $\varphi^{-1} \times \mathrm{id}$ to get a diffeomorphism

$$TM|_U := \cup_{p\in U} T_pM \to \varphi(U) \times \mathbf{R}^n \to U \times \mathbf{R}^n.$$

Of course each $T_pM$ is abstractly isomorphic to $\mathbf{R}^n$, where $n = \dim M$, but here we are saying that the part of $TM$ consisting of fibers lying over points in $U$ is diffeomorphic to a product $U \times \mathbf{R}^n$ in such a way that $T_pM$ corresponds to $\{p\} \times \mathbf{R}^n$. This is the idea of a vector bundle.

**2.5.1 Definition** A (smooth) *vector bundle* of rank $k$ over a smooth manifold $M$ is a smooth manifold $E$, called the *total space*, together with a smooth projection $\pi : E \to M$ such that:

*a.* $E_p := \pi^{-1}(p)$ is a vector space of dimension $k$ for all $p \in M$;

*b.* $M$ can be covered by open sets $U$ such that there exists a diffeomorphism $E|_U = \pi^{-1}(U) \to U \times \mathbf{R}^k$ mapping $E_p$ linearly and isomorphically onto $\{p\} \times \mathbf{R}^k$ for all $p \in U$.

The *trivial vector bundle* of rank $k$ over $M$ is the direct product $M \times \mathbf{R}^k$ with the projection onto the first factor. A vector bundle of rank $k = 1$ is also called a *line bundle*.

An equivalent definition of vector bundle, more similar in spirit to the definition of smooth manifold, is as follows.

**2.5.2 Definition** A (smooth) *vector bundle* of rank $k$ over a smooth manifold $M$ is a set $E$, called the *total space*, together with a projection $\pi : E \to M$ with the following properties:

*a.* $M$ admits a covering by open sets $U$ such that there exists a bijection $\varphi_U : E|_U = \pi^{-1}(U) \to U \times \mathbf{R}^k$ satisfying $\pi = \pi_1 \circ \varphi_U$, where $\pi_1 : U \times \mathbf{R}^k \to U$ is the projection onto the first factor. Such a $\varphi_U$ is called a *local trivialization*.

*b.* Given local trivializations $\varphi_U$, $\varphi_V$ with $U \cap V \neq \varnothing$, the change of local trivialization or transition function

$$\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbf{R}^k \to (U \cap V) \times \mathbf{R}^k$$

has the form

$$(x, a) \mapsto (x, g_{UV}(x)a)$$

where

$$g_{UV} : U \cap V \to \mathbf{GL}(k, \mathbf{R})$$

is smooth.

**2.5.3 Exercise** Prove that the family of transition functions $\{g_{UV}\}$ in Definition 2.5.2 satisfies the *cocycle conditions*:

$$\begin{aligned} g_{UU}(x) &= \mathrm{id} \qquad (x \in U) \\ g_{UV}(x)g_{VW}(x)g_{WU}(x) &= \mathrm{id} \qquad (x \in U \cap V \cap W) \end{aligned}$$

**2.5.4 Exercise** Let $M$ be a smooth manifold.

*a.* Prove that for a vector bundle $\pi : E \to M$ as in Definition 2.5.2, the total space $E$ has a natural structure of smooth manfifold such that $\pi$ is smooth and the local trivializations are diffeomorphisms.

*b*. Prove that Definitions 2.5.1 and 2.5.2 are equivalent.

**2.5.5 Example** In this example, we construct a very important example of vector bundle which is not a tensor bundle, called the *tautological (line) bundle* over $\mathbf{R}P^n$. Recall that a point $p$ in real projective space $M = \mathbf{R}P^n$ is a 1-dimensional subspace of $\mathbf{R}^{n+1}$ (Example 1.2.9). Set $E = \dot\cup_{p\in M} E_p$ where $E_p$ is the subspace of $\mathbf{R}^{n+1}$ corresponding to $p$, namely, $E_p$ consists of vectors $v \in \mathbf{R}^{n+1}$ such that $v \in p$. Let $\pi : E \to M$ map $E_p$ to $p$. We will prove that this is a smooth vector bundle by constructing local trivializations and using Definition 2.5.2. Recall the atlas $\{\varphi_i\}_{i=1}^{n+1}$ of Example 1.2.9. Set

$$\tilde\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbf{R} \qquad v \mapsto (\pi(v), x_i(v)).$$

This is a bijection and the cocycle

$$g_{ij}(x_1, \ldots, x_{n+1}) = x_i/x_j \in \mathbf{GL}(1, \mathbf{R}) = \mathbf{R} \setminus \{0\}$$

is smooth on $U_i \cap U_j$, as wished.

## 2.6 Problems

§ 2.1

**1** Let $V$ be a vector space and let $\iota : V^n \to \otimes^n V$ be defined as $\iota(v_1, \ldots, v_n) = v_1 \otimes \cdots \otimes v_n$, where $V^n = V \times \cdots \times V$ ($n$ factors on the right hand side). Prove that $\otimes^n V$ satisfies the following universal property: for every vector space $U$ and every $n$-multilinear map $T : V^n \to U$, there exists a unique linear map $\tilde T : \otimes^n V \to U$ such that $\tilde T \circ \iota = T$.

$$\begin{tikzcd}
\otimes^n V \arrow[dr, dotted, "\tilde T"] & \\
V^n \arrow[u, "\iota"] \arrow[r, "T"'] & U
\end{tikzcd}$$

**2** Prove that $\otimes^n V$ is canonically isomorphic to the dual space of the space $n$-multilinear forms on $V^n$. (Hint: Use Problem 1.)

**3** Let $V$ be a vector space. An $n$-multilinear map $T : V^n \to U$ is called *alternating* if $T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (\operatorname{sgn}\sigma) T(v_1, \ldots, v_n)$ for every $v_1, \ldots, v_n \in V$ and every permutation $\sigma$ of $\{1, \ldots, n\}$, where $\operatorname{sgn}$ denotes the sign $\pm 1$ of the permutation.

Let $\iota : V^n \to \Lambda^n(V)$ be defined as $\iota(v_1, \ldots, v_n) = v_1 \wedge \cdots \wedge v_n$. Note that $\iota$ is alternating. Prove that $\Lambda^n V$ satisfies the following universal property: for every vector space $U$ and every alternating $n$-multilinear map $T : V^n \to U$, there exists a unique linear map $\tilde{T} : \Lambda^n(V) \to U$ such that $\tilde{T} \circ \iota = T$.

$$\begin{tikzcd}
\Lambda^n V \arrow[dr, dotted, "\tilde{T}"] & \\
V^n \arrow[u, "\iota"] \arrow[r, "T"'] & U
\end{tikzcd}$$

**4** Denote the vector space of all alternating multilinear forms $V^n \to \mathbf{R}$ by $A_n(V)$. Prove that $\Lambda^n(V)$ is canonically isomorphic to $A_n(V)^*$.

**5** Prove that $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$.

**6** Let $V$ and $W$ be vector spaces and let $T : V \to W$ be a linear map.

*a.* Show that $T$ naturally induces a linear map $\Lambda^k(T) : \Lambda^k(V) \to \Lambda^k(W)$. (Hint: Use Problem 3.)

*b.* Show that the maps $\Lambda^k(V)$ for various $k$ induce an algebra homomorphism $\Lambda(T) : \Lambda(V) \to \Lambda(W)$.

*c.* Let now $V = W$ and $n = \dim V$. The operator $\Lambda^n(T)$ is multiplication by a scalar, as $\dim \Lambda^n(V) = 1$; define the *determinant* of $T$ to be this scalar. Any $n \times n$ matrix $A = (a_{ij})$ can be viewed as the representation of a linear operator on $\mathbf{R}^n$ with respect to the canonical basis. Prove that

$$\det A = \sum_{\sigma} (\operatorname{sgn} \sigma)\, a_{i,\sigma(i)} \cdots a_{n,\sigma(n)},$$

where $\operatorname{sgn} \sigma$ is the sign of the permutation $\sigma$ and $\sigma$ runs over the set of all permutations of the set $\{1, \ldots, n\}$. Prove also that the determinant of the product of two matrices is the product of their determinants.

*d.* Using Problem 7(a) below, prove that the transpose map $\Lambda^k(T)^* = \Lambda^k(T^*)$.

**7** Let $V$ be vector space.

*a.* Prove that there is a canonical isomorphism

$$\Lambda^k(V^*) \to \Lambda^k(V)^*$$

given by

$$v_1^* \wedge \cdots \wedge v_k^* \mapsto (\ u_1 \wedge \cdots \wedge u_k \mapsto \det(v_i^*(u_j))\ ).$$

*b.* Let $\alpha$, $\beta \in V^* \cong \Lambda^1(V^*) \cong A_1(V)$. Show that $\alpha \wedge \beta \in \Lambda^2(V^*)$, viewed as an element of $\Lambda^2(V)^* \cong A_2(V)$ is given by

$$\alpha \wedge \beta\,(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

for all $u$, $v \in V$.

**8** Let $V$ be an Euclidean vector space, that is, a vector space equipped with a (positive-definite) inner product $\langle , \rangle$. Prove that there is an induced inner product on $\Lambda^k(V)$ given by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle \mapsto \det \left( \langle v_i, u_j \rangle \right).$$

**9** Let $V$ be a vector space.

*a.* In analogy with the exterior algebra, construct the *symmetric algebra* $Sym(V)$, a commutative graded algebra, as a quotient of $T(V)$.

*b.* Determine a basis of the homogeneous subspace $Sym^n(V)$.

*c.* State and prove that $Sym^n(V)$ satisfies a certain universal property.

*d.* Show that the $Sym^n(V)$ is canonically isomorphic to the dual of the space $S_n(V)$ of symmetric $n$-multilinear forms $V^n \to \mathbf{R}$.

In view of (d), $Sym(V^*)$ is usually defined to be the space $\mathcal{P}(V)$ of *polynomials* on $V$.

**10** An element of $\Lambda^n(V)$ is called *decomposable* if it lies in the subset $\Lambda^1(V) \wedge \cdots \wedge \Lambda^1(V)$ ($n$ factors).

*a.* Show that in general not every element of $\Lambda^n(V)$ is decomposable.

*b.* Show that, for $\dim V \leq 3$, every homogeneous element in $\Lambda(V)$ is decomposable.

*c.* Let $\omega$ be a differential form. Is $\omega \wedge \omega = 0$?

**11** Let $V$ be an oriented vector space equipped with a non-degenerate symmetric bilinear form (we do not require positive-definiteness from the outset). Let $\dim V = n$.

*a.* Prove there exists an element $\omega \in \Lambda^n(V)$ such that

$$\omega = e_1 \wedge \cdots \wedge e_n$$

for every positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of $V$ (here *orthonormal* means that $e_i \cdot e_j = \pm\delta_{ij}$ (delta of Kronecker)).

*b.* Check that the bilinear form on $V$ induces an isomorphism $V \to V^*$, which induces an isomorphism $\Lambda^k(V) \to \Lambda^k(V^*)$ via Problem 6(a).

*c.* Show that the bilinear map

$$\Lambda^k(V) \times \Lambda^{n-k}(V) \to \Lambda^n(V), \qquad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

together with the isomorphism

$$\mathbf{R} \to \Lambda^n(V), \qquad a \mapsto a\,\omega$$

define a canonical isomorphism

$$(\Lambda^k(V))^* \to \Lambda^{n-k}(V).$$

*d*. Combine the isomorphims of (b) and (c) with that in Problem 7(a) to get a linear isomorphism

$$* : \Lambda^k(V) \to \Lambda^{n-k}(V)$$

for $0 \leq k \leq n$, called the *Hodge star*.

*e*. Show that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \omega$$

for all $\alpha$, $\beta \in \Lambda^k(V)$, where we use the inner product of Problem 8.

*f*. Assume the inner product is positive definite and let $\{e_1, \ldots, e_n\}$ be a positively oriented orthonormal basis of $V$. Show that

$$*1 = e_1 \wedge \cdots \wedge e_n, \quad *(e_1 \wedge \cdots \wedge e_n) = 1,$$

and

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n.$$

Show also that

$$** = (-1)^{k(n-k)}$$

on $\Lambda^k(V)$.

§ 2.2

**12** Let $M$ be a smooth manifold. A *Riemannian metric* $g$ on $M$ is an assignment of positive definite inner product $g_p$ on each tangent space $T_pM$ which is smooth in the sense that $g(X,Y)(p) = g_p(X(p), Y(p))$ defines a smooth function for every $X$, $Y \in \mathfrak{X}(M)$. A *Riemannian manifold* is a smooth manifold equipped with a Riemannian metric.

*a*. Show that a Riemannian metric $g$ on $M$ is the same as a tensor field $\tilde{g}$ of type $(0,2)$ which is *symmetric*, in the sense that $\tilde{g}(Y,X) = \tilde{g}(X,Y)$ for every $X$, $Y \in \mathfrak{X}(M)$, with the additional property of positive-definiteness at each point.

*b*. Fix a local coordinate system $(U, x_1, \ldots, x_n)$ on $M$.

(i) Let $g$ be a Riemannian metric on $M$. Show that $g|_U = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \in C^\infty(U)$, $g_{ij} = g_{ji}$ and the matrix $(g_{ij})$ is everywhere positive definite.

(ii) Conversely, given functions $g_{ij} = g_{ji} \in C^\infty(U)$ such that the matrix $(g_{ij})$ is positive definite everywhere in $M$, show how to define a Riemannian metric on $U$.

*c*. Use part (b)(ii) and a partition of unity to prove that every smooth manifold can be equipped with a Riemannian metric.

*d*. On a Riemannian manifold $M$ there exists a natural diffeomorphism $TM \approx T^*M$ taking fibers to fibers. (Hint: There exist linear isomorphisms $v \in T_pM \mapsto g_p(v, \cdot) \in T_pM^*$).

§ 2.3

**13** Consider $\mathbf{R}^3$ with coordinates $(x, y, z)$. In each case, decide whether $d\omega = 0$ or there exists $\eta$ such that $d\eta = \omega$.

*a.* $\omega = yzdx + xzdy + xydz$.

*b.* $\omega = xdx + x^2y^2dy + yzdz$.

*c.* $\omega = 2xy^2dx \wedge dy + zdy \wedge dz$.

**14 (The operator $d$ on $\mathbf{R}^3$)** Identify 1- and 2-forms on $\mathbf{R}^3$ with vector fields on $\mathbf{R}^3$, and 0- and 3-forms on $\mathbf{R}^3$ with smooth functions on $\mathbf{R}^3$, and check that:

$d$ on 0-forms is the gradient;
$d$ on 1-forms is the curl;
$d$ on 2-forms is the divergent.

Also, interpret $d^2 = 0$ is those terms.

§ 2.4

**15** Let $M$ and $N$ be smooth manifolds where $M$ is connected, and consider the projection $\pi : M \times N \to N$ onto the second factor. Prove that a $k$-form $\omega$ on $M \times N$ is of the form $\pi^*\eta$ for some $k$-form $\eta$ on $N$ if and only if $\iota_X\omega = L_X\omega = 0$ for every $X \in \mathfrak{X}(M \times N)$ satisfying $d\pi \circ X = 0$.

**16** Let $M$ be a smooth manifold.

*a.* Prove that $\iota_X\iota_X = 0$ for every $X \in \mathfrak{X}(M)$.

*b.* Prove that $\iota_{[X,Y]}\omega = L_X\iota_Y\omega - \iota_Y L_X\omega$ for every $X$, $Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$.

§ 2.5

**17** The *Whitney sum* $E_1 \oplus E_2$ of two vector bundles $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ is a vector bundle $\pi : E = E_1 \oplus E_2 \to M$ where $E_p = (E_1)_p \oplus (E_2)_p$ for all $p \in M$.

*a.* Show that $E_1 \oplus E_2$ is indeed a vector bundle by expressing its local trivializations in terms of those of $E_1$ and $E_2$ and checking the conditions of Definition 2.5.2.

*b.* Similarly, construct the tensor product bundle $E_1 \otimes E_2$ and the dual bundle $E^*$.

CHAPTER 3

# Lie groups

Lie groups are amongst the most important examples of smooth manifolds. At the same time, almost all usually encountered examples of smooth manifolds are related to Lie groups, in a way or another. A Lie group is a smooth manifold with an additional, compatible structure of group. Here compatibility refers to the fact that the group operations are smooth (another point of view is to regard a Lie group as a group with an additional structure of manifold...). The reader can keep in mind the matrix group $\mathbf{GL}(n, \mathbf{R})$ of non-singular real $n \times n$ matrices (Examples 1.2.7) in which the $n^2$ matrix coefficients form a global coordinate system. The conjuction of the smooth and the group structures allows one to give a more explicit description of the differential invariants attached to a manifold. For this reason, Lie groups form a class of manifolds suitable for testing general hypotheses and conjectures. The same remarks apply to homogeneous spaces, which are certain quotients of Lie groups.

## 3.1 Basic definitions and examples

A *Lie group* $G$ is a smooth manifold endowed with a group structure such that the group operations are smooth. More concretely, the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are required to be smooth.

**3.1.1 Examples** (a) The Euclidean space $\mathbf{R}^n$ with its additive vector space structure is a Lie group. Since the multiplication is commutative, this is an example of a *Abelian* (or *commutative*) Lie group.

(b) The multiplicative group of nonzero complex numbers $\mathbf{C}^\times$. The subgroup of unit complex numbers is also a Lie group, and as a smooth manifold it is diffeomorphic to the circle $S^1$. This is also an Abelian Lie group.

(c) If $G$ and $H$ are Lie groups, the direct product group structure turns the product manifold $G \times H$ into a Lie group.

(d) It follows from (b) and (c) that the *$n$-torus* $T^n = S^1 \times \cdots \times S^1$ ($n$ times) is a Lie group. Of course, $T^n$ is a compact connected Abelian Lie group. Conversely, we will see in Theorem 3.5.3 that every compact connected Abelian Lie group is an $n$-torus.

(e) If $G$ is a Lie group, the connected component of the identity of $G$, denoted by $G^\circ$, is also a Lie group. Indeed, $G^\circ$ is open in $G$, so it inherits a smooth structure from $G$ just by restricting the local charts. Since $\mu(G^\circ \times G^\circ)$ is connected and $\mu(1,1) = 1$, we must have $\mu(G^\circ \times G^\circ) \subset G^\circ$. Similarly, $\iota(G^\circ) \subset G^\circ$. Since $G^\circ \subset G$ is an open submanifold, it follows that the group operations restricted to $G^\circ$ are smooth.

(f) Any finite or countable group endowed with the discrete topology becomes a 0-dimensional Lie group. Such examples are called *discrete Lie groups*.

(g) We now turn to some of the classical matrix groups. The general linear group $\mathbf{GL}(n, \mathbf{R})$ is a Lie group since the entries of the product of two matrices is a quadratic polynomial on the entries of the two matrices, and the entries of inverse of a non-singular matrix is a rational function on the entries of the matrix.

Similarly, one defines the *complex general linear group of order $n$*, which is denoted by $\mathbf{GL}(n, \mathbf{C})$, as the group consisting of all nonsingular $n \times n$ complex matrices, and checks that it is a Lie group. Note that $\dim \mathbf{GL}(n, \mathbf{C}) = 2n^2$ and $\mathbf{GL}(1, \mathbf{C}) = \mathbf{C}^\times$.

We have already encountered the orthogonal group $\mathbf{O}(n)$ as a closed embedded submanifold of $\mathbf{GL}(n, \mathbf{R})$ in 1.4.14. Since $\mathbf{O}(n)$ is an embedded submanifold, it follows from Theorem 1.4.9 that the group operations of $\mathbf{O}(n)$ are smooth, and hence $\mathbf{O}(n)$ is a Lie group.

Similarly to $\mathbf{O}(n)$, one checks that the

$$\begin{aligned}
\mathbf{SL}(n, \mathbf{R}) &= \{A \in \mathbf{GL}(n, \mathbf{R}) \mid \det(A) = 1\} \quad (\textit{real special linear group}) \\
\mathbf{SL}(n, \mathbf{C}) &= \{A \in \mathbf{GL}(n, \mathbf{C}) \mid \det(A) = 1\} \quad (\textit{complex special linear group}) \\
\mathbf{U}(n) &= \{A \in \mathbf{GL}(n, \mathbf{C}) \mid AA^* = I\} \quad (\textit{unitary group}) \\
\mathbf{SO}(n) &= \{A \in \mathbf{O}(n) \mid \det(A) = 1\} \quad (\textit{special orthogonal group}) \\
\mathbf{SU}(n) &= \{A \in \mathbf{U}(n) \mid \det(A) = 1\} \quad (\textit{special unitary group})
\end{aligned}$$

are Lie groups, where $A^*$ denotes the complex conjugate transpose matrix of $A$. Note that $\mathbf{U}(1) = S^1$.

### Lie algebras

For an arbitrary smooth manifold $M$, the space $\mathfrak{X}(M)$ of smooth vector fields on $M$ is an infinite-dimensional vector space over $\mathbf{R}$. In addition, we have already encountered the Lie bracket, a bilinear map $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying:

$a.$ $[Y, X] = -[X, Y]$;

*b.* $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ (Jacobi identity);
for every $X, Y \in \mathfrak{X}(M)$. In general, a vector space with a bilinear operation satisfying (a) and (b) above is called a *Lie algebra*. So $\mathfrak{X}(M)$ is Lie algebra over **R**.

It turns out in case of a Lie group $G$, we can single out a finite dimensional subalgebra of $\mathfrak{X}(M)$. For that purpose, let us first introduce translations in $G$. The *left translation* defined by $g \in G$ is the map $L_g : G \to G$, $L_g(x) = gx$. It is a diffeomorphism of $G$, its inverse being given by $L_{g^{-1}}$. Similarly, the *right translation* defined by $g \in G$ is the map $R_g : G \to G$, $R_g(x) = xg$. It is also a diffeomorphism of $G$, and its inverse is given by $R_{g^{-1}}$.

The translations in $G$ define canonical identifications between the tangent spaces to $G$ at different points. For instance, $dL_g : T_hG \to T_{gh}G$ is an isomorphism for every $g$, $h \in G$. This allows us to consider invariant tensors, the most important case being that of vector fields. A vector field $X$ on $G$ is called *left-invariant* if $d(L_g)_x(X_x) = X_{gx}$ for every $g$, $x \in X$. This condition is simply $dL_g \circ X = X \circ L_g$ for every $g \in G$; equivalently, $X$ is $L_g$-related to itself, or yet $L_{g*}X = X$ (since $L_g$ is a diffeomorphism), for all $g \in G$. We can similarly define *right-invariant* vector fields, but most often we will be considering the left-invariant variety. Note that left-invariance and right-invariance are the same property in case of an Abelian group.

**3.1.2 Lemma** *Every left invariant vector field $X$ in $G$ is smooth.*

*Proof.* Let $f$ be a smooth function defined on a neighborhood of $1$ in $G$, and let $\gamma : (-\epsilon, \epsilon) \to G$ be a smooth curve with $\gamma(0) = 1$ and $\gamma'(0) = X_1$. Then the value of $X$ on $f$ is given by

$$X_g(f) = dL_g(X_1)(f) = X_1(f \circ L_g) = \frac{d}{dt}\Big|_{t=0} f(g\gamma(t)) = \frac{d}{dt}\Big|_{t=0} f \circ \mu(g, \gamma(t)),$$

and hence, it is a smooth function of $g$. □

Let $\mathfrak{g}$ denote the set of left invariant vector fields on $G$. It follows that $\mathfrak{g}$ is a vector subspace of $\mathfrak{X}(M)$. Further, $\mathfrak{g}$ is a subalgebra of $\mathfrak{X}(M)$, for given $X, Y \in \mathfrak{g}$, we have by Proposition 1.6.18 that

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y],$$

for every $g \in G$. Finally, we explain why $\mathfrak{g}$ is finite-dimensional: the map $X \in \mathfrak{g} \mapsto X_1$ defines a linear isomorphism between $\mathfrak{g}$ and the tangent space to $G$ at the identity $T_1G$, since any left invariant vector field is completely defined by its value at the identity.

The discussion above shows that to any Lie group $G$ is naturally associated a (real) finite-dimensional Lie algebra $\mathfrak{g}$ of the same dimension as $G$, consisting of the left invariant vector fields on $G$. This Lie algebra is the infinitesimal object associated to $G$ and, as we shall see, completely determines its local structure.

**3.1.3 Examples** (*The Lie algebras of some known Lie groups*)

(i) The left-invariant vector fields on $\mathbf{R}^n$ are precisely the constant vector fields, namely, the linear combinations of coordinate vector fields (in the canonical coordinate system) with constant coefficients. The bracket of two constant vector fields on $\mathbf{R}^n$ is zero. It follows that the Lie algebra of $\mathbf{R}^n$ is $\mathbf{R}^n$ itself with the null bracket. In general, a vector space equipped with the null bracket is called an *Abelian* Lie algebra.

(ii) The Lie algebra of the direct product $G \times H$ is the direct sum of Lie algebras $\mathfrak{g} \oplus \mathfrak{h}$, where the bracket is taken componentwise.

(iii) Owing to the skew-symmetry of the Lie bracket, every one-dimensional Lie algebra is Abelian. In particular, the Lie algebra of $S^1$ is Abelian. It follows from (ii) that also the Lie algebra of $T^n$ is Abelian.

(iv) $G$ and $G^\circ$ have the same Lie algebra.

(v) The Lie algebra of a discrete group is $\{0\}$.

**3.1.4 Examples** (*Some abstract Lie algebras*)

(i) Let $A$ be any real associative algebra and set $[a, b] = ab - ba$ for $a$, $b \in A$. It is easy to see that $A$ becomes a Lie algebra.

(ii) The cross-product $\times$ on $\mathbf{R}^3$ is easily seen to define a Lie algebra structure.

(iii) If $V$ is a two-dimensional vector space and $X$, $Y \in V$ are linearly independent, the conditions $[X, X] = [Y, Y] = 0$, $[X, Y] = X$ define a Lie algebra structure on $V$.

(iv) If $V$ is a three-dimensional vector space spanned by $X$, $Y$, $Z$, the conditions $[X, Y] = Z$, $[Z, X] = [Z, Y] = 0$ define a Lie algebra structure on $V$, called the *(3-dimensional) Heisenberg algebra*. It can be realized as a Lie algebra of smooth vector fields on $\mathbf{R}^3$ as in Example 1.6.15(b).

**3.1.5 Exercise** Check the assertions of Examples 3.1.3 and 3.1.4.

## 3.2 The exponential map

For a Lie group $G$, we have constructed its most basic invariant, its Lie algebra $\mathfrak{g}$. Our next step will be to present the fundamental map that relates $G$ and $\mathfrak{g}$, namely, the exponential map $\exp : \mathfrak{g} \to G$.

### Matrix exponential

Recall that the exponential of a matrix $A \in \mathbf{M}(n, \mathbf{R})$ (or $\mathbf{M}(n, \mathbf{C})$) is given by the formula:

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \end{aligned}$$

Since $||\sum_{k=0}^{n}\frac{1}{k!}A^k|| \leq e^{||A||}$ for all $n \geq 0$, the series is absolutely convergent on the entire $\mathbf{M}(n,\mathbf{R})$; here $||\cdot||$ denotes the usual Euclidean norm in $\mathbf{M}(n,\mathbf{R}) = \mathbf{R}^{n^2}$. In case $n = 1$, we recover the usual exponential map on the line. In general, note that:

*a.* $e^0 = I$;

*b.* $e^{A+B} = e^A e^B$ if $A$ and $B$ commute.

Indeed, to check (b) notice that one can compute the product of $e^A$ and $e^B$ by multiplying the individual terms and rearranging, by absolute convergence. In particular:

*c.* $e^{(s+t)A} = e^{sA}e^{tB}$ for all $s, t \in \mathbf{R}$;

*d.* $e^A$ is invertible and $(e^A)^{-1} = e^{-A}$.

View $t \in \mathbf{R} \mapsto e^{tA}$ as a curve in $\mathbf{M}(n,\mathbf{R})$. The last property worth mentioning is

*e.* $\frac{d}{dt}\big|_{t=t_0} e^{tA} = A\, e^{t_0 A} = e^{t_0 A}\, A$.

**Flow of left-invariant vector fields**

Let $G$ be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra.

**3.2.1 Proposition** *Every left-invariant vector field is complete.*

*Proof.* Given $X \in \mathfrak{g}$, there exists a maximal integral curve $\gamma_X : (a,b) \to G$ of $X$ with $0 \in (a,b)$ ($a$, $b \in [-\infty,\infty]$) and $\gamma_X(0) = 1$; namely, $\gamma_X'(t) = X_{\gamma_X(t)}$. Since

$$\frac{d}{dt}\Big|_{t=t_0} L_g(\gamma_X(t)) = d(L_g)(X_{\gamma_X(t_0)}) = X_{L_g(\gamma_X(t_0))},$$

we have that $L_g \circ \gamma_X$ is an integral curve of $X$ starting at $g$. In particular, if $b < \infty$, by taking $g = \gamma(s)$ with $s$ very close to $b$, this shows that $\gamma_X$ can be extended beyond $b$, leading to a contradiction. Similarly, one sees that $a = -\infty$. Hence $X$ is complete. ☐

Now the integral curve $\gamma_X$ of any $X \in \mathfrak{g}$ starting at the identity is defined on $\mathbf{R}$. The *exponential map* of $G$ is the map $\exp : \mathfrak{g} \to G$ defined by $\exp X = \gamma_X(1)$.

Note that $\frac{d}{ds}\big|_{s=s_0}\gamma_X(ts) = t\gamma_X'(ts_0) = tX(\gamma_X(ts_0))$. This implies $\gamma_X(ts) = \gamma_{tX}(s)$ for all $s, t \in \mathbf{R}$ and therefore

$$\begin{aligned} \gamma_X(t) &= \gamma_{tX}(1) \\ &= \exp(tX), \end{aligned} \tag{3.2.2}$$

namely, every integral curve of a left-invariant vector field through the identity factors through the exponential map.

**3.2.3 Exercise** Check that the flow $\{\varphi_t\}$ of a left-invariant vector field $X$ is given by $\varphi_t = R_{\exp tX}$ (recall that $R_g$ denotes a right-translation). What is the corresponding result for right-invariant vector fields?

Moreover, we state:

**3.2.4 Proposition** *The exponential map* $\exp : \mathfrak{g} \to G$ *is smooth and it is a local diffeomorphism at* $0$*.*

*Proof.* Smoothness follows from general properties of flows, namely, smooth dependence on parameters of solutions of ODE's. Moreover, $d\exp_0 : T_0\mathfrak{g} \cong \mathfrak{g} \to T_1G \cong \mathfrak{g}$ is the identity, since

$$d\exp_0(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX) = \varphi_X'(0) = X.$$

Thus, $\exp$ is a diffeomorphism from a neighborhood of $0$ in $\mathfrak{g}$ onto a neighborhood of $1$ in $G$ by the Inverse Function Theorem (1.3.8). □

Recall that the identity component $G^\circ$ is an open subgroup of $G$.

**3.2.5 Proposition** $G^\circ$ *is generated as a group by any neighborhood* $U$ *of* $1$ *in* $G^\circ$*, namely,*

$$G^\circ = \bigcup_{n\geq 1} U^n,$$

*where* $U^n$ *denotes the set of* $n$*-fold products of elements in* $U$*. In particular,* $G^\circ$ *is generated by* $\exp[\mathfrak{g}]$*.*

*Proof.* By replacing $U$ by $U \cap U^{-1}$, if necessary, we may assume that $U = U^{-1}$. Define $V = \cup_{n\geq 0} U^n$ and consider the relation in $G^\circ$ given by $g \sim g'$ if and only if $g^{-1}g' \in V$. Note that this is an equivalence relation, and equivalence classes are open as $g' \sim g$ implies $g'U \sim g$, where $g'U$ is an open neighborhood of $g'$. Hence $V = G^\circ$. □

### The case of $\mathbf{GL}(n, \mathbf{R})$

Recall that $G = \mathbf{GL}(n, \mathbf{R})$ inherits its manifold structure as an open subset of the Euclidean space $\mathbf{M}(n, \mathbf{R})$. In particular, the tangent space at the identity $T_IG = \mathbf{M}(n, \mathbf{R})$. Let $A \in \mathbf{M}(n, \mathbf{R})$ and denote by $\tilde{A} \in \mathfrak{g}$ the corresponding left-invariant vector field on $G$. For any $g \in G$, we have $\tilde{A}_g = (dL_g)(A) = gA$ (matrix multiplication on the right hand side).

Using property (e) of the matrix exponential,

$$\frac{d}{dt}\Big|_{t=t_0} e^{tA} = e^{t_0A}A = \tilde{A}_{e^{t_0A}}$$

shows that $t \mapsto e^{tA}$ is the integral curve of $\tilde{A}$ through the identity, namely

$$\exp \tilde{A} = e^A$$

for all $A \in \mathbf{M}(n, \mathbf{R})$.

Finally, to determine the Lie bracket in $\mathfrak{g}$, we resort to (1.6.22). Let $A$, $B \in M(n, \mathbf{R})$, denote by $\tilde{A}$, $\tilde{B}$ the corresponding left-invariant vector fields on $G$, let $\{\varphi_t = R_{e^{tA}}\}$ be the flow of $\tilde{A}$ (cf. Exercise 3.2.3):

$$\begin{aligned}
[A, B] &= [\tilde{A}, \tilde{B}]_I \\
&= (L_{\tilde{A}}\tilde{B})_I \\
&= \frac{d}{dt}\Big|_{t=0} d\varphi_{-t}(\tilde{B}_{\varphi_t(I)}) \\
&= \frac{d}{dt}\Big|_{t=0} e^{tA}Be^{-tA} \\
&= AB - BA.
\end{aligned}$$

Note that the Lie algebra structure in $\mathbf{M}(n, \mathbf{R})$ is induced from its associative algebra structure as in Example 3.1.4(i). The space $\mathbf{M}(n, \mathbf{R})$ with this Lie algebra structure will be denoted by $\mathfrak{gl}(n, \mathbf{R})$.

The case of $\mathbf{GL}(n, \mathbf{C})$ is completely analogous.

## 3.3 Homomorphisms and Lie subgroups

A *(Lie group) homomorphism* between Lie groups $G$ and $H$ is map $\varphi : G \to H$ which is both a group homomorphism and a smooth map. $\varphi$ is called an *isomorphism* if, in addition, it is a diffeomorphism. An *automorphism* of a Lie group is an isomorphism of the Lie group with itself. A *(Lie algebra) homomorphism* between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ which preserves brackets. $\Phi$ is called an *isomorphism* if, in addition, it is bijective. An *automorphism* of a Lie algebra is an isomorphism of the Lie algebra with itself.

**3.3.1 Exercise** For a homomorphism $\varphi : G \to H$, check that $L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$ for all $g \in G$.

A homomorphism $\varphi : G \to H$ between Lie groups induces a linear map $d\varphi_1 : T_1G \to T_1H$ and hence a linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Indeed, if $X$ is a left invariant vector field on $G$, let $X'$ be the unique left invariant vector field on $H$ such that $X'_1 = d\varphi_1(X_1)$ and put $d\varphi(X) = X'$.

**3.3.2 Proposition** *If $\varphi : G \to H$ is a homomorphism between Lie groups then $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism between the corresponding Lie algebras.*

*Proof.* Let $X \in \mathfrak{g}$. We first claim that $X$ and $X' := d\varphi(X)$ are $\varphi$-related. In fact,

$$X'_{\varphi(g)} = d(L_{\varphi(g)})_1(X'_1) = d(L_{\varphi(g)} \circ \varphi)_1(X_1) = d(\varphi \circ L_g)_1(X_1) = d\varphi_g(X_g),$$

proving the claim. Now, if $Y \in \mathfrak{g}$, then $Y$ and $\varphi(Y)$ are $\varphi$-related. Therefore $[X,Y]$ and $[d\varphi(X), d\varphi(Y)]$ are $\varphi$-related and thus

$$d\varphi([X,Y]_1) = [d\varphi(X), d\varphi(Y)]_{\varphi(1)},$$

or

$$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)].$$

This shows that $d\varphi$ is a Lie algebra homomorphism. □

Let $G$ be a Lie group. A *Lie subgroup* of $G$ is an immersed submanifold $(H, \varphi)$ of $G$ such that $H$ is a Lie group and $\varphi : H \to G$ is a homomorphism.

**3.3.3 Remark** Similarly as in the case of immersed submanifolds (Problem 19 in Chapter 1), we consider two Lie subgroups $(H_1, \varphi_1)$ and $(H_2, \varphi_2)$ of $G$ equivalent if there exists a Lie group isomorphism $\alpha : H_1 \to H_2$ such that $\varphi_1 = \varphi_2 \circ \alpha$. This is an equivalence relation in the class of Lie subgroups of $G$ and each equivalence class contains a unique representative of the form $(A, \iota)$, where $A$ is a subset of $G$ (an actual subgroup) and $\iota : A \to G$ is the inclusion. So we lose no generality in assuming that a Lie subgroup of $G$ is an abstract subgroup $H$ of $G$ which is an immersed submanifold of $G$ and a Lie group with respect to the operations induced from $G$; namely, the multiplication and inversion in $G$ must restrict to smooth maps $H \times H \to H$ and $H \to H$, respectively.

**3.3.4 Example** The skew-line $(\mathbf{R}, f)$ in $T^2$ (Example 1.4.2) is an example of a Lie subgroup of $T^2$ which is not closed.

If $\mathfrak{g}$ is a Lie algebra, a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Lie subalgebra* if $\mathfrak{h}$ is closed under the bracket of $\mathfrak{g}$.

Let $H$ be a Lie subgroup of $G$, say, $\iota : H \to G$ is the inclusion map. Since $\iota$ is an immersion, $d\iota : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras, and we may and will view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$. Conversely, as our most important application of Frobenius' theorem, we have:

**3.3.5 Theorem (Lie)** *Let $G$ be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup $H$ of $G$ such that the Lie algebra of $H$ is $\mathfrak{h}$.*

*Proof.* We have that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and so defines a subspace $\mathfrak{h}(1) := \{X(1) \mid X \in \mathfrak{h}\}$ of $T_1G$. Let $\mathcal{D}$ be the left-invariant distribution on $G$ defined by $\mathfrak{h}$, namely, $\mathcal{D}_g = dL_g(\mathfrak{h}(1))$ for all $g \in G$. Then $\mathcal{D}$ is a smooth distribution, as it is globally generated by left-invariant vector fields $X_1, \ldots, X_k$ in $\mathfrak{h}$. The fact that $\mathcal{D}$ is involutive follows from (and is equivalent to) $\mathfrak{h}$ being a

Lie subalgebra of $\mathfrak{g}$. In fact, suppose $X$ and $Y$ lie in $\mathcal{D}$ over the open subset $U$ of $G$. Write $X = \sum_i a_i X_i$, $Y = \sum_j b_j X_j$ for $a_i, b_j \in C^\infty(U)$. Then

$$[X,Y] = \sum_{i,j} a_i b_j [X_i, Y_j] + a_i X_i(b_j) X_j - b_j X_j(a_i) X_i$$

also lies in $\mathcal{D}$, as $[X_i, Y_j] \in \mathfrak{h}$.

By Frobenius theorem (1.7.10), there exists a unique maximal integral manifold of $\mathcal{D}$ passing through 1, which we call $H$. Since $\mathcal{D}$ is left-invariant, for every $h \in H$, $L_{h^{-1}}(H) = h^{-1}H$ is also a maximal integral manifold of $\mathcal{D}$, and it passes through through $h^{-1}h = 1$. This implies $h^{-1}H = H$, by uniqueness. It follows that $H$ is a subgroup of $G$. The operations induced by $G$ on $H$ are smooth because $H$ is an initial submanifold, due to Proposition 1.7.3. This proves that $H$ is a Lie group. Its Lie algebra is $\mathfrak{h}$ because $\mathfrak{h}$ consists precisely of the elements of $\mathfrak{g}$ whose value at 1 lies in $\mathcal{D}_1 = T_1 H$, and these are exacly the elements of the Lie algebra of $H$.

Suppose now $H'$ is another Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. Then $H'$ must also be an integral manifold of $\mathcal{D}$ through 1. By the maximality of $H$, we have $H' \subset H$, and the inclusion map is smooth by Proposition 1.7.3 and thus an immersion. Now $H'$ is an open submanifold of $H$ and contains a neighborhood of 1 in $H$. Owing to Propostion 3.2.5, $H' = H$. □

**3.3.6 Corollary** *There is a bijective correspondence between connected Lie subgroups of a Lie group and subalgebras of its Lie algebra.*

**3.3.7 Example** Let $G$ be a Lie group. A subgroup $H$ of $G$ which is an embedded submanifold of $G$ is a Lie subgroup of $G$ by Proposition 1.4.9. It follows from Example 1.4.14(b) that $\mathbf{O}(n)$ is a closed Lie subgroup of $\mathbf{GL}(n, \mathbf{R})$. Similarly, the other matrix groups listed in Examples 3.1.1(g) are closed Lie subgroups of $\mathbf{GL}(n, \mathbf{R})$, except that $\mathbf{SL}(n, \mathbf{C})$ is a closed Lie subgroup of $\mathbf{GL}(n, \mathbf{C})$. In particular, the Lie bracket in those subgroups is given by $[A, B] = AB - BA$.

**3.3.8 Exercise** Show that the Lie algebras of the matrix groups listed in Examples 3.1.1(g) are respectively as follows:

$$\begin{aligned}
\mathfrak{o}(n) &= \{A \in \mathfrak{gl}(n, \mathbf{R}) \mid A + A^t = 0\} \\
\mathfrak{sl}(n, \mathbf{R}) &= \{A \in \mathfrak{gl}(n, \mathbf{R}) \mid \operatorname{trace}(A) = 0\} \\
\mathfrak{sl}(n, \mathbf{C}) &= \{A \in \mathfrak{gl}(n, \mathbf{C}) \mid \operatorname{trace}(A) = 0\} \\
\mathfrak{u}(n) &= \{A \in \mathfrak{gl}(n, \mathbf{C}) \mid A + A^* = 0\} \\
\mathfrak{so}(n) &= \mathfrak{o}(n) \\
\mathfrak{su}(n) &= \{A \in \mathfrak{u}(n) \mid \operatorname{trace}(A) = 0\}
\end{aligned}$$

A Lie group homomorphism $\varphi : \mathbf{R} \to G$ is called a *(smooth) one-parameter subgroup*. Note that such a $\varphi$ is the integral curve of $X := d\varphi(1) \in \mathfrak{g}$, and we have seen in (3.2.2) that $\varphi(t) = \exp(tX)$ for all $t \in \mathbf{R}$.

More generally, let $\varphi : G \to H$ be a homomorphism between Lie groups. Then, for a left invariant vector field $X$ on $G$, $t \mapsto \varphi(\exp^G(tX))$ is a one-parameter subgroup of $H$ with $\frac{d}{dt}\big|_{t=0}\varphi(\exp^G tX) = d\varphi(X_1)$. In view of the above,

$$\varphi \circ \exp^G X = \exp^H \circ d\varphi(X), \tag{3.3.9}$$

for every $X \in \mathfrak{g}$. In particular, if $K$ is a Lie subgroup of $G$, then the inclusion map $\iota : K \to G$ is a Lie group homomorphism, so that the exponential map of $G$ restricts to the exponential map of $K$, and the connected component of $K$ is generated by $\exp^G[\mathfrak{k}]$, where $\mathfrak{k}$ is the Lie algebra of $K$. It follows also that

$$\mathfrak{k} = \{X \in \mathfrak{g} : \exp^G(tX) \in K, \text{ for all } t \in \mathbf{R}\}. \tag{3.3.10}$$

Indeed, let $X \in \mathfrak{g}$ with $\exp^G(tX) \in K$ for all $t \in \mathbf{R}$. Since $K$ is an integral manifold of an involutive distribution (compare Theorem 3.3.5), $t \mapsto \exp^G(tX)$ defines a smooth map $\mathbf{R} \to K$ and thus a one-parameter subgroup of $K$. Therefore $\exp^G(tX) = i \circ \exp^K(tX')$ for some $X' \in \mathfrak{k}$, and hence $X = di(X')$.

## 3.4 Covering Lie groups

Let $G$ be a connected Lie group. Consider the universal covering $\pi : \tilde{G} \to G$. By Problem 5 in Chapter 1 or the results in Appendix A, $\tilde{G}$ has a unique smooth structure for which $\pi$ is a local diffeomorphism.

**3.4.1 Theorem** *Every connected Lie group $G$ has a simply-connected covering $\pi : \tilde{G} \to G$ such that $\tilde{G}$ is a Lie group and $\pi$ is a Lie group homomorphism.*

*Proof*. Consider the smooth map $\alpha : \tilde{G} \times \tilde{G} \to G$ given by $\alpha(\tilde{g}, \tilde{h}) = \pi(\tilde{g})\pi(\tilde{h})^{-1}$. Choose $\tilde{1} \in \pi^{-1}(1)$. As $\tilde{G}$ is simply-connected, so is $\tilde{G} \times \tilde{G}$. By the lifting criterion, there exists a unique map smooth $\tilde{\alpha} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$. Put

$$\tilde{g}^{-1} := \tilde{\alpha}(\tilde{1}, \tilde{g}), \qquad \tilde{g}\tilde{h} := \tilde{\alpha}(\tilde{g}, \tilde{h}^{-1})$$

for $\tilde{g}$, $\tilde{h} \in \tilde{G}$. These operations are shown to make $\tilde{G}$ into a group by use of the uniqueness part in the lifting criterion. As an example,

$$\pi(\tilde{g}\tilde{1}) = \pi\tilde{\alpha}(\tilde{g}, \tilde{1}^{-1}) = \alpha(\tilde{g}, \tilde{1}^{-1}) = \pi(\tilde{g})\pi(\tilde{1}^{-1})^{-1} = \pi(\tilde{g}) \tag{3.4.2}$$

since $\tilde{1}^{-1} = \tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$ and $\pi(\tilde{1}) = 1$. Identity (3.4.2) shows that $\tilde{g} \mapsto \tilde{g}\tilde{1}$ is a lifting of $\tilde{g} \mapsto \pi(\tilde{g})$, $\tilde{G} \to G$, to a map $\tilde{G} \to \tilde{G}$ which takes $\tilde{1}$ to $\tilde{1} \cdot \tilde{1} =$

$\tilde{\alpha}(\tilde{1}, \tilde{1}^{-1}) = \tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$. However, the identity map of $\tilde{G}$ is also a lifting of $\tilde{g} \mapsto \pi(\tilde{G})$ which takes $\tilde{1}$ to $\tilde{1}$. By uniqueness, both liftings coincide and $\tilde{g}\tilde{1} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$.

Now $\tilde{G}$ is a group. Since $\tilde{\alpha}$ is smooth, $\tilde{G}$ is a Lie group. Finally,

$$\pi(\tilde{g}^{-1}) = \pi\tilde{\alpha}(\tilde{1}, \tilde{g}) = \alpha(\tilde{1}, \tilde{g}) = \pi(\tilde{1})\pi(\tilde{g})^{-1} = \pi(\tilde{g})^{-1}$$

and

$$\pi(\tilde{g}\tilde{h}) = \pi\tilde{\alpha}(\tilde{g}, \tilde{h}^{-1}) = \alpha(\tilde{g}, \tilde{h}^{-1}) = \pi(\tilde{g})\pi(\tilde{h}^{-1})^{-1} = \pi(\tilde{g})\pi(\tilde{h}).$$

Hence, $\pi : \tilde{G} \to G$ is a Lie group homomorphism. □

**3.4.3 Remark** It follows from Lemma 3.4.4(c) and Theorem 3.7.7 that the structure of Lie group on the universal covering $\tilde{G}$ of $G$ is unique, up to isomorphism.

**3.4.4 Lemma** *Let $\varphi : G \to H$ be a homomorphism between Lie groups. Consider the induced homomorphism between the corresponding Lie algebras $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Then:*

*a. $d\varphi$ is injective if and only if the kernel of $\varphi$ is discrete.*

*b. $d\varphi$ is surjective if and only if $\varphi(G^\circ) = H^\circ$.*

*c. $d\varphi$ is bijective if and only if $\varphi$ is a topological covering (here we assume $G$ and $H$ connected).*

*Proof.* (a) If $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is injective, then $\varphi$ is an immersion at $1$ and thus everywhere by Exercise 3.3.1. Therefore $\varphi$ is locally injective and hence $\ker\varphi$ is discrete. Conversely, if $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is not injective, $\ker d\varphi_g$ is positive-dimensional for all $g \in G$ and thus defines a smooth distribution $\mathcal{D}$. Note that $X$ lies in $\mathcal{D}$ if and only if $X$ is $\varphi$-related to the null vector field on $H$. It follows that $\mathcal{D}$ is involutive. The maximal integral manifold of $\mathcal{D}$ through the identity is collapsed to a point under $\varphi$ implying that $\ker\varphi$ is not discrete.

(b) Since $\varphi \circ \exp = \exp \circ d\varphi$ and $G^\circ$ is generated by $\exp[\mathfrak{g}]$, we have that $\varphi(G^\circ)$ is the subgroup of $H^\circ$ generated by $\exp[d\varphi(\mathfrak{g})]$, thus $\varphi(G^\circ) = H^\circ$ if $d\varphi$ is surjective. On the other hand, if $d\varphi$ is not surjective, $d\varphi(\mathfrak{g})$ is a proper subalgebra of $\mathfrak{h}$ to which there corresponds a connected, proper subgroup $K$ of $H^\circ$, and $\exp[d\varphi(\mathfrak{g})]$ generates $K$.

(c) Assume $G$, $H$ connected. If $\varphi$ is a covering then $\ker d\varphi$ is discrete and $\varphi$ is surjective, so $d\varphi$ is an isomorphism by (a) and (b). Conversely, suppose that $d\varphi : \mathfrak{g} \to \mathfrak{h}$ an isomorphism. Then $\varphi$ is surjective by (b). Let $U$ be a neighborhood of $1$ in $G$ such that $\varphi : U \to \varphi(U) := V$ is a diffeomorphism. We can choose $U$ so that $U \cap \ker d\varphi = \{1\}$ by (a). Then $\varphi^{-1}(V) = \cup_{n\in\ker\varphi} nU$, and this is a (disjoint union) for $ng = n'g'$ with $n$, $n' \in \ker\varphi$ and $g$, $g' \in U$ implies $gg'^{-1} = n^{-1}n' \in \ker\varphi$ and so $\varphi(g) = \varphi(g')$ and then $g = g'$. Since $\varphi \circ L_n = \varphi$ for $n \in \ker\varphi$, we also have that $\varphi|nU$ is a diffeomorphism onto

$V$. This shows that $V$ is an evenly covered neighborhood of 1. Now $hV$ is an evenly covered neighborhood of any given $h \in H$, and thus $\varphi$ is a covering. □

**3.4.5 Theorem** *Let $G_1$, $G_2$ be Lie groups, and assume that $G_1$ is connected and simply-connected. Then, given a homomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between the Lie algebras, there exists a unique homomorphism $\varphi : G_1 \to G_2$ such that $d\varphi = \Phi$.*

*Proof.* The graph of $\Phi$, $\mathfrak{h} = \{(X, \Phi(X)) : X \in \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let $H$ be the subgroup of $G_1 \times G_2$ defined by $\mathfrak{h}$ (Theorem 3.3.5). Consider the projections

$$\Phi_i : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_i, \qquad \varphi_i : G_1 \times G_2 \to G_i,$$

for $i = 1, 2$. Since $\Phi_1|\mathfrak{h} : \mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism, we have that $\Phi = \Phi_2 \circ (\Phi_1|\mathfrak{h})^{-1}$ and $\varphi_1 : H \to G_1$ is a covering. Since $G_1$ is simply-connected, $\varphi_1|H : H \to G_1$ is an isomorphism of Lie groups, and we can thus define $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$. This proves the existence part. The uniqueness part comes from the fact that $d\varphi = \Phi$ specifies $\varphi$ in a neighborhood of 1 (by using the exponential map as in (3.3.9)), and $G_1$ is generated by this neighborhood. □

## 3.5 The adjoint representation

Let $G$ be a Lie group, and denote its Lie algebra by $\mathfrak{g}$. The noncommutativity of $G$ is organized by the adjoint representation. In order to introduce it, let $g \in G$, and define a map $\mathrm{Inn}_g : G \to G$ by $\mathrm{Inn}_g(x) = gxg^{-1}$. Then $\mathrm{Inn}_g$ is an automorphism of $G$, which is called the *inner automorphism defined by* $g$. The differential $d(\mathrm{Inn}_g) : \mathfrak{g} \to \mathfrak{g}$ defines an automorphism of $\mathfrak{g}$, which we denote by $\mathrm{Ad}_g$. Then

$$\mathrm{Ad}_g X = \frac{d}{dt}\Big|_{t=0} \mathrm{Inn}(g)(\exp tX) = \frac{d}{dt}\Big|_{t=0} g \exp tX g^{-1}.$$

**3.5.1 Example** *In case $G = GL(n, \mathbf{R})$, $\mathrm{Inn_g}$ is the restriction of the linear map $\mathbf{M}(n, \mathbf{R}) \to \mathbf{M}(n, \mathbf{R})$, $X \mapsto gXg^{-1}$, so $\mathrm{Ad}_g X = d(\mathrm{Inn}_g)_1(X) = gXg^{-1}$.*

This defines a homomorphism

$$\mathrm{Ad} : g \in G \to \mathrm{Ad}_g \in \mathbf{GL}(\mathfrak{g}),$$

which is called the *adjoint representation* of $G$ on $\mathfrak{g}$.

We have

$$\begin{aligned}(\mathrm{Ad}_g X)_1 &= (dL_g)(dR_{g^{-1}})X_1 \\ &= (dR_{g^{-1}})(dL_g)X_1 \\ &= (dR_{g^{-1}})(X_g) \\ &= (dR_g^{-1}\circ X\circ R_g)(1) \\ &= \big((R_{g^{-1}})_*X\big)_1.\end{aligned}$$

Recall that $\mathbf{GL}(\mathfrak{g})$ is itself a Lie group isomorphic to $\mathbf{GL}(n,\mathbf{R})$, where $n=\dim\mathfrak{g}$. Its Lie algebra consists of all linear endomorphisms of $\mathfrak{g}$ with the bracket $[A,B]=AB-BA$ and it is denoted by $\mathfrak{gl}(\mathfrak{g})$. Note that $\mathrm{Ad}_g = D_2F(g,1)$, where $F: G\times G\to G$ is the smooth function $F(g,x)=gxg^{-1}$, so the linear endomorphism $\mathrm{Ad}_g$ of $\mathfrak{g}$ depends smoothly on $g$. Now $\mathrm{Ad}: g\in G\to \mathrm{Ad}_g\in\mathbf{GL}(\mathfrak{g})$ is homomorphism of Lie groups and its differential $d(\mathrm{Ad})$ defines the *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad}: X\in\mathfrak{g}\to\mathrm{ad}_X=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}_{\exp tX}\in\mathfrak{gl}(\mathfrak{g}).$$

Since $\varphi_t = R_{\exp tX}$ is the flow of $X$, we get

$$\mathrm{ad}_XY=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}_{\exp tX}Y=\frac{d}{dt}\Big|_{t=0}\big((R_{\exp(-tX)})_*Y\big)_1=(L_XY)_1=[X,Y].$$

As an important special case of (3.3.9), we have

$$\begin{aligned}\mathrm{Ad}_{\exp X} &= e^{\mathrm{ad}_X} \\ &= I+\mathrm{ad}_X+\frac{1}{2}\mathrm{ad}_X^2+\frac{1}{3!}\mathrm{ad}_X^3+\cdots\end{aligned}$$

for all $X\in\mathfrak{g}$.

**3.5.2 Lemma** *For given $X$, $Y\in\mathfrak{g}$, we have that $[X,Y]=0$ if and only if $\exp X\exp Y=\exp Y\exp X$. In that case, $\exp(t(X+Y))=\exp tX\exp tY$ for all $t\in\mathbf{R}$. It follows that a connected Lie group is Abelian if and only if its Lie algebra is Abelian.*

*Proof.* The first assertion is a special case of Proposition 1.6.23 using that $\varphi_t=R_{\exp tX}$ is the flow of $X$ and $\psi_s=R_{\exp sY}$ is the flow of $Y$. The second one follows from noting that both $t\mapsto\exp(t(X+Y))$ and $t\mapsto\exp tX\exp tY$ are one-parameter groups with initial speed $X+Y$. Finally, we have seen that $\mathfrak{g}$ is Abelian if and only if $\exp[\mathfrak{g}]$ is Abelian, but the latter generates $G^\circ$. □

**3.5.3 Theorem** *Every connected Abelian Lie group $G$ is isomorphic to $\mathbf{R}^{n-k}\times T^k$. In particular, a simply-connected Abelian Lie group is isomorphic to $\mathbf{R}^n$ and a compact connected Abelian Lie group is isomorphic to $T^n$.*

*Proof.* It follows from Lemma 3.5.2 that $\mathfrak{g}$ is Abelian and $\exp : \mathfrak{g} \to G$ is a homomorphism, where $\mathfrak{g} \cong \mathbf{R}^n$ as a Lie group, thus $\exp$ is a smooth covering by Lemma 3.4.4(c). Hence $G$ is isomorphic to the quotient of $\mathbf{R}^n$ by the discrete group $\ker \exp$. □

## 3.6 Homogeneous manifolds

Let $G$ be a Lie group and let $H$ be a closed subgroup. Consider the set $G/H$ of left cosets of $H$ in $G$ equipped with the quotient topology with respect to the projection $\pi : G \to G/H$. Note also that left multiplication in $G$ induces a map $\lambda : G \times G/H \to G/H$, namely, $\lambda(g, xH) = (gx)H$, and that

$$\pi \circ L_g = \lambda_g \circ \pi \tag{3.6.1}$$

for all $g \in G$, where $\lambda_g(p) = \lambda(g, p)$ for $p \in G/H$.

**3.6.2 Lemma** *A closed Lie subgroup $H$ of a Lie group $G$ must have the induced topology.*

*Proof.* We need to prove that the inclusion map $\iota : H \to G$ is an embedding. Since $\iota$ commutes with left translations, it suffices to find an open subset $V$ of $H$ such that the restriction $\iota|_V$ is an embedding into $G$. By the proof of Theorem 3.3.5, there exists a distinguished chart $(U, \varphi = (x_1, \ldots, x_n))$ of $G$ around $1$ such that $H \cap U$ consists of at most countably many plaques, each plaque being a slice of the form

$$x_{k+1} = c_{k+1}, \ \ldots, \ x_n = c_n$$

for some $c_{k+1}, \ldots, c_n \in \mathbf{R}$, where $k = \dim H$. Denote by $\tau : \mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ the projection. Let $\tilde{U}$ be a compact neighborhood of $1$ contained in $U$. Now $H \cap \tilde{U}$ is compact, so $\tau(H \cap \tilde{U})$ is a non-empty closed countable subset of $\mathbf{R}^{n-k}$ which by the Baire category theorem must have an isolated point. This point specifies a isolated plaque $V$ of $H$ in $U$ along which $\iota$ is an open mapping and hence a homeomorphism onto its image, as desired. □

**3.6.3 Theorem** *If $G$ is a Lie group and $H$ is a closed subgroup of $G$, then there is a unique smooth structure on the topological quotient $G/H$ such that $\lambda : G \times G/H \to G/H$ is smooth. Moreover, $\pi : G \to G/H$ is a surjective submersion and* $\dim G/H = \dim G - \dim H$.

*Proof.* For an open set $V$ of $G/H$ we have that $\pi^{-1}(\pi(V)) = \bigcup_{g \in G} gV$ is a union of open sets and thus open. This shows that $\pi$ is an open map and hence the projection of a countable basis of open sets of $G$ yields a countable basis of open sets of $G/H$. To prove that $G/H$ is Hausdorff, we use closedness of $H$. Indeed it implies that the equivalence relation $\mathcal{R} \subset$

$G \times G$, defined by specifying that $g \sim g'$ if and only if $g^{-1}g' \in H$, is a closed subset of $G \times G$. Now if $gH \neq g'H$ in $G/H$ then $(g, g') \notin \mathcal{R}$ and there exist open neighborhoods $W$ of $g$ and $W'$ of $g'$ in $G$ such that $(W \times W') \cap \mathcal{R} = \varnothing$. It follows that $\pi(W)$ and $\pi(W')$ are disjoint neighborhoods of $g$ and $g'$ in $G/H$, respectively.

We first construct a local chart of $G/H$ around $p_0 = \pi(1) = 1H$. Recall from Proposition 3.2.4 and (3.3.9) that the exponential map $\exp = \exp^G$ gives a parametrization of $G$ around the identity element and restricts to the exponential map of $\mathfrak{h}$. Denote the Lie algebras of $G$ and $H$ by $\mathfrak{g}$ and $\mathfrak{h}$, resp., and choose a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. We can choose a product neighborhood of $0$ in $\mathfrak{g}$ of the form $U_0 \times V_0$, where $U_0$ is a neighborhood of $0$ in $\mathfrak{h}$, $V_0$ is a neighborhood of $0$ in $\mathfrak{m}$ such that the map

$$f : V_0 \times U_0 \to G, \qquad f(X, Y) = \exp X \exp Y$$

is a diffeomorphism from onto its image (apply the Inverse Function Theorem 1.3.8 to $f$). Owing to Lemma 3.6.2, $H$ has the topology induced from $G$, so we may choose a neighborhood $W$ of $1$ in $G$ such that $W \cap H = \exp(U_0)$. We also shrink $V_0$ so that $(\exp V_0)^{-1} \exp V_0 \subset W$. Now we claim that $\pi \circ \exp|_{V_0}$ is injective. Indeed, if $\pi(\exp X) = \pi(\exp X')$ for some $X, X' \in V_0$, then $(\exp X)^{-1} \exp X' \in H \cap W = \exp(U_0)$, so $\exp X' = \exp X \exp Y$ for some $Y \in U_0$. Since $f$ is injective on $U_0 \times V_0$, this implies that $X' = X$ and $Y = 0$ and proves the claim. Note $\exp V_0 \exp U_0$ is open in $G$, so the image $\pi(\exp V_0) = \pi(\exp V_0 \exp U_0)$ is open in $G/H$. We have shown that $\pi \circ \exp$ defines a homeomorphism from $V_0$ onto the open neighborhood $V = \pi(\exp V_0)$ of $p$ in $G/H$, whose inverse can then be used to define a local chart $(V, \psi)$ of $G/H$ around $p_0$.

Now the collection $\{(V^g, \psi^g)\}_{g \in G}$ defines an atlas of $G/H$, where $V^g = gV$ and $\psi^g = \psi \circ L_{g^{-1}}$, and we need to check the its smoothness. Suppose $g$, $g' \in G$ are such that $V^g \cap V^{g'} \neq \varnothing$, and that $p = (g \exp X)H = (g' \exp X')H$ is an element there, namely, $\psi^g(p) = X$ and $\psi^{g'}(p) = X'$. Then $\exp X' = (g')^{-1} g \exp X h \in \exp V_0$ for some $h \in H$, so there exists a neighborhood $\tilde{V_0}$ of $X$ in $V_0$ such that $(g')^{-1} g (\exp \tilde{V_0}) h \subset V_0$, and thus $\psi^{g'} \circ (\psi^g)^{-1}|_{\tilde{V_0}}$ can be written as the composite map

$$\tau \circ \log \circ R_h \circ L_{(g')^{-1}g} \circ \exp,$$

where $\log$ denotes the inverse map of $\exp : U_0 \times V_0 \to \exp(U_0 \times V_0)$, and $\tau : \mathfrak{g} \to \mathfrak{m}$ denotes the projection along $\mathfrak{h}$. Hence the change of charts $\psi^{g'} \circ (\psi^g)^{-1}$ is smooth.

The local representation of $\pi$ is $\tau$ in the above charts is $\tau$, namely, there is a commutative diagram

$$\begin{tikzcd} g\exp V_0 \exp U_0 \arrow[r, "\pi"] \arrow[d, "f^{-1}\circ L_{g^{-1}}"'] & V^g \\ U_0 \times V_0 \arrow[r, "\tau"'] & V_0 \arrow[u, "L_g \circ \pi \circ \exp"'] \end{tikzcd}$$

which shows that $\pi$ is a submersion. Similarly, the commutative diagram

$$\begin{tikzcd} G \times g\exp V_0 \arrow[r] \arrow[d, "\mathrm{id}\times\pi"'] & G \arrow[d, "\pi"] \\ G \times V^g \arrow[r, "\lambda|_{G\times V^g}"'] & G/H \end{tikzcd}$$

proves that $\lambda$ is smooth. The uniqueness of the smooth structure follows from Proposition 3.6.4 below. $\square$

Let $M$ be a smooth manifold and let $G$ be a Lie group. An *action* of $G$ on $M$ is a smooth map $\mu : G\times M \to M$ such that $\mu(1,p) = p$ and $\mu(g, \lambda(g', p)) = \mu(gg', p)$ for all $p \in M$ and $g$, $g' \in G$. For brevity of notation, in case $\mu$ is fixed and clear from the context, we will simply write $\mu(g,p) = gp$.

An action of $G$ is $M$ is called *transitive* if for every $p$, $q \in M$ there exists $g \in G$ such that $gp = q$. In this case, $M$ is called *homogeneous under $G$*, *$G$-homogeneous*, or simply a *homogeneous manifold*. Examples of homogeneous manifolds are given by the quotients $G/H$, where $H$ is closed Lie subgroup of $G$, according to Theorem 3.6.3. Conversely, the next proposition that every homogeneous manifold is of this form. For an action of $G$ on $M$ and $p \in M$, the *isotropy group* at $p$ is the subgroup $G_p$ of $G$ consisting of elements that fix $p$, namely, $G_p = \{ g \in G \mid gp = p \}$. Plainly, $G_p$ is a closed subgroup of $G$, and so a Lie subgroup of $G$, owing to Theorem 3.7.1 below.

**3.6.4 Proposition** *Let $\mu : G \times M \to M$ be a transitive action of a Lie group $G$ on a smooth manifold $M$. Fix $p_0 \in M$ and let $H = G_{p_0}$ be the isotropy group at $p_0$. Define a map*

$$f : G/H \to M, \qquad f(gH) = \mu(g, p_0).$$

*Then $f$ is well-defined and a diffeomorphism.*

*Proof*. It is easy to see that $f$ is well-defined, bijective and smooth. We can write $f \circ \pi = \omega$, where $\omega : G \to M$ is the "orbit map" $\omega(g) = gp_0$. For $X \in \mathfrak{g}$, we have

$$d\omega_1(X) = \frac{d}{ds}\Big|_{s=0} (\exp sX)p_0 = d(\exp(-sX))\frac{d}{dt}\Big|_{t=s} (\exp tX)p_0,$$

so $X \in \ker d\omega_1$ if and only if $\exp tX \in H$ for all $t \in \mathbf{R}$ if and only if $X$ belongs to the Lie algebra $\mathfrak{h}$ of $H$, due to (3.3.10). Since $df_{1H} \circ d\pi_1 = d\omega_1$ and $\ker d\pi_1 = \mathfrak{h}$, this implies that $f$ is an immersion at $1H$, and thus an immersion everywhere by the equivariance property $f \circ \lambda_g = \mu_g \circ f$ for all $g \in G$.

This already implies that $\dim G/H \leq \dim M$ and that $(G/H, f)$ is a submanifold of $M$, but the strict inequality cannot hold as $f$ is bijective and the image of a smooth map from a smooth manifold into a strictly higher dimensional smooth manifold has null measure (this result follows from the statement that the image of a smooth map $\mathbf{R}^n \to \mathbf{R}^{n+k}$ with $k > 0$ has null measure, and the second countability of smooth manifolds). It follows that $f$ is a local diffeomorphism and hence a diffeomorphism. $\square$

**3.6.5 Examples** (a) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$ and view elements of $\mathbf{R}^n$ as column-vectors ($n \times 1$ matrices). Then $GL(n, \mathbf{R})$ acts on $\mathbf{R}^n$ by left-multiplication:

$$GL(n, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n \tag{3.6.6}$$

The basis $\{e_i\}$ is orthonormal with respect to the standard scalar product in $\mathbf{R}^n$. The orthogonal group $O(n)$ precisely consists of those elements of $GL(n, \mathbf{R})$ whose action on $\mathbf{R}^n$ preserves the lengths of vectors. In particular, the action (3.6.6) restricts to an action

$$O(n) \times S^{n-1} \to S^{n-1} \tag{3.6.7}$$

which is smooth, since $S^{n-1}$ is an embedded submanifold of $\mathbf{R}^n$. The action (3.6.7) is transitive due to the facts that any unit vector can be completed to an orthonormal basis of $\mathbf{R}^n$, and any two orthonormal bases of $\mathbf{R}^n$ differ by an orthogonal transformation. The isotropy group of (3.6.7) at $e_1$ consists of transformations that leave the orthogonal complement $e_1^\perp$ invariant, and indeed any orthogonal transformation of $e_1^\perp \cong \mathbf{R}^{n-1}$ can occur. It follows that the isotropy group is isomorphic to $O(n-1)$ and hence

$$S^{n-1} = O(n)/O(n-1)$$

presents the unit sphere as a homogeneous space, where a the diffeomorphism is given by $gO(n-1) \mapsto g(e_1)$. If we use only orientation-preserving transformations on $\mathbf{R}^n$, also the elements of the isotropy group will act by orientation-preserving transformations and hence

$$S^{n-1} = SO(n)/SO(n-1).$$

(b) The group $SO(n)$ also acts transtively on the set of lines through the origin in $\mathbf{R}^n$. Besides the orthogonal transformations of $e_1^\perp$, the isotropy group at the line $\mathbf{R}e_1$ now also contains transformations that map $e_1$ to $-e_1$. It follows that

$$\mathbf{R}P^n = SO(n)/O(n-1)$$

where $O(n-1)$ is identified with the subgroup of $SO(n)$ consisting of matrices of the form

$$\begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}$$

where $A \in O(n-1)$.

(c) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{C}^n$. It is a unitary basis with respect to the standard Hermitian inner product in $\mathbf{C}^n$. Similarly to (a), one shows that $U(n)$ and $SU(n)$ act transitively on the set of unit vectors of $\mathbf{C}^n$, namely, the sphere $S^{2n-1}$. More interesting is to consider the set $\mathbf{C}P^{n-1}$ of one-dimensional complex subspaces of $C^n$. This set is homogeneous under $SU(n)$ and the isotorpy group at the line $\mathbf{C}e_1$ consists of matrices of the form

$$\begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix}$$

where $A \in U(n-1)$. It follows from Theorem 3.6.3 that $\mathbf{C}P^{n-1}$ is a smooth manifold and

$$\mathbf{C}P^{n-1} = SU(n)/U(n-1)$$

as a homogeneous manifold, called *complex proejctive space*

(d) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$, and let $V_k(\mathbf{R}^n)$ be the set of orthonormal $k$-frames in $\mathbf{R}^n$, that is, ordered $k$-tuples of orthonormal vectors in $\mathbf{R}^n$. There is an action

$$O(n) \times V_k(\mathbf{R}^n) \to V_k(\mathbf{R}^n), \quad g \cdot (v_1, \ldots, v_k) = (gv_1, \ldots, gv_k)$$

which is clearly transitive. The isotropy group at $(e_1, \ldots, e_k)$ is the subgroup of $O(n)$ consisting of matrices of the form

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \tag{3.6.8}$$

where $A \in O(n-k)$. The resulting homogeneous space

$$V_k(\mathbf{R}^n) = O(n)/O(n-k)$$

is called the *Stiefel manifold* of $k$-frames in $\mathbf{R}^n$. Note that the restricted action of $SO(n)$ on $V_k(\mathbf{R}^n)$ is also transitive and

$$V_k(\mathbf{R}^n) = SO(n)/SO(n-k).$$

## 3.7 Additional results

In this section, we state without proofs some important, additional results about basic Lie theory, and add some remarks.

### Closed subgroups

**3.7.1 Theorem** *Let $G$ be a Lie group, and let $A$ be a closed (abstract) subgroup of $G$. Then $A$ admits a unique manifold structure whch makes it into a Lie group; moreover, the topology in this manifold structure must be the relative topology.*

**3.7.2 Corollary** *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then $A = \ker \varphi$ is a closed Lie subgroup of $G$ with Lie algebra $\mathfrak{a} = \ker d\varphi$.*

*Proof.* $A$ is a closed subgroup and hence a Lie subgroup of $G$ by Theorem 3.7.1. The rest follows from (3.3.9) and (3.3.10). ☐

### Continuous homomorphisms

**3.7.3 Theorem** *Let $\varphi : G \to H$ be a* continuous *homomorphism between Lie groups. Then $\varphi$ is smooth.*

**3.7.4 Definition** A *topological group* is an abstract group equipped with a topology such that the group operations are continuous maps.

**3.7.5 Corollary** *A Hausdorff second countable locally Euclidean group $G$ can have at most one smooth structure making it into a Lie group.*

*Proof.* Let $[\mathcal{A}]$ and $[\mathcal{B}]$ two such smooth structures on $G$. The identity map $(G, [\mathcal{A}]) \to (G, [\mathcal{B}])$ is a homomorphism and a homeomorphism, and hence a diffeomorphism by Theorem 3.7.3. ☐

*Hilbert's fifth problem* is the fifth mathematical problem posed by David Hilbert in his famous address to the International Congress of Mathematicians in 1900. One (restricted) interpretation of the problem in modern language asks whether a connected (Hausdorff second countable) locally Euclidan group admits a smooth structure which makes it into a Lie group. In 1952, A. Gleason proved that a locally compact group satisfying the "no-small subgroups" (NSS) condition (compare Problem 12) is a Lie group, and then immediately afterwards Montgomery and Zippin used Gleason's result to prove inductively that locally Euclidean groups of any dimension satisfy NSS. The two papers appeared together in the same issue of the Annals of Mathematics. Here one says that a topological group satisfies *NSS* if there exists a neighborhood of the identity which contains no subgroups other than the trivial group. (Actually, the above is not quite the full story; Gleason assumed a weak form of finite dimensionality in his original argument that NSS implies Lie, but shortly thereafter Yamabe showed that finite dimensionality was not needed in the proof.)

### Theorem of Ado

A (real) *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbf{R})$; if, in addition, $\varphi$ is injective, it is called a *faithful* representation.

A faithful representation of a Lie algebra $\mathfrak{g}$ can be thought of a "linear picture" of $\mathfrak{g}$ and allows one to view $\mathfrak{g}$ as a Lie algebra of matrices.

**3.7.6 Theorem (Ado)** *Every Lie algebra admits a faithful representation.*

**3.7.7 Theorem** *There is a bijective correspondence between isomorphism classes of Lie algebras and isomorphism classes of simply-connected Lie groups.*

*Proof.* If $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbf{R})$ by Theorem 3.7.6. Owing to Theorem 3.3.5, there is a connected Lie subgroup of $\mathbf{GL}(n, \mathbf{R})$ with Lie algebra $\mathfrak{g}$. Due to Theorem 3.4.1 and Lemma 3.4.4(c), there is also a simply-connected Lie group with Lie algebra $\mathfrak{g}$. Two simply-connected Lie groups with isomorphic Lie algebras are isomorphic in view of Theorem 3.4.5. $\square$

### Theorem of Yamabe

**3.7.8 Theorem (Yamabe)** *An arcwise connected subgroup of a Lie group is a Lie subgroup.*

**3.7.9 Corollary** *Let $G$ be a connected Lie group and let $A$ and $B$ be connected Lie subgroups. Then the subgroup $(A, B)$ generated by the commutators*

$$S = \{aba^{-1}b^{-1} \;:\; a \in A,\; b \in B\}$$

*is a Lie subgroup of $G$. In particular, the commutator of $G$, $(G, G)$, is a Lie subgroup of $G$.*

*Proof.* As a continuous image of $A \times B$, $S$ is arcwise connected, and so is $T = S \cup S^{-1}$, since $S \cap S^{-1} \ni 1$. As a continuous image of $T \times \cdots \times T$ ($n$ factors) also $T^n$ is arcwise connected and hence so is $(A, B) = \cup_{n \geq 1} T^n$, since it is an increasing union of arcwise connected subsets. The result follows from Yamabe's theorem 3.7.8. $\square$

**3.7.10 Example** In general, the subgroup $(A, B)$ does not have to be closed for closed connected subgroups $A$ and $B$ of $G$, even if $G$ is simply-connected. Indeed, take $G$ to be the simply-connected covering of $\mathbf{SL}(4, \mathbf{R})$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be one-dimensional and respectively spanned by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Then $A$ and $B$ are closed one-dimensional subgroups isomorphic to $\mathbf{R}$ but their commutator is a dense line in a torus.

## 3.8 Problems

§ 3.1

**1** Let $\alpha$, $\beta : (-\epsilon, \epsilon) \to G$ be smooth curves in a Lie group $G$ such that $\alpha(0) = \beta(0) = 1$, and consider $\gamma(t) = \alpha(t)\beta(t)$. Prove that $\dot\gamma(0) = \dot\alpha(0) + \dot\beta(0)$. (Hint: consider the multiplication map $\mu : G \times G \to G$ and show that $d\mu(v, w) = d\mu((v, 0) + (0, w)) = v + w$ for $v$, $w \in T_1G$.)

**2** $a$. Show that

$$\mathbf{SO}(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;:\; a,\ b \in \mathbf{R},\ a^2 + b^2 = 1 \right\}.$$

Deduce that $\mathbf{SO}(2)$ is diffeomorphic to $S^1$.

$b$. Show that

$$\mathbf{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \;:\; \alpha,\ \beta \in \mathbf{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Deduce that $\mathbf{SU}(2)$ is diffeomorphic to $S^3$.

**3** Let

$$H^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \quad x, y, z \in \mathbf{R} \right\}.$$

$a$. Prove that $H^3$ is closed under matrix multiplication and it has the structure of a Lie group (the so called *Heisenberg group*).

$b$. Show that $A = \frac{\partial}{\partial x}$, $B = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $C = \frac{\partial}{\partial z}$ are left-invariant vector fields. Compute their Lie brackets.

$c$. Describe the Lie algebra of $H^3$.

**4** In this problem, we classify all real Lie algebras of dimension two and three.

$a$. Show that a non-Abelian two-dimensional Lie algebra contains two vectors $X, Y$ such that $[X, Y] = X$.

$b$. For an arbitrary Lie algebra $\mathfrak{g}$, denote by $[\mathfrak{g}, \mathfrak{g}]$ the subspace spanned by all elements of the form $[X, Y]$ for $X$, $Y \in \mathfrak{g}$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is a subalgebra of $\mathfrak{g}$ (this is called the *derived subalgebra* of $\mathfrak{g}$).

Throughout the remainder of this problem, we let $\mathfrak{g}$ be a three-dimensional Lie algebra. Put $n = \dim[\mathfrak{g}, \mathfrak{g}]$. Note that $n = 0$ if and only if $\mathfrak{g}$ is Abelian.

*c.* Assume $n = 3$. Fix a non-zero $\omega \in \Lambda^3\mathfrak{g}$ and show that

$$X \wedge Y \wedge Z = \langle [X, Y], Z\rangle\, \omega$$

for all $X$, $Y$, $Z \in \mathfrak{g}$ defines an inner product on $\mathfrak{g}$ such that $\mathrm{ad}_X$ is skew-symmetric for all $X \in \mathfrak{g}$. Conclude that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(3)$ or $\mathfrak{sl}(2, \mathbf{R})$.

*d.* Now consider $n = 2$. Choose linearly independent $X$, $Y \in [\mathfrak{g}, \mathfrak{g}]$. Use (a) and the Jacobi identity to show that $[X, Y] = 0$. Show also that it is possible to chose a non-zero $Z \notin [\mathfrak{g}, \mathfrak{g}]$ such that one of the following holds:

(i) $[Z, X] = X$, $[Z, Y] = X + Y$.
(ii) $[Z, X] = X$, $[Z, Y] = \lambda Y$, $\lambda \in \mathbf{R} \setminus \{0\}$.
(iii) $[Z, X] = aX + Y$, $[Z, Y] = -X + aY$, $a \in \mathbf{R}$.

(Hint: Consider the Jordan canonical form of $\mathrm{ad}_Z : [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{g}, \mathfrak{g}]$.)

*e.* Finally, show that $\mathfrak{g}$ is either the Heisenberg algebra or a certain product algebra in case $n = 1$.

*f.* Conclude that the above reasoning classifies real Lie algebras in dimension 3 (Bianchi 1898). Can you find corresponding Lie groups?

**5** Let $G = \mathbf{O}(n)$.

*a.* Show that $G^\circ \subset \mathbf{SO}(n)$.

*b.* Prove that any element in $\mathbf{SO}(n)$ is conjugate in $G$ to a matrix of the form

$$\begin{pmatrix} R_{t_1} & & & & & \\ & \ddots & & & & \\ & & R_{t_p} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where $R_t$ is the $2 \times 2$ block

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and $t_1, \ldots, t_p \in \mathbf{R}$.

*c.* Deduce from the above that $\mathbf{SO}(n)$ is connected. Conclude that $\mathbf{O}(n)$ has two connected components and $\mathbf{SO}(n)$ is the identity component.

**6** Prove that Lie groups are parallelizable manifolds (cf. Problem 35 in Chapter 1).

§ 3.2

**7** Show that

$$\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in \mathbf{R}$.

**8** Give examples of matrices $A$, $B \in \mathfrak{gl}(2, \mathbf{R})$ such that $e^{A+B} \neq e^A e^B$.

**9** In this problem, we show that the exponential map in a Lie group does not have to be surjective.

*a.* Show that every element $g$ in the image of $\exp : \mathfrak{g} \to G$ has a square root, namely, there is $h \in G$ such that $h^2 = g$.

*b.* Prove that $\operatorname{trace} A^2 \geq -2$ for any $A \in \mathbf{SL}(2, \mathbf{R})$ (Hint: $A$ satisfies its characteristic polynomial $X^2 - 2(\operatorname{trace} X)X + (\det X)I = 0$.)

*c.* Deduce from the above that $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ does not lie in the image of $\exp : \mathfrak{sl}(2, \mathbf{R}) \to \mathbf{SL}(2, \mathbf{R})$.

**10** Let $X \in \mathfrak{sl}(2, \mathbf{R})$. Show that

$$e^X = \begin{cases} \cosh(-\det X)^{1/2} I + \frac{\sinh(-\det X)^{1/2}}{(-\det X)^{1/2}} X & \text{if } \det X < 0, \\ \cos(\det X)^{1/2} I + \frac{\sin(\det X)^{1/2}}{(\det X)^{1/2}} X & \text{if } \det X > 0, \\ I + X & \text{if } \det X = 0. \end{cases}$$

**11** (*Polar decomposition of matrices*)

*a.* Prove that any $g \in \mathbf{GL}(n, \mathbf{R})$ can be written as $g = hk$ where $h \in \mathbf{O}(n)$ and $k$ is a positive-definite symmetric matrix.

*b.* Prove that the exponential map defines a bijection between the space of real symmetric matrices and the set of real positive-definite symmetric matrices. (Hint: Prove it first for diagonal matrices.)

*c.* Deduce from the above that $\mathbf{GL}(n, \mathbf{R})$ is diffeomorphic to $\mathbf{O}(n) \times \mathbf{R}^{\frac{n(n+1)}{2}}$.

**12** Let $G$ be a Lie group. Prove that it does not have small subgroups; i.e., prove the existence of an open neighborhood of $1$ such that $\{1\}$ is the only subgroup of $G$ entirely contained in $U$.

**13** For a connected Lie group, prove that the second-countability of its topology is a consequence of the other conditions in the definition of a Lie group. (Hint: Use Proposition 3.2.5).

§ 3.3

**14** Check that

$$A + iB \in \mathbf{GL}(n, \mathbf{C}) \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbf{GL}(2n, \mathbf{R})$$

defines an injective homomorphism $\varphi$ of $\mathbf{GL}(n, \mathbf{C})$ onto a closed subgroup of $\mathbf{GL}(2n, \mathbf{R})$. Check also that $\varphi$ restricts to an injective homomorphism of $\mathbf{U}(n)$ onto a closed subgroup of $\mathbf{SO}(2n)$.

**15** Prove that a discrete normal subgroup of a connected Lie group is central.

**16** Determine the center of $\mathbf{SU}(n)$.

**17** Construct a diffeomorphism between $\mathbf{U}(n)$ and $S^1 \times \mathbf{SU}(n)$. Is it an isomorphism of Lie groups?

§ 3.4

**18** Consider $G = \mathbf{SU}(2)$ and its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$.

*a.* Check that

$$\mathfrak{g} = \left\{ \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} : x, y, z \in \mathbf{R} \right\}.$$

*b.* Identify $\mathfrak{g}$ with $\mathbf{R}^3$ and check that $\det : \mathfrak{g} \to \mathbf{R}$ corresponds to the usual quadratic form on $\mathbf{R}^3$. Check also that $\mathrm{Ad}_g$ preserves this quadratic form for all $g \in G$.

*c.* Deduce form the above that there is a smooth homomorphism $\mathbf{SU}(2) \to \mathbf{SO}(3)$ which is the simply-connected covering of $\mathbf{SO}(3)$.

§ 3.5

**19** Prove that the kernel of the adjoint representation of a connected Lie group coincides with its center.

**20** Let $A$ be a connected subgroup of a connected Lie group $G$. Prove that $A$ is a normal subgroup of $G$ if and only if the Lie algebra $\mathfrak{a}$ of $A$ is an ideal of the Lie algebra $\mathfrak{g}$ of $G$.

§ 3.6

**21** *a.* Let $\mathrm{Gr}_k(\mathbf{R}^n)$ be the set of $k$-dimensional subspaces of $\mathbf{R}^n$. Prove that

$$\mathrm{Gr}_k(\mathbf{R}^n) = SO(n)/S(O(k) \times O(n-k)).$$

This is called the *Grassmann manifold* of $k$-planes in $\mathbf{R}^n$.

$b$. Consider now the set $\mathrm{Gr}_k^+(\mathbf{R}^n)$ of *oriented* $k$-dimensional subspaces of $\mathbf{R}^n$, and prove that

$$\mathrm{Gr}_k^+(\mathbf{R}^n) = SO(n)/SO(k) \times SO(n-k).$$

This is called the Grassmann manifold of oriented $k$-planes in $\mathbf{R}^n$.

$c$. Define the Grassmann manifold $\mathrm{Gr}_k(\mathbf{C}^n)$ of $k$-planes in $\mathbf{C}^n$ and prove that

$$\begin{aligned} \mathrm{Gr}_k(\mathbf{C}^n) &= U(n)/\left[U(k) \times U(n-k)\right] \\ &= SU(n)/S(U(k) \times SU(n-k)). \end{aligned}$$

CHAPTER 4

# Integration

## 4.1 Orientation

Recall the formula for change of variables in a multiple integral

$$\int_{\varphi(D)} f(y_1,\dots,y_n)dy_1\cdots dy_n$$

$$(4.1.1)\qquad = \int_D f(\varphi(x_1,\dots,x_n))|J\varphi(x_1,\dots,x_n)|dx_1\cdots dx_n$$

Here $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ are two sets of coordinates on $\mathbf{R}^n$ related by a diffeomorphism $\varphi : U \to V$ between open subsets of $\mathbf{R}^n$, $D$ is a bounded domain of integration in $U$, $f$ is a real continuous function on $D$,

$$J\varphi = \det\left(\frac{\partial(y_i\circ\varphi)}{\partial x_j}\right)$$

is the Jacobian determinant of $\varphi$, and $\int$ refers to the Riemann integral. Let us interpret (4.1.1) in terms of differential forms. We have

$$d\varphi\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \sum_i \frac{\partial(y_i\circ\varphi)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_i}\Big|_{\varphi(p)}$$

and

$$(d\varphi)^*(dy_i|_p) = \sum_j \frac{\partial(y_i\circ\varphi)}{\partial x_j}\Big|_p dx_j|_p,$$

so, in view of Exercise 6 in Chapter 2,

$$\varphi^*(dy_1\wedge\cdots\wedge dy_n) = (J\varphi)\,dx_1\wedge\cdots\wedge dx_n.$$

If we define, as we do, the left hand side of (4.1.1) as the integral of the $n$-form $\omega = f dy_1\wedge\cdots\wedge dy_n$ over $\varphi(D)$, that formula says that

$$(4.1.2)\qquad \int_{\varphi(D)}\omega = \pm\int_D \varphi^*\omega$$

where the sign is positive or negative according to whether the sign of the Jacobian determinant is positive or negative throughout $D$. In general, a diffeomorphism between open subsets of $\mathbf{R}^n$ is called *orientation-preserving* if its Jacobian determinant is everywhere positive. The above discussion shows that integration of $n$-forms on bounded domains is not invariant under diffeomorphisms in general, but only under those that preserve orientation. This suggests that if we want to transfer these ideas to smooth manifolds via local charts, and define integration of $n$-forms there in a manner independent of local coordinates, we should try to sort out a consistent sign for the transition maps.

Let $M$ be a smooth manifold. A smooth atlas for $M$ is called *oriented* if all the transition maps are orientation-preserving, and $M$ is called *orientable* if it admits an oriented atlas. If $M$ is orientable, two oriented atlases are said to define the same orientation if their union is an oriented atlas; this defines an equivalence relation on the set of oriented atlases, and a choice of equivalence class is called an *orientation* for $M$.

If $M$ is orientable, an oriented atlas for $M$ defines an orientation on each tangent space induced from the canonical orientation of $\mathbf{R}^n$ via the local charts. For these reason, an orientation on $M$ can also be viewed as a coherent choice of orientations on the tangent spaces to $M$.

**4.1.3 Exercise** Recall that an *orientation* on a vector space $V$ is an equivalence class of (ordered) bases, where two bases are said to be equivalent if the matrix of change from one basis to the other has positive determinant. Clearly, a vector space admits exactly two orientations. Show that for any non-zero element $\omega \in \Lambda^n(V^*)$ ($n = \dim V$) and any basis $(e_1, \ldots, e_n)$ of $V$, the number $\omega(e_1, \ldots, e_n)$ is not zero and its sign is constant in each equivalence class of bases. Deduce that the components of $\Lambda^n(V^*) \setminus \{0\} \cong \mathbf{R} \setminus \{0\}$ naturally correspond to the orientations in $V$.

**4.1.4 Proposition** *A smooth manifold $M$ of dimension $n$ is orientable if and only if it has a nowhere vanishing $n$-form.*

*Proof.* Let $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$ be the canonical $n$-form on $\mathbf{R}^n$. The basic fact we need is that a diffeomorphism $\tau$ of $\mathbf{R}^n$ is orientation-preserving if and only if $\tau^*\omega_0 = f\,\omega_0$ for a everywhere positive smooth function $f$.

Assume first $\omega$ is a nowhere vanishing $n$-form on $M$. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for $M$ where each $U_\alpha$ is connected. For all $\alpha$, $\varphi_\alpha^*\omega_0 = f_\alpha\omega$ where $f_\alpha$ is a nowhere zero smooth function on $U_\alpha$. Thus $f_\alpha$ is everywhere positive or everywhere negative on $U_\alpha$; in the latter case, we replace $\varphi_\alpha$ by $\psi_\alpha = \tau \circ \varphi_\alpha$ where $\tau(x_1, \ldots, x_n) = (-x_1, \ldots, x_n)$. Since $\psi_\alpha^*\omega_0 = \varphi_\alpha^*\tau^*\omega_0 = -\varphi_\alpha^*\omega_0 = -f_\alpha\omega$, this shows that, by replacing $\mathcal{A}$ with an equivalent atlas, we may assume that $f_\alpha > 0$ for all $\alpha$. Now $(\varphi_\beta\varphi_\alpha^{-1})^*\omega_0 = (f_\beta \circ \varphi_\alpha^{-1})/(f_\alpha \circ \varphi_\alpha^{-1})\omega_0$ with $f_\beta/f_\alpha > 0$ for all $\alpha$, $\beta$, which proves that $\mathcal{A}$ is oriented.

Conversely, assume $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an oriented atlas for $M$. Define $\omega_\alpha = \varphi_\alpha^*\omega_0$. Then $\omega_\alpha$ is a nowhere vanishing $n$-form on $U_\alpha$, and $\omega_\alpha$, $\omega_\beta$ are positive multiples of one another on $U_\alpha \cap U_\beta$. It follows that $\omega := \sum_\alpha \rho_\alpha\omega_\alpha$ is a well defined, nowhere vanishing $n$-form on $M$, where $\{\rho_\alpha\}$ is a partition of unity strictly subordinate to $\{U_\alpha\}$. $\square$

In view of the proof of Proposition 4.1.4, on an orientable manifold $M$ of dimension $n$, there exists a bijection between equivalence classes of oriented atlases and equivalence classes of nowhere vanishing $n$-forms, where two nowhere vanishing $n$-forms on $M$ are said to be equivalent if they differ by a positive smooth function. On a connected manifold, the sign of a nowhere zero function cannot change, so on a connected orientable manifold there are exactly two possible orientations.

**4.1.5 Example** Let $M$ be the pre-image of a regular value of a smooth map $f : \mathbf{R}^{n+1} \to \mathbf{R}$. Then $M$ is an (embedded) submanifold of $\mathbf{R}^{n+1}$ and we show in the following that $M$ is orientable by constructing a nowhere vanishing $n$-form on $M$. Let $U_i = \{p \in M \mid \frac{\partial f}{\partial x_i}(p) \neq 0\}$ for $i = 1, \ldots, n+1$. Then $\{U_i\}$ is an open cover of $M$ and we can take $(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$ as local coordinates on $U_i$. Define a nowhere vanishing $n$-form on $U_i$ by

$$\omega_i = (-1)^i \left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Since $f$ is constant on $M$, $\sum_k \frac{\partial f}{\partial x_k} dx_k = 0$, so we have on $U_j$ that

$$dx_j = -\left(\frac{\partial f}{\partial x_j}\right)^{-1} \sum_{k \neq j} \frac{\partial f}{\partial x_k} dx_k.$$

Now one easily checks that

$$\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$$

and hence the $\omega_i$ can be pieced together to yield a global $n$-form on $M$.

Let $M$ be an orientable smooth manifold and fix an orientation for $M$, say given by an oriented atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$. We want to define the integral of a compactly supported $n$-form $\omega$ on $M$. For that purpose, consider first the special case in which the support of $\omega$ is contained in the domain of some local chart, say, $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$. Then we set

$$\int_M \omega = \int_{U_\alpha} \omega = \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*\omega$$

Note that choosing another local chart in $\mathcal{A}$ whose domain contains the support of $\omega$ yields the same result due to (4.1.2). In the general case, we choose a partition of unity $\{\rho_i\}$ subordinate to $\{U_\alpha\}$, $\operatorname{supp}\rho_i \subset U_{\alpha(i)}$, and put

$$\int_M \omega = \sum_i \int_{U_{\alpha(i)}} \rho_i\omega.$$

Note that only finitely many terms in this sum are nonzero as $\operatorname{supp}\omega$ is compact and $\{\operatorname{supp}\rho_i\}$ is locally finite. Let us check that this definition is independent of the choices made. Namely, let $\{(V_\beta, \psi_\beta)\}$ be another oriented atlas defining the same orientation, and let $\{\lambda_j\}$ be a partition of unity subordinate to $\{V_j\}$, namely, $\operatorname{supp}\lambda_j \subset V_{\beta(j)}$. Note that $\rho_i\lambda_j\omega$ has support contained in $U_{\alpha(i)} \cap V_{\beta(j)}$, so, by the special case,

$$\int_{U_{\alpha(i)}} \rho_i\lambda_j\omega = \int_{V_{\beta(j)}} \rho_i\lambda_j\omega.$$

It follows that

$$\begin{aligned} \sum_i \int_{U_{\alpha(i)}} \rho_i\omega &= \sum_{i,j} \int_{U_{\alpha(i)}} \rho_i\lambda_j\omega \\ &= \sum_{i,j} \int_{V_{\beta(j)}} \rho_i\lambda_j\omega \\ &= \sum_j \int_{V_{\beta(j)}} \lambda_j\omega, \end{aligned}$$

as we wished, where we have used that $\sum_i \rho_i = \sum_j \lambda_j = 1$.

**4.1.6 Exercise** Let $f : M \to N$ be a diffeomorphism between connected oriented manifolds of dimension $n$, and let $\omega$ be a compactly supported $n$-form on $N$. Prove that

$$\int_M f^*\omega = \pm \int_N \omega$$

where the sign is "$+$" if $f$ is orientation-preserving and "$-$" if $f$ is orientation-reversing. (Hint: Use (4.1.2).)

**4.1.7 Exercise** Let $M$ be a connected orientable manifold of dimension $n$ and denote by $-M$ the same manifold with the opposite orientation. Show that

$$\int_{-M} \omega = -\int_M \omega$$

for every compactly supported $n$-form $\omega$ on $M$.

## 4.2 Stokes' theorem

Stokes' theorem for manifolds is the exact generalization of the classical theorems of Green, Gauss and Stokes of Vector Calculus. In order to proceed, we need to develop a notion of boundary.

### Manifolds with boundary

In the same way as manifolds are modeled on $\mathbf{R}^n$, manifolds with boundary are modeled on the *upper half space*

$$\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}.$$

A *smooth manifold with boundary* of dimension $n$ is given by a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ where $\varphi_\alpha$ maps $U_\alpha$ homeomorphically onto an open subset of $\mathbf{R}^n_+$ and the transition maps are diffeomorphisms between open subsets of $\mathbf{R}^n_+$. Recall a function $f$ from an arbitrary subset $A$ of $\mathbf{R}^n$ is called *smooth* if it admits a smooth extension $\tilde{f}$ to an open neighborhood of $A$. In case $A$ is an open subset of $\mathbf{R}^n_+$, by continuity all partial derivatives of $\tilde{f}$ at points in $\partial\mathbf{R}^n_+$ are determined by the values of $f$ in the interior of $\mathbf{R}^n_+$, and therefore in particular are independent of the choice of extension.

Of course, $\mathbf{R}^n_+$ is itself a manifold with boundary. There is a natural decomposition of $\mathbf{R}^n_+$ into the *boundary*

$$\partial\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n = 0\}$$

and its complement, the *interior*, and both are smooth manifolds in the previous (restricted) sense, with a natural diffeomorphism $\partial\mathbf{R}^n_+ \approx \mathbf{R}^{n-1}$. For an open subset $U$ of $\mathbf{R}^n_+$, we also put $\partial U = U \cap \partial\mathbf{R}^n_+$.

**4.2.1 Lemma** *Let $\tau : U \to V$ be a diffeomorphism between open subsets of $\mathbf{R}^n_+$ with everywhere positive Jacobian determinant. Then $\tau$ restricts to a diffeomorphism $\partial\tau : \partial U \to \partial V$ with everywhere positive Jacobian determinant.*

*Proof.* A diffeomorphism between open sets of Euclidean space is an open map, so $\tau(U \setminus \partial U) \subset V \setminus \partial V$; applying this to $\tau^{-1}$, we get equality and hence $\tau(\partial U) = \partial V$.

Write $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. By assumption the Jacobian matrix of $\tau = (\tau_1, \ldots, \tau_n)$ at $(x', 0) \in \partial U$ has positive determinant and block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$C = \left(\frac{\partial\tau_n}{\partial x_1}(x', 0), \ldots, \frac{\partial\tau_n}{\partial x_{n-1}}(x', 0)\right) = (0, \ldots, 0)$$

since $\tau_n(x', 0) = 0$ for all $x'$, and

$$D = \frac{\partial \tau_n}{\partial x_n}(x', 0) > 0$$

since $\tau$ maps the upper half space into itself. It follows that $A$, which is the Jacobian of $\partial\tau$ at $(x', 0)$, also has positive determinant, as desired. $\square$

Let $M$ be a smooth manifold with boundary. It follows from Lemma 4.2.1 that the *boundary* of $M$, namely, the subset $\partial M$ consisting of points of $M$ mapped to $\partial\mathbf{R}^n_+$ under coordinate charts, is well defined. Moreover, it is a smooth manifold of dimension $(n-1)$, and an (oriented) atlas for $M$ induces an (oriented) atlas for $\partial M$ by restricting the coordinate charts. Note also that $M \setminus \partial M$ is a smooth manifold of dimension $n$.

**4.2.2 Examples** (a) The closed unit ball $\bar{B}^n$ in $\mathbf{R}^n$ is a smooth manifold with boundary $S^{n-1}$.

(b) The Möbius band is smooth manifold with boundary a circle $S^1$.

In general, for an oriented smooth manifold with boundary, we will always use the so called *induced orientation* on its boundary. Namely, if in $\mathbf{R}^n_+$ we use the standard orientation given by $dx_1 \wedge \cdots \wedge dx_n$, then the induced orientation on $\partial\mathbf{R}^n_+$ is specified by $(-1)^n\, dx_1 \wedge \cdots \wedge dx_{n-1}$ (the sign is required to make the statement of Stokes' theorem right). On an oriented smooth manifold with boundary $M$, for any local chart $(U, \varphi)$ in an oriented atlas of $M$, we declare the restriction of $\varphi$ to $\partial U \to \partial\mathbf{R}^n_+$ to be orientation-preserving.

A $0$-manifold $M$ is just a countable discrete collection of points. In this case, an orientation for $M$ is an assignment of sign $\sigma(p) = \pm 1$ for each $p \in M$ and $\int_M f = \sum_{p\in M} \sigma(p) f(p)$ for any $0$-form $f \in C^\infty(M)$ with compact support.

As it is, the closed interval $[a, b] \subset \mathbf{R}$ ($a < b$) admits an orientation given by the nowhere vanishing 1-form $dx_1$, but no oriented atlas consisting of local charts with values on $\mathbf{R}^1_+$! (Note that in the proof of Proposition 4.1.4, we used the fact that if $(x_1, \ldots, x_n)$ are local coordinates on our manifold, then so are $(-x_1, \ldots, x_n)$.) To remedy this situation, we introduce a slight modification in the definition of manifold with boundary in case $n = 1$ and also allow local charts with values on the *left-line* $\mathbf{R}^1_-$. Accordingly, for the standard orientation $dx_1$ of $\mathbf{R}^1_-$, the induced orientation is on $\partial\mathbf{R}^n_+$ is specified by $+1$. With such conventions, the induced orientation at $a$ is $-1$ and that at $b$ is $+1$.

**4.2.3 Remark** A smooth manifold $M$ in the old sense is a smooth manifold with boundary with $\partial M = \varnothing$. Indeed, we can always find an atlas for $M$ whose local charts have images in $\mathbf{R}^n_+ \setminus \partial\mathbf{R}^n_+$.

Let $M$ be a smooth manifold with boundary of dimension $n$. The tangent space to $M$ at a point $p$ is an $n$-dimensional vector space defined in the same way as in the case of a smooth manifold (even in case $p \in \partial M$). The definition of the tangent bundle also works, and $TM$ is itself a manifold with boundary. More generally, tensor bundles and differential forms are also defined. If $M$ is in addition oriented, the integral of compactly supported $n$-forms is defined similarly to above.

### Statement and proof of the theorem

**4.2.4 Theorem** *Let $\omega$ be an $(n-1)$-form with compact support on an oriented smooth $n$-manifold $M$ with boundary and give $\partial M$ the induced orientation. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In the right hand side of Stokes' theorem, $\omega$ is viewed as $\iota^*\omega$, where $\iota : \partial M \to M$ is the inclusion, and the integral vanishes if $\partial M = \varnothing$. In the case $n = 1$, the integral on the right hand side is a finite sum and the result reduces to the Fundamental Theorem of Calculus.

*Proof of Theorem 4.2.4.* We first consider two special cases.

Case 1: $M$ is an open subset $U$ of $\mathbf{R}^n$. View $\omega$ as an $(n-1)$-form on $\mathbf{R}^n$ which is zero on the complement of $U$. Write $\omega = \sum_i a_i \, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$. Then $d\omega = \sum_i (-1)^{i-1} \frac{\partial a_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$. By Fubini's theorem,

$$\begin{aligned}
\int_U d\omega &= \int_{\mathbf{R}^n} d\omega \\
&= \sum_i (-1)^{i-1} \int_{\mathbf{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} \, dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_n \\
&= 0
\end{aligned}$$

because

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} \, dx_i \\
&= a_i(\ldots, x_{i-1}, \infty, x_{i+1}, \ldots) - a_i(\ldots, x_{i-1}, -\infty, x_{i+1}, \ldots) \\
&= 0,
\end{aligned}$$

as $a_i$ has compact support. Since $M$ has no boundary, this case is settled.

Case 2: $M$ is an open subset $U$ of $\mathbf{R}^n_+$. View $\omega$ as an $(n-1)$-form on $\mathbf{R}^n_+$ which is zero on the complement of $U$. Write $\omega = \sum_i a_i \, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$ as before, but note that while the $a_i$ are smooth on (a neighborhood) of $\mathbf{R}^n_+$, the linear forms $dx_i$ are defined on the entire $\mathbf{R}^n$. Since $a_i$ has compact support, $\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i}\, dx_i = 0$ for $i < n$, so by Fubini's theorem

$$\begin{aligned}
\int_U d\omega &= \int_{\mathbf{R}^n_+} d\omega \\
&= (-1)^{n-1} \int_{\mathbf{R}^{n-1}} \left( \int_0^{\infty} \frac{\partial a_n}{\partial x_n}\, dx_n \right) dx_1 \cdots dx_{n-1} \\
&= (-1)^{n-1} \int_{\mathbf{R}^{n-1}} -a_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} \\
&= \int_{\partial \mathbf{R}^n_+} \omega \\
&= \int_{\partial U} \omega,
\end{aligned}$$

finishing this case.

General case: $M$ is an arbitrary manifold with boundary of dimension $n$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas for $M$ such that each $U_\alpha$ has compact closure and let $\{\rho_\alpha\}$ be a partition of unity strictly subordinate to $\{U_\alpha\}$. Then $\omega = \sum_\alpha \rho_\alpha \omega$ where each term has compact support. By linearity, it suffices to prove Stokes' formula for $\rho_\alpha \omega$ which has support contained in $U_\alpha$. Since $U_\alpha$ is diffeomorhic to an open set in $\mathbf{R}^n$ or $\mathbf{R}^n_+$, cases 1 and 2 imply that the formula holds on $U_\alpha$, so

$$\int_M d\rho_\alpha \omega = \int_{U_\alpha} d\rho_\alpha \omega = \int_{\partial U_\alpha} \rho_\alpha \omega = \int_{\partial M} \rho_\alpha \omega,$$

which concludes the proof of the theorem. □

## 4.3 De Rham Cohomology

De Rham theory, named after Georges de Rham, is a cohomology theory in the realm of smooth manifolds and "constitutes in some sense the most perfect example of a cohomology theory" (Bott and Tu). The de Rham complex of a smooth manifold is defined as a differential invariant, but turns out to be a topological invariant (we will not prove that, but in the next section we shall see that its an invariant of the *smooth* homotopy type).

The most basic invariant of a topological space $X$ is perhaps its number of connected components. In terms of continuous functions, a component is characterized by the property that on it every locally constant continuous function is globally constant. If we define $H^0(X)$ to be the vector space of real valued locally constant continuous functions on $X$, then $\dim H^0(X)$ is the number of connected components of $X$. Of course, in case $X = M$ is a smooth manifold and we define $H^0(M)$ to be the vector space of real valued locally constant *smooth* functions on $M$, again $\dim H^0(X)$ is the number of connected components of $M$.

In seeking to define $H^k(M)$ for $k > 0$, assume for simplicity $M$ is an open subset of $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$. In this case, the locally constant smooth functions $f$ on $M$ are exactly those satisfying

$$df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i = 0.$$

Therefore $H^0(M)$ appears as the space of solutions of a differential equation. In case $k > 0$, points and functions are replaced by $k$-dimensional submanifolds and $k$-forms, respectively. For instance, if $k = 1$, a 1-form $\omega = \sum_i a_i, dx_i$ defines a function on smooth paths

$$\gamma \mapsto \int_\gamma \omega$$

and we look for locally constant functions, namely, those left unchanged under a small perturbation of $\gamma$ keeping the endpoints fixed. In general, if we homotope $\gamma$ to a nearby curve with endpoints fixed, the difference between the line integrals is given by the integral of $d\omega$ along the spanned surface, owing to Stokes' theorem. Therefore the condition of local constancy is here $d\omega = 0$ or, equivalently, the system of partial differential equations

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0 \tag{4.3.1}$$

for all $i$, $j$. On the other hand, $\int_\gamma df = f(q) - f(p)$ where $p$, $q$ are the endpoints of $\gamma$, so 1-forms of type $df$ yield trivial solutions of (4.3.1). This suggest that $H^1(M)$ be defined as the vector space of locally constant line integrals modulo the trivially constant ones, and similarly for bigger $k$.

**4.3.2 Definition** Let $M$ be a smooth manifold. A $k$-form $\omega$ on $M$ is called *closed* if $d\omega = 0$, and it is called *exact* if $\omega = d\eta$ for some $(k-1)$-form $\eta$ on $M$. These conditions define subspaces of the real vector space of $k$-forms on $M$. Since $d^2 = 0$, every exact form is closed. The *k-th de Rham cohomology space* of $M$ is the quotient vector space

$$H^k(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\}.$$

**4.3.3 Examples** (a) For any smooth manifold $M$ of dimension $n$, there are no exact 0-forms and all $n$-forms are closed. Moreover $H^0(M) = \mathbf{R}^p$ where $p$ is the number of connected components of $M$, and $H^k(M) = 0$ for $k > n$ since in this case there are no nonzero $k$-forms.

(b) Let $\omega = f(x)dx$ be a 1-form on $\mathbf{R}$. Then $\omega = dg$ where $g(x) = \int_0^x f(t)\, dt$. Therefore every 1-form on $\mathbf{R}$ is exact and hence $H^1(\mathbf{R}) = 0$. It follows from Poincaré lemma to be proved in the next section that $H^k(\mathbf{R}^n) = 0$ for all $k > 0$.

(c) Owing to Stokes' theorem, an $n$-form $\omega$ on an $n$-dimensional oriented manifold $M$ (without boundary) can be of the form $d\eta$ for a *compactly supported* $(n-1)$-form $\eta$ only if $\int_M \omega = 0$; in particular, if $M$ is compact, $\omega$ can be exact only if $\int_M \omega = 0$. On the other hand, if $M$ is compact and orientable, let $(U, x_1, \ldots, x_n)$ be a positively oriented local coordinate system and let $f$ be a non-negative smooth function with compact support contained in $U$. Then $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ defines an $n$-form on $M$ with $\int_M \omega > 0$ and hence $H^n(M) \neq 0$. We will see later that "integration over $M$" defines an isomorphism $H^n(M) \cong \mathbf{R}$ for compact connected orientable $M$.

(d) The 1-form
$$\omega = \frac{-y\, dx + x\, dy}{x^2 + y^2}$$
on $M = \mathbf{R}^2 \setminus \{(0,0)\}$ is easily checked to be closed by a direct calculation. Let $\iota : S^1 \to M$ be the unit circle. If $\omega$ is exact, $\omega = df$ for some $f \in C^\infty(M)$, then $d(\iota^* f) = \iota^* df = \iota^* \omega$, and also $\iota^*\omega$ is exact, but $\int_{S^1} \iota^*\omega = 2\pi \neq 0$, so this cannot happen, owing to (c). It follows that $H^1(M) \neq 0$.

(e) Consider $M = S^1$. The polar cooordinate function $\theta$ on $S^1$ is defined only locally, but any two determinations of the angle differ by a constant multiple of $2\pi$, so its differential is a well defined 1-form called the "angular form" and usually denoted by $d\theta$, although it is not globally exact (be careful!). It is easily seen that $\iota^*\omega = d\theta$, where $\omega$ is as in (d), and so $H^1(S^1) \neq 0$. We next show that $\int_{S^1} : \Omega^1(S^1) \to \mathbf{R}$ induces an isomorphism $H^1(S^1) \to \mathbf{R}$. Every 1-form is closed, so we need only to identify its kernel with the exact 1-forms. Since $d\theta$ never vanishes, any 1-form $\alpha$ on $S^1$ can be written as $\alpha = f\, d\theta$ where $f \in C^\infty(S^1)$. Now $\int_{S^1} \alpha = 0$ says that $\int_0^{2\pi} f(e^{it})\, dt = 0$, so
$$\tilde{g}(t) = \int_0^t f(e^{is})\, ds$$
is a smooth, $2\pi$-periodic function on $\mathbf{R}$ which induces $g \in C^\infty(S^1)$ such that $g(e^{it}) = \tilde{g}(t)$ for all $t \in \mathbf{R}$. It is clear that $dg = \alpha$, completing the argument.

**4.3.4 Exercise** Prove that the restriction of $\omega$ from Example 4.3.3(d) to the half-plane $x > 0$ is exact.

### Induced maps in cohomology

Let $f : M \to N$ be smooth. Since $d(f^*\omega) = f^*(d\omega)$ for any $\omega \in \Omega^*(N)$, $f^*\omega$ is closed if $\omega$ is closed, and it is exact if $\omega$ is exact. Thus there is an induced homomorphism
$$f^* : H^k(N) \to H^k(M)$$

for each $k \geq 0$. In addition, if $g : N \to P$ is smooth, then

$$(g \circ f)^* = f^* \circ g^*.$$

Of course, the identity map $\mathrm{id} : M \to M$ induces the identity map in cohomology. Such properties show that de Rham cohomology defines a family of contravariant functors and, in particular, a diffeomorphism $f : M \to N$ induces an isomorphism between all the corresponding cohomology spaces. Thus de Rham cohomology is a differential invariant of smooth manifolds. We will prove later that it is a homotopy invariant.

## 4.4 Homotopy-invariance of cohomology

Let $f, g : M \to N$ be smooth maps between smooth manifolds. A (smooth) *homotopy* between $f$ and $g$ is a smooth map $F : M \times [0,1] \to N$ such that

$$\begin{cases} F(p,0) & = & f(p) \\ F(p,1) & = & g(p) \end{cases}$$

for $p \in M$. If there exists a homotopy between $f$ and $g$, we say that they are *homotopic*.

**4.4.1 Proposition** *Let $f$, $g$ be homotopic maps. Then the induced maps in de Rham cohomology*

$$f^*, g^* : H^k(N) \to H^k(M)$$

*are equal.*

The proof of this propositon is given below. First, we need to make some remarks. For $t \in [0,1]$, consider the inclusions $i_t$ given by

$$i_t(p) = (p,t)$$

for $p \in M$, and consider the natural projection $\pi : M \times [0,1] \to M$ given by $\pi(p,t) = p$. Then, obviously,

$$\pi \circ i_t = \mathrm{id}_M$$

implying that

$$i_t^* \pi^* = \mathrm{id} \quad \text{in } \Omega^k(M) \text{ and } H^k(M).$$

We consider the projection $t : M \times [0,1] \to [0,1]$. Then there exist a "vertical" vector field $\frac{\partial}{\partial t}$ and a 1-form $dt$ on $M \times [0,1]$. Note that $\ker d\pi$ is spanned by $\frac{\partial}{\partial t}$.

**4.4.2 Lemma** *Let $\omega \in \Omega^k(M \times [0,1])$. Then we can write*

$$\omega = \zeta + dt \wedge \eta \tag{4.4.3}$$

*where $\zeta \in \Omega^k(M \times [0,1])$ has the property that it vanishes if some of its arguments belongs to $\ker d\pi$, and $\eta \in \Omega^{k-1}(M \times [0,1])$ has the same property.*

*Proof*. Set $\eta = i_{\frac{\partial}{\partial t}}\omega$ and $\zeta = \omega - dt \wedge \eta$. Since

$$i_{\frac{\partial}{\partial t}}\eta = i_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial t}}\omega = 0,$$

it is clear that $\eta$ has the claimed property. Similarly,

$$\begin{aligned} i_{\frac{\partial}{\partial t}}\zeta &= i_{\frac{\partial}{\partial t}}\omega - i_{\frac{\partial}{\partial t}}(dt \wedge \eta) \\ &= \eta - i_{\frac{\partial}{\partial t}} dt \wedge \eta + dt \wedge i_{\frac{\partial}{\partial t}}\eta \\ &= \eta - \eta + 0 \\ &= 0, \end{aligned}$$

as desired, where we have used that interior multiplication is an anti-derivation. □

We define the homotopy operator

$$H_k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$$

by the formula

$$(H_k\omega)_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_{(p,t)}(di_t(v_1), \ldots, di_t(v_{k-1}))\, dt,$$

where $\omega$ is decomposed as in (4.4.3) and $p \in M$, $v_1, \ldots, v_{k-1} \in T_pM$. Note that $H_k$ is "integration along the fiber of $\pi$". For simplicity, we henceforth drop the subscript and just write $H$ for the homotopy operator.

*Proof of Propostion 4.4.1*. Let $\omega \in \Omega^k(M \times [0,1])$. We first claim that

$$dH\omega + Hd\omega = i_1^*\omega - i_0^*\omega. \tag{4.4.4}$$

The proof is by direct computation: since this is a pointwise identity, we can work in a coordinate system. Let $(U, x_1, \ldots, x_n)$ be a coordinate system in $M$. Then $(U \times [0,1], x_1 \circ \pi, \ldots, x_n \circ \pi, t)$ is a coordinate system in $M \times [0,1]$ and we can write

$$\omega|_{U\times[0,1]} = \sum_I a_I dx_I + dt \wedge \sum_J b_J dx_J$$

where $a_I$, $b_J$ are smooth functions on $U \times [0,1]$ and $I$, $J$ are increasing multi-indices. In $U \times [0,1]$, we have:

$$H\omega = \sum_J \left(\int_0^1 b_J dt\right) dx_J,$$

$$dH\omega = \sum_{J,i} \left(\int_0^1 \frac{\partial b_J}{\partial x_i} dt\right) dx_i \wedge dx_J,$$

$$d\omega = \sum_{I,i} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I + \sum_I \frac{\partial a_I}{\partial t} dt \wedge dx_I - dt \wedge \sum_{J,i} \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_J,$$

$$Hd\omega = \sum_I \left( \int_0^1 \frac{\partial a_I}{\partial t} dt \right) dx_I - \sum_{J,i} \left( \int_0^1 \frac{\partial b_J}{\partial x_i} dt \right) dx_i \wedge dx_J.$$

It follows that

$$\begin{aligned} dH\omega + Hd\omega|_p &= \sum_I \left( \int_0^1 \frac{\partial a_I}{\partial t}(p,t)\, dt \right) dx_I \\ &= \sum_I (a_I(p,1) - a_I(p,0)) dx_I \\ &= i_1^*\omega - i_0^*\omega|_p, \end{aligned}$$

as claimed.

Suppose now that $F : M \times [0,1] \to N$ is a homotopy between $f$ and $g$. Let $\alpha$ be a closed $k$-form in $N$ representing the cohomology class $[\alpha] \in H^k(N)$. Applying identity (4.4.4) to $\omega = F^*\alpha$ yields

$$dHF^*\alpha + HF^*d\alpha = i_1^*F^*\alpha - i_0^*F^*\alpha.$$

Since $d\alpha = 0$ and $F \circ i_0 = f$, $F \circ i_1 = g$, we get

$$d\,(HF^*\alpha) = g^*\alpha - f^*\alpha.$$

Hence $g^*\alpha$ and $f^*\alpha$ are cohomologous. □

Two smooth manifolds $M$ and $N$ are said to have the same *homotopy type* (in the smooth sense) and are called *homotopy equivalent* (in the smooth sense) if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ and $f \circ g$ are smoothly homotopic to the identity maps on $M$ and $N$, respectively. Each of the maps $f$ and $g$ is then called a *homotopy equivalence*, and $f$ and $g$ are called *inverses up to homotopy* or *homotopy inverses*. A manifold homotopy equivalent to a point is called *contractible*.

**4.4.5 Corollary** *A homotopy equivalence between smooth manifolds induces an isomorphism in de Rham cohomology.*

**4.4.6 Corollary (Poincaré Lemma)** *The de Rham cohomology of $\mathbf{R}^n$ (or a star-shaped open subset of $\mathbf{R}^n$) is $\mathbf{R}$ in dimension zero and zero otherwise:*

$$H^k(\mathbf{R}^n) = \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

Consider an inclusion $\iota : A \to M$. A map $r : M \to A$ satisfying $r \circ \iota = \mathrm{id}_A$ is called a *retraction*. A special case of homotopy equivalence is the case in which $\iota \circ r : M \to M$ is homotopic to $\mathrm{id}_M$; if that happens, $r$ is called a *deformation retraction* of $M$ onto $A$ and $A$ is called a *deformation retract* of $M$.

**4.4.7 Exercise** Check that $r : \mathbf{R}^2 \setminus \{0\} \to S^1$ given by $r(x) = \frac{x}{||x||}$ is a deformation retraction. Compare with Examples 4.3.3(d) and (e).

**4.4.8 Lemma** *There exists no smooth retraction $r : \bar{B}^n \to \partial\bar{B}^n$ from the closed ball onto its boundary.*

*Proof.* The case $n = 1$ is easy as a retraction is surjective, the closed interval $\bar{B}^1$ is connected and its boundary is disconnected. Assume $n \geq 2$ and suppose, to the contrary, that such a retraction $r$ exists. From $r \circ \iota = \mathrm{id}_{\partial\bar{B}^n}$ we deduce that $\iota^* r^* = \mathrm{id}$ and thus that $r^* : H^{n-1}(\partial\bar{B}^n) \to H^{n-1}(\bar{B}^n)$ is injective. However $\partial\bar{B}^n = S^{n-1}$ and $H^{n-1}(S^{n-1}) \neq 0$ (Example 4.3.3(c)) whereas $H^{n-1}(\bar{B}^n) = 0$ (Corollary 4.4.6), which is a contradiction. □

**4.4.9 Theorem (Smooth Brouwer's fixed point theorem)** *Let $f : \bar{B}^n \to \bar{B}^n$ be a smooth map. Then there exists $p \in \bar{B}^n$ such that $f(p) = p$. In other words, every smooth self-map of the closed $n$-ball admits a fixed point.*

*Proof.* Suppose, on the contrary, that $f(x) \neq x$ for all $x \in \bar{B}^n$. The half-line originating at $f(x)$ and going through $x$ meets $\partial\bar{B}^n$ at a unique point; call it $r(x)$. It is easy to see that this defines a smooth retraction $r : \bar{B}^n \to \partial\bar{B}^n$ which is prohibited by Lemma 4.4.8. □

**4.4.10 Remark** The theorem is not true in the case of the open $n$-ball, as is easily seen.

For the next result, consider the unit sphere $\iota : S^n \to \mathbf{R}^{n+1}$. It is useful to have an explicit expression for a non-zero element in $H^n(S^n)$ (Example 4.1.5):

$$\omega = (-1)^i \frac{1}{x_i}\, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots dx_{n+1} \tag{4.4.11}$$

on $x_i \neq 0$ for $i = 1, \ldots, n+1$.

**4.4.12 Theorem (Hairy ball theorem)** *Let $X$ be a smooth vector field on $S^{2m}$. Then there exists $p \in S^{2m}$ such that $X_p = 0$. In other words, every smooth vector field on an even-dimensional sphere has a zero.*

*Proof.* Suppose, on the contrary, that $X$ never vanishes. By rescaling, we may assume that $X$ is a unit vector field with respect to the metric induced from Euclidean space. Set

$$F_t : S^{2m} \to S^{2m}, \quad F_t(p) = \cos t\, p + \sin t\, X(p).$$

It is clear that $F_t$ defines a homotopy between the identity map and the antipodal map of $S^{2m}$:

$$F_0 = \mathrm{id}_{S^{2m}} \quad \text{and} \quad F_\pi = -\mathrm{id}_{S^{2m}}.$$

Note that

$$F_\pi^*(x_i \circ \iota) = -x_i \circ \iota.$$

It follows that

$$F_\pi^* \omega = (-1)^{2m+1}\omega = -\omega,$$

where $\omega$ is as in (4.4.11). On the other hand,

$$F_0^*\omega = \omega,$$

and by Proposition 4.4.1, $F_0^*\omega$ and $F_\pi^*\omega$ are cohomologous, which contradicts the fact that $\omega$ is not cohomologous to zero. □

**4.4.13 Corollary** *The even-dimensional spheres cannot admit a structure of Lie group compatible with its standard topology and smooth structure.*

*Proof.* It follows from Problem 35 in Chapter 1 and Problem 6 in Chapter 3. Indeed it is known that the only parallelizable spheres are $S^1$, $S^3$ and $S^7$, and the only ones that are Lie groups are the first two. □

**4.4.14 Remark** Theorems 4.4.9 and 4.4.12 can be extended to the continuous category by using appropriate approximation results.

We close this section computing the de Rham cohomology of the $n$-sphere. The argument is a nice presentation of the "Mayer-Vietoris principle" in a very special case.

**4.4.15 Proposition** *The de Rham cohomology of $S^n$ vanishes except in dimensions $0$ and $n$.*

*Proof.* We may assume $n > 1$. We prove first that $H^1(S^n) = 0$. Let $\omega$ be a closed 1-form on $S^n$. We must show that $\omega$ is exact. Decompose $S^n$ into the union of two open sets $U$ and $V$, where $U$ in a neighborhood of the northern hemisphere diffeomorphic to an open $n$-ball, $V$ is a neighborhood of the southern hemisphere diffeomorphic to an open $n$-ball, and $U \cap V$ is a neighborhood of the equator which is diffeomorphic to $S^{n-1} \times (-1, 1)$. Since $U$ and $V$ are contractible, $\omega|_U = df$ for a smooth function $f$ on $U$ and $\omega|_V = dg$ for a smooth function $g$ on $V$. In general on $U \cap V$, $f$ and $g$ do not agree, but the difference $h := f|_{U\cap V} - g|_{U\cap V}$ has $dh = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$. Since $n > 1$, $S^{n-1}$ is connected and thus $h$ is a constant. Setting

$$k := \begin{cases} f & \text{on } U, \\ g + h & \text{on } V, \end{cases}$$

defines a smooth function on $S^n$ such that $dk = \omega$, as we wished.

We proceed by induction. Let $\omega$ be a closed $k$-form on $S^n$ for $1 < k < n$. We shall prove that $\omega$ is exact using the same decomposition $S^n = U \cap V$ as above and the induction hypothesis. As above, $\omega|_U = d\alpha$ for a $(k-1)$-form $\alpha$ on $U$ an $\omega|_V = d\beta$ for a $(k-1)$-form $\beta$ on $V$. Let $\gamma = \alpha|_{U\cap V} - \beta|_{U\cap V}$. Then $d\gamma = 0$. Since $\gamma$ is a closed $(k-1)$-form on $U \cap V$ and $U \cap V$ is homotopy equivalent to $S^{n-1}$, by the induction hypothesis, $\gamma = d\xi$ for a $(k-2)$-form on $U \cap V$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. Setting

$$\eta := \begin{cases} \alpha - d(\rho_V \xi) & \text{on } U, \\ \beta + d(\rho_U \xi) & \text{on } V, \end{cases}$$

defines a $(k-1)$-form on $S^n$ such that $d\eta = \omega$. This completes the induction step and the proof of the theorem. $\square$

**4.4.16 Remark** The "Mayer-Vietoris principle" indeed yields a long exact sequence in cohomology. One nice application is to show that the de Rham cohomology spaces of a compact manifold are always finite-dimensional.

## 4.5 Degree theory

Our first aim is to prove that the top dimensional de Rham cohomology of a compact connected orientable smooth manifold is one-dimensional. We start with a lemma in Calculus.

**4.5.1 Lemma** *Let $f$ be a smooth function on $\mathbf{R}^n$ with support in the open cube $C^n = (-1,1)^n$ and*

$$\int_{\mathbf{R}^n} f \, dx_1 \cdots dx_n = 0.$$

*Then there exist smooth functions $f_1, \ldots, f_n$ on $\mathbf{R}^n$ with support in $C^n$ such that*

$$f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

*Proof*. By induction on $n$. If $n = 1$, we simply define $f_1(x_1) = \int_{-\infty}^{x_1} f(t)\, dt$. If $n \geq 2$, define a smooth function $g$ on $\mathbf{R}^{n-1}$ by

$$g(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{+\infty} f(x_1, \ldots, x_{n-1}, t)\, dt.$$

Then $g$ has total integral zero by Fubini's theorem, and clearly support contained in $C^{n-1}$, so by the induction hypothesis we can write

$$g = \sum_{i=1}^{n-1} \frac{\partial g_i}{\partial x_i}$$

for smooth functions $g_i$ on $\mathbf{R}^{n-1}$ with support in $C^{n-1}$. Now choose a smooth function $\rho$ on $\mathbf{R}$ with support in $(-1,1)$ and total integral 1, and define $f_j \mathbf{R}^n \to \mathbf{R}$ by

$$f_j(x_1,\ldots,x_{n-1},x_n) = g_j(x_1,\ldots,x_{n-1})\rho(x_n)$$

for $j = 1,\ldots,n-1$. Clearly the $f_j$ have support in $C^n$. Set

$$h = f - \sum_{i=1}^{n-1} \frac{\partial f_j}{\partial x_j}$$

and

$$f_n(x_1,\ldots,x_{n-1},x_n) = \int_{-\infty}^{x_n} h(x_1,\ldots,x_{n-1},t)\,dt$$

Clearly $h$ has support in $C^n$, so the same is true of $f_n$ and we are done. $\square$

**4.5.2 Lemma** *Let $\omega$ be an $n$-form on $\mathbf{R}^n$ with support contained in the open cube $C$ such that $\int_{\mathbf{R}^n} \omega = 0$. Then there exists an $(n-1)$-form $\eta$ on $\mathbf{R}^n$ with support contained in $C$ such that $d\eta = \omega$.*

*Proof.* The Poincaré lemma yields $\eta$ with $d\eta = \omega$ but does not give information about the support of $\eta$. Instead, write $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ for $f \in C^\infty(\mathbf{R}^n)$. Then $\mathrm{supp} f \subset C$ and $\int_{\mathbf{R}^n} f\,dx_1 \cdots dx_n = 0$, so $f = \sum_i \frac{\partial f_i}{\partial x_i}$ as in Lemma 4.5.1, and thus $\omega = d\eta$ where $\eta = \sum_i (-1)^{i-1} f_i\,dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$. $\square$

**4.5.3 Proposition** *If $M$ is a compact connected orientable smooth manifold of dimension $n$, then $H^n(M) = \mathbf{R}$.*

*Proof.* By compactness, there is a finite cover $\{U_1,\ldots,U_m\}$ by coordinate neighborhoods diffeomorphic to the open cube $C$. Let $\omega_0$ be a bump $n$-form as in Example 4.3.3(c) with support contained in $U_1$ and total integral 1. Then $\omega_0$ defines a non-zero cohomology class in $H^n(M)$. We shall prove that any $n$-form $\omega$ on $M$ is cohomologous to a multiple of $\omega_0$, namely, $\omega = c\,\omega_0 + d\eta$ for some $c \in \mathbf{R}$ and some $(n-1)$-form $\eta$. Using a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, we can write $\omega = \sum_{i=1}^m \rho_i\omega$ where $\rho_i\omega$ is an $n$-form with support in $U_i$. By linearity, it suffices to prove the result for $\rho_i\omega$, so we may assume from the outset that the support of $\omega$ is contained in $U_k$, for some $k = 1,\ldots,m$.

Owing to the connectedness of $M$, we can find a chain $U_{i_1},\ldots,U_{i_r}$ such that $U_{i_1} = U_1$, $U_{i_r} = U_k$ and $U_{i_j} \cap U_{i_{j+1}} \neq \varnothing$ for all $j = 1,\ldots,r-1$. For all $j = 1,\ldots,r-1$, choose an $n$-form $\omega_j$ with support in $U_{i_j} \cap U_{i_{j+1}}$ and total integral 1. Now $\omega_0 - \omega_1$ has support in $U_{i_1} = U_1$ and total integral zero, so by Lemma 4.5.2, there exists $\eta_1$ with support in $U_1$ such that

$$\omega_0 - \omega_1 = d\eta_1.$$

Next, $\omega_1-\omega_2$ has support in $U_{i_2}$ and total integral zero, so the lemma yields $\eta_2$ with support in $U_{i_2}$ such that

$$\omega_1-\omega_2=d\eta_2.$$

Continuing, we find $\eta_j$ with support in $U_{i_j}$ such that

$$\omega_{j-1}-\omega_j=d\eta_j$$

for all $j=1,\ldots,r-1$. Adding up, we get

$$\omega_0-\omega_{r-1}=d\eta$$

where $\eta=\sum_{j=1}^{r-1}\eta_j$. Now $U_{i_r}=U_k$ contains the support of $\omega$ and $\omega_{r-1}$, and $\omega-c\omega_{r-1}$ has total integral zero, where $c=\int_M\omega$. By applying the lemma again,

$$\omega-c\omega_{r-1}=d\zeta$$

and hence

$$\omega=c\omega_0+d(\zeta-c\eta)$$

as required. □

**4.5.4 Corollary** *Let $M$ be a compact connected oriented smooth manifold of dimension $n$. Then "integration over $M$"*

$$\int_M:H^n(M)\to\mathbf{R}$$

*is a well defined linear isomorphism which is positive precisely on the cohomology classes defined by nowhere vanishing $n$-forms belonging to the orientation of $M$.*

*Proof.* By Stokes' formula, the integral of an exact form is zero, so the integral of an $n$-form depends only on its cohomology class and thus the map is well defined. By the theorem, $H^n(M)$ is one dimensional and there exist bump $n$-forms with non-zero integral, so the map is an isomorphism.

Let $\omega$ be a nowhere vanishing $n$-form belonging to the orientation of $M$, choose an oriented atlas $\{(U_\alpha,\varphi_\alpha=(x_1^\alpha,\ldots,x_n^\alpha))\}$ and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. Then $\omega=\sum_\alpha\rho_\alpha\omega$, where $\rho_\alpha\omega$ has support in $U_\alpha$ and on which its local representation is of the form $f_\alpha\,dx_1^\alpha\wedge\cdots\wedge dx_n^\alpha$ for a non-negative smooth function $f_\alpha$ on $U_\alpha$. It follows that

$$\int_M\omega=\sum_\alpha\int_{\varphi_\alpha(U_\alpha)}(f_\alpha\circ\varphi_\alpha^{-1})\,dx_1\cdots dx_n>0$$

since $f_\alpha\geq 0$ and it is positive somewhere. Conversely, if $\omega'$ is an $n$-form with $\int_M\omega'>0$, then $\omega'$ is cohomologous to $c\omega$, where $c=\int_M\omega'/\int_M\omega>0$, and $c\omega$ and $\omega$ are nowhere vanishing $n$-forms defining the same orientation on $M$. $\square$

Let $f : M \to N$ be a smooth map between compact connected oriented manifolds of the same dimension $n$. Let $\omega_M$, $\omega_N$ be $n$-forms on $M$, $N$, respectively, with total integral one. Then $f^* : H^n(N) \to H^n(M)$ carries $[\omega_N]$ to a multiple of $[\omega_M]$; this number is called the *degree* of $f$, denoted $\deg f$. It follows from Proposition 4.4.1 that homotopic maps have the same degree.

**4.5.5 Remark** In case $N = S^n$, Hopf's degree theorem [GP10] asserts that non-homotopic maps have different degrees. For the case $n = 1$, see Problem 19.

**4.5.6 Proposition** *Let $f : M \to N$ be a smooth.*
*a. The degree of $f$ is an integer.*
*b. For all $\omega \in \Omega^n(N)$,*

$$\int_M f^*\omega = (\deg f) \int_N \omega$$

*c. If $q \in N$ is a regular value of $f$, then*

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sgn}(\det df_p) \qquad \textit{(finite sum)}$$

*Proof.* (b) follows from the commutativity of the diagram

$$\begin{tikzcd} H^n(N) \arrow[r, "f^*"] \arrow[d, "\int_N"'] & H^n(M) \arrow[d, "\int_M"] \\ \mathbf{R} \arrow[r, "\deg f"'] & \mathbf{R} \end{tikzcd}$$

and (a) follows from (c). Let us prove (c).

Consider first the case in which $q$ is a point outside the image of $f$. Since $f(M)$ is compact, we can find a bump $n$-form $\alpha$ on $N$ with total integral one and support disjoint from $f(M)$. It follows from (b) that $\deg f = \int_M f^*\alpha = 0$. Since $f^{-1}(q) = \varnothing$, (c) is proved in this case.

Suppose now $q$ lies in the image of $f$. Since $q$ is a regular value and $\dim M = \dim N$, $f$ is a local diffeomorphism at each $p \in f^{-1}(q)$. In particular, $f^{-1}(q)$ is discrete and thus finite, due to the compactness of $M$. Write $f^{-1}(q) = \{p_1, \ldots, p_m\}$ and choose open neighborhoods $\tilde{U}_i$ of $p_i$ and $V_i$ of $q$ such that $f : \tilde{U}_i \to V_i$ is a diffeomorphism for all $i = 1, \ldots, m$. Setting $V = \cap_{i=1}^m V_i$ and $U_i = \tilde{U}_i \cap f^{-1}(V)$, now $f : U_i \to V$ is a diffeomorphism for all $i$. Moreover, $f(M \setminus \cup_{i=1}^m \tilde{U}_i)$ is a compact subset of $N$ disjoint from $q$, so by further shrinking $V$ we can ensure that $f^{-1}(V) = \cup_{i=1}^m U_i$.

Let $\alpha$ be an $n$-form on $N$ with total integral one and support contained in $V$. Then $f^*\alpha$ is an $n$-form on $M$ with support in $\cup_{i=1}^m U_i$. In view of Exercise 4.1.6

$$\int_{U_i} f^*\alpha = \operatorname{sgn}(\det df_{p_i}) \int_V \alpha = \operatorname{sgn}(\det df_{p_i})$$

where we consider the determinant of the Jacobian matrix of $f$ at $p_i$ relative to orientation-preserving local charts around $p_i$ and $q$, so its sign is $+1$ if $df_{p_i} : T_{p_i}M \to T_qN$ preserves orientation and $-1$ if it reverses orientation. It follows that

$$\deg f = \int_M f^*\alpha = \sum_{i=1}^p \int_{U_i} f^*\alpha = \sum_{i=1}^p \operatorname{sgn}(\det df_{p_i}),$$

as desired. □

**4.5.7 Corollary** *The degree of a non-surjective map is zero.*

**4.5.8 Remark** There always exists a regular value of $f$ by Sard's theorem [GP10].

More generally, if $M$ has finitely many connected components $M_1, \ldots, M_r$, the degree of $f : M \to N$ can still be defined as the sum of the degrees of the restrictions $f : M_i \to N$, and Proposition 4.5.6 remains true.

**4.5.9 Example** Consider $S^1$ as the set of unit complex numbers. Then $f : S^1 \to S^1$ given by $f(z) = z^n$ is smooth and has degree $n$, which we can show as follows. Recall the angular form $d\theta$ generates $H^1(S^1)$. Removal of one point does not change the integral below on the left hand side, and $h : (0, 2\pi) \to S^1 \setminus \{1\}$, $h(x) = e^{ix}$ is an orientation-preserving diffeomorphism, so

$$\int_{S^1} f^*d\theta = \int_0^{2\pi} h^*f^*d\theta = \int_0^{2\pi} (f \circ h)^*d\theta$$

where $(f \circ h)^*d\theta$ is exact on $(0, 2\pi)$ and in fact equal to

$$d(f \circ h)^*\theta = d(\theta \circ f \circ h) = n\,dx$$

therefore

$$\int_{S^1} f^*d\theta = \int_0^{2\pi} n\,dx = 2\pi n = n \cdot \int_{S^1} d\theta,$$

as we wished.

**4.5.10 Example** Let $f : S^1 \to \mathbf{R}^2$ be a smooth map. Its image is a circle in the plane. Fix a point $q$ not in this circle. The *winding number* $W(f, q)$ of $f$ with respect to $q$ is the degree of the map $u : S^1 \to S^1$ given by

$$u(x) = \frac{x - q}{||x - q||}.$$

Note that $W(f, q_1) = W(f, q_2)$ if $q_1$ and $q_2$ lie in the same connected component of the complement of the image of $f$.

Introducing the complex variable $z = x + iy$ we have

$$\frac{-y\,dx + x\,dy}{x^2 + y^2} = \Im\left\{\frac{1}{z}dz\right\}$$

(compare Examples 4.3.3(d)). Using this formula, it is easy to arrive at the complex integral for the winding number,

$$W(f, q) = \frac{1}{2\pi i}\int_C \frac{dz}{z - q}\,dz, \tag{4.5.11}$$

where $C$ is the image of $f$ (Cauchy 1825).

**4.5.12 Example** Let $f$, $g : S^1 \to \mathbf{R}^3$ be two smooth maps. Their images yield two circles in $\mathbf{R}^3$ which we suppose to be disjoint. The *linking number* $\mathrm{Lk}(f, g)$ is the degree of the map $F : S^1 \times S^1 \to S^2$ given by

$$F(x, y) = \frac{f(x) - g(y)}{||f(x) - g(y)||}.$$

If $f_t$, $g_t : S^1 \to \mathbf{R}^3$ are homotopies of $f$, $g$ such that $f_t$ and $g_t$ have disjoint images for all $t$, then $\mathrm{Lk}(f_t, g_t)$ is independent of $t$.

In case $f$, $g : S^1 \to S^3$, one chooses $q \in S^3$ not in the image of those maps and performs stereographic projection $S^3 \setminus \{q\} \to \mathbf{R}^3$ to define their linking number. Moving $q$ continuously yields homotopies of $f$, $g$, so since the union of the images of $f$ and $g$ does not disconnect $S^3$, this definition does not depend on the choice of $q$.

According to Problems 6 and 10, the volume form of $S^2$, normalized to have total integral 1, is

$$dA = \frac{1}{4\pi}\left(x_1\,dx_2 \wedge dx_3 + x_2\,dx_3 \wedge dx_1 + x_3\,dx_1 \wedge dx_2\right).$$

Since

$$F^*dA = \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial y},$$

an easy calculation yields the formula for the linking number (Gauss 1833)

$$\mathrm{Lk}(f, g) = \int_{S^1}\int_{S^1} \frac{f(x) - g(y)}{||f(x) - g(y)||^3} \cdot \frac{df}{dx} \times \frac{dg}{dy}\,dxdy \tag{4.5.13}$$

**4.5.14 Example** We can generalize Example 4.5.10 as follows. Let $f : M \to \mathbf{R}^{n+1}$ be a smooth map from a compact, connected oriented manifold $M$ of dimension $n$. If $q \in \mathbf{R}^{n+1}$ does not lie in the image of $f$, the winding number $W(f,q)$ of $f$ with respect to $q$ is the degree of the map $u : M \to S^n$ given by

$$u(x) = \frac{f(x) - q}{||f(x) - q||}.$$

It records how $f$ "wraps" around $q$.

**4.5.15 Exercise** Check formulae (4.5.11) and (4.5.13).

## 4.6 The Borsuk-Ulam theorem

The Borsuk-Ulam theorem is one of the theorems in topology with most applications in practice. It was conjectured by Ulam at the Scottish Cafe in Lvov. The theorem proven in one form by Borsuk in 1933 has several other equivalent formulations and many different proofs. One, well-known of these was first proven by Lyusternik and Shnirel´man in 1930. A host of extensions and generalizations, and numerous interesting applications to areas that include combinatorics, differential equations and even economics add to its importance.

**4.6.1 Lemma** *Let $F : \bar{B}^n \to \mathbf{R}^n$ be a smooth map. Denote the restriction of $F$ to the boundary $\partial\bar{B}^n$ by $f$ and let $q \in \mathbf{R}^n$ be a point that does not lie in the image of $f$. Then the winding number $W(f,q)$ equals the number of preimages of $q$ under $F$ counted with signs according to whether $F$ preserves or reverses orientation at the point, as in Proposition 4.5.6.*

*Proof.* Suppose first that $q$ does not lie in the image of $F$. Let $F_t : S^{n-1} \to \mathbf{R}^n$ be defined by $F_t(x) = F((1-t)x)$ for $0 \le t \le 1$. Then $f_0 = f$ and

$$u_t(x) = \frac{F_t(x) - q}{||F_t(x) - q||}$$

defines an homotopy from $u_0$ to the constant map $u_1$. This shows that $W(f,q) = \deg(u_0) = \deg(u_1) = 0$.

Suppose next that $F^{-1}(q) = \{p_1, \ldots, p_k\}$, and let $B_i$ be a small ball around $p_i$ such that the $B_i$'s are disjoint one another and from the boundary of $\bar{B}^n$. Let $f_i : \partial B_i \to \mathbf{R}^n$ be the restriction of $F$. Note that $W(f_i, q) = \pm 1$ according to whether $F$ preserves or reverses orientation at $p_i$. On the other hand, set $X = \bar{B}^n \setminus \bigcup_{i=1}^k B_i$. The map

$$u(x) = \frac{F(x) - q}{||F(x) - q||}$$

is well defined and smooth on $X$. By Problem 20, $\deg u|_{\partial X} = 0$. It follows that

$$\begin{aligned} W(f,a) &= \deg u|_{\partial \bar{B}^n} \\ &= \sum_{i=1}^{k} \deg u|_{\partial B_i} \\ &= \sum_{i=1}^{k} W(f_i, q) \\ &= \sum_{i=1}^{k} \operatorname{sgn}(\det dF_{p_i}), \end{aligned}$$

as we wished. □

A map $f : S^n \to \mathbf{R}^{n+1}$ will be called *odd* or *antipode-preserving* if $f(-x) = -f(x)$ for all $x \in S^n$, where $-x$ denotes the antipodal point of $x$.

**4.6.2 Theorem (Borsuk-Ulam)** *An odd smooth map $f : S^n \to S^n$ has odd degree.*

*Proof.* We proceed by induction on $n$. The initial case $n = 1$ is Problem 24. Next assume the result true for $n-1$ and let $f : S^n \to S^n$ be an odd map.

Let $g : S^{n-1} \to S^n$ be the restriction of $f$ to the equator. By Sard's theorem, there is $q \in S^n$ which is a regular value of both $f$ and $g$. This means that $q$ is not in the image of $g$ (by dimensional reasons) and the oriented number of preimages of $q$ under $f$ is the degree $d$ of $f$.

By composing $f$ with a rotation, we may assume that $q$ is the north pole. Since $f$ is odd (and $f$ does not hit $q$ along the equator), the south pole $-q$ is also a regular value of $f$, and $f$ hits $q$ in the southern hemisphere as many times as it hits $-q$ in the northern hemisphere $S^n_+$. Let $f_+$ denote the restriction of $f$ to $S_n^+$. Now $d$ is the oriented number of preimages of $\{\pm q\}$ under $f_+$. Another way is to consider the orthogonal projection $\pi : S^n_+ \to \bar{B}^n$ to the equatorial plane and note that $d$ is the oriented number of preimages of $0$ under $\pi \circ f_+$. Since $0$ does not lie in the image of $\pi \circ g$, Lemma 4.6.1 implies that $d = W(\pi \circ g, 0) = \deg(\frac{\pi \circ g}{||\pi \circ g||})$ which, by the induction hypothesis, is odd as $\frac{\pi \circ g}{||\pi \circ g||} : S^{n-1} \to S^{n-1}$ is an odd map. □

**4.6.3 Corollary** *Let $f_1, \ldots, f_n$ be smooth functions on $S^n$. Then there is a pair of antipodal points $\pm p \in S^n$ such that*

$$f_1(p) = f_1(-p), \ldots, f_n(p) = f_n(-p).$$

*Proof.* Let $f : S^n \to \mathbf{R}^n$ have components $f_i$ and suppose, to the contrary, that $g(x) = f(x) - f(-x)$ never vanishes. Then $h : S^n \to S^n$ defined by

$$h(x) = \left( \frac{g(x)}{||g(x)||}, 0 \right)$$

is an odd smooth map that never hits the points $(0, \ldots, 0, \pm 1) \in S^n$. By Corollart 4.5.7, $\deg h = 0$ contradicting Theorem 4.6.2. $\square$

A popular illustration of Corollary 4.6.3 in case $n = 2$ is that if a baloon is deflated and laid flat on the floor then at least two antipodal points end up on top of one another. A meteorological formulation states that at any given time there are two antipodal points on the surface of Earth with identical temperature and pressure (although anyone who has ever touched a griddle-hot stove knows that temperature needs not be a continuous function!)

## 4.7 Maxwell's equations

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and mathematician James Clerk Maxwell who published an early form of those equations between 1861 and 1862.

The *electric field*

$$\vec{E}(t) = (E_1, E_2, E_3)$$

and the *magnetic field*

$$\vec{B}(t) = (B_1, B_2, B_3)$$

are vector fields on $\mathbf{R}^3$. *Maxwell's equations* are

$$\begin{aligned} \operatorname{div} \vec{E} &= 4\pi\rho \\ \operatorname{div} \vec{B} &= 0 \\ \operatorname{curl} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \operatorname{curl} \vec{B} &= \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{J} \end{aligned}$$

where $\rho$ is the *electric charge density* and $\vec{J} = (J_1, J_2, J_3)$ is the *electric current density*.

*Minkowski spacetime* is $\mathbf{R}^4$ with coordinates $(t, x_1, x_2, x_3)$ and an inner product of signature $(- + ++)$. The *electromagnetic field* is $F \in \Omega^2(\mathbf{R}^4)$ given by

$$\begin{aligned} F = (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) \wedge dt \\ + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2 \end{aligned}$$

We use the Hodge star (Problem 11 in Chapter 2) to write

$$\begin{aligned} *F = -(B_1 dx_1 + B_2 dx_2 + B_3 dx_3) \wedge dt \\ + E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2 \end{aligned}$$

The *source* is $\mathcal{J} \in \Omega^3(\mathbf{R}^4)$ given by

$$\begin{aligned} \mathcal{J} &= *(-\rho\, dt + J_1\, dx_1 + J_2\, dx_2 + J_3\, dx_3) \\ &= \rho dx_1 \wedge dx_2 \wedge dx_3 \\ &\quad - dt \wedge (J_1 dx_2 \wedge dx_3 + J_2 dx_3 \wedge dx_1 + J_3 dx_1 \wedge dx_2). \end{aligned} \tag{4.7.1}$$

Now Maxwell's equations are equivalent to

$$\begin{aligned} dF &= 0 \\ d * F &= 4\pi \mathcal{J} \end{aligned}$$

The second equation says in particular that $\mathcal{J}$ is exact, thus $d\mathcal{J} = 0$. Computing $d\mathcal{J}$ from (4.7.1) we get the *law of conservation of charge*

$$\frac{d\rho}{dt} + \operatorname{div} \vec{J} = 0.$$

Integrating throughout over a compact domain $W$ in $\mathbf{R}^3$ with smooth boundary, and using the Divergence theorem (see Problem 10), we obtain

$$\int_{\partial W} (\vec{J} \cdot \vec{n})\, dA = -\frac{d}{dt} \int_W \rho\, dx dy dz.$$

The left-hand side represents the total amount of charge flowing outwards through the surface $\partial W$ per unit time. The right-hand side represents the amont by which the charge is decreasing inside the region $W$ per unit time. In other words, charge does not disappear into or is created of out of nothingness — it decreases in a region of space only becase it flows into other regions. This is an important test of Maxwell's equations since all experimental evidence points to charge conservation.

The geometrization of Maxwell's equations on the twentieth century lead to a vast generalization in the form of the so called Yang-Mills equations, which describe not only electromagnetism but also the strong and weak nuclear forces, but this is much beyond the scope of these modest notes.

## 4.8 Problems

§ 4.1

**1** Let $M$ be a smooth manifold of dimension $n$ and let $f : M \to \mathbf{R}^{n+1}$ be an immersion. Prove that $M$ is orientable if and only if there exists a nowhere vanishing smooth vector field $X$ along $f$ (see page 23) such that $X_p$ is normal to $df_p(T_pM)$ in $\mathbf{R}^{n+1}$ for all $p \in M$.

**2** Prove that $\mathbf{R}P^n$ is orientable if and only if $n$ is odd.

**3** Show that the global $n$-form constructed in Example 4.1.5 in the case of $S^n$ can be given as the restriction of

$$\alpha = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

to $S^n$, up to a constant multiple.

**4** Prove that a parallelizable manifold is orientable.

**5 (Integration on a Riemannian manifold)** Let $(M, g)$ be a Riemannian manifold of dimension $n$.

*a.* On any coordinate neighborhood $U$, construct a *local orthonormal frame* $E_1, \ldots, E_n$, that is, a set of $n$ smooth vector fields on $U$ which is orthonormal at every point of $U$. (Hint: Apply the Gram-Schmidt process to the coordinate vector fields.)

*b.* Let $\omega_1, \ldots, \omega_n$ be the 1-forms dual to an orthonormal frame on $U$. This is called a *local orthonormal coframe* on $U$. Suppose now $\omega'_1, \ldots, \omega'_n$ is a local orthonormal coframe on $U'$. Prove that

$$\omega_1 \wedge \cdots \wedge \omega_n = \pm \omega'_1 \wedge \cdots \wedge \omega'_n$$

at each point of $U \cap U'$.

*c.* Deduce that in case $M$ is orientable, the locally defined $n$-forms $\omega_1 \wedge \cdots \wedge \omega_n$ can be pieced together to yield a globally defined nowhere vanishing $n$-form $\mathrm{vol}_M$ on $M$ satisfying

$$\mathrm{vol}_M(E_1, \ldots, E_n) = 1$$

for every positive local orthonormal frame $E_1, \ldots, E_n$. This form is called the *volume form* of the oriented Riemannian manifold $M$ and its integral is called the *volume* of $M$.

*d.* Show that for a positively oriented basis $v_1, \ldots, v_n$ of $T_pM$, we have

$$(\mathrm{vol}_M)_p(v_1, \ldots, v_n) = \sqrt{\det\left(g_p(v_i, v_j)\right)}.$$

Deduce that, in local coordinates $(U, \varphi = (x^1, \ldots, x^n))$,

$$\mathrm{vol}_M = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

**6** Consider the unit sphere $S^n$ in $\mathbf{R}^{n+1}$ as a Riemannian manifold where, for each $p \in S^n$, the inner product on the tangent space $T_pS^n$ is obtained by restriction of the standard scalar product in $\mathbf{R}^{n+1}$. Recall the $n$-form $\alpha$ on $S^n$ given in Exercise 3. Let $X$ be the outward unit normal vector field along $S^n$.

*a.* Show that

$$\alpha_p = \iota_{X_p}(dx_1 \wedge \cdots \wedge dx_{n+1}|_p)$$

for all $p \in S^n$.

*b.* Deduce from (a) that $\alpha$ is the volume form of $S^n$ with respect to some orientation.

*c.* In case $n = 2$, compute the volume of $S^2$.

§ 4.2

**7** Let $\gamma : [a, b] \to M$ be a smooth curve, and let $\gamma(a) = p$, $\gamma(b) = q$. Show that if $\omega = df$ for a smooth function $f$ on $M$, then

$$\int_a^b \gamma^*\omega = f(q) - f(p).$$

**8** Let $\gamma : [a, b] \to M$ be a smooth curve, and let $h : [c, d] \to [a, b]$ a smooth map with $h(c) = a$ and $h(d) = b$. Show that

$$\int_a^b \gamma^*\omega = \int_c^d (\gamma \circ h)^*\omega$$

for every 1-form $\omega$ on $M$.

**9** A *closed curve* in $M$ is a smooth map $\gamma : S^1 \to M$. For a 1-form $\omega$ on $M$, define the *line integral* of $\omega$ around $\gamma$ as

$$\int_\gamma \omega := \int_{S^1} \gamma^*\omega.$$

*a.* Write the line integral in local coodinates in case the image of $\gamma$ lies in a coordinate neighborhood of $M$.

*b.* Show that

$$\int_\gamma \omega = \int_0^{2\pi} (\gamma \circ h)^* \omega$$

where $h : [0, 2\pi] \to S^1$ is given by $h(t) = e^{it}$.

**10** Let $S$ be an orientable smooth manifold of dimension 2, let $f : S \to \mathbf{R}^3$ be an immersion, and let $\vec{n}$ be a unit normal vector field along $f$ as in Problem 1. Consider the Riemannian metric induced by the immersion $f$, that is,

$$g_p(u, v) = df_p(u) \cdot df_p(v)$$

for all $p \in M$ and $u$, $v \in T_pM$.

*a.* Prove that the volume form (see Problem 5) of $(S, g)$ is given by

$$dA = n_1\, dx_2 \wedge dx_3 + n_2\, dx_3 \wedge dx_1 + n_3\, dx_1 \wedge dx_2$$

where $n_1$, $n_2$, $n_3$ are the components of $\vec{n}$ in $\mathbf{R}^3$ and each $dx_i$ is restricted to $S$.

*b.* Assume $f$ is an inclusion, $S$ is the boundary of a a compact domain $W$ in $\mathbf{R}^3$, and $\vec{F}$ is a smooth vector field on $W$. Show that Stokes' formula 4.2.4 specializes to the classical Divergence theorem:

$$\int_S (\vec{F} \cdot \vec{n})\, dA = \int_W (\mathrm{div}\vec{F})\, dx_1 dx_2 dx_3.$$

§ 4.3

**11** Let $\alpha$ and $\beta$ be closed differential forms. Show that $\alpha \wedge \beta$ is closed. In addition, if $\beta$ is exact, show that $\alpha \wedge \beta$ is exact.

**12** Let $\alpha = (2x + y \cos xy)\, dx + (x \cos xy)\, dy$ be a 1-form on $\mathbf{R}^2$. Show that $\alpha$ is exact by finding a smooth function $f$ on $\mathbf{R}^2$ such that $df = \alpha$.

**13** Prove that $T^2$ and $S^2$ are not diffeomorphic by using de Rham cohomology.

§ 4.4

**14** *a.* Prove that every closed 1-form on the open subset $A$ in $\mathbf{R}^3$ given by

$$1 < \left( \sum_{i=1}^{3} x_i^2 \right)^{1/2} < 2$$

is exact.

*b.* Give an example of a 2-form on $A$ which is closed but not exact.

*c.* Prove that $A$ is not diffeomorphic to the open ball in $\mathbf{R}^3$.

**15** Assume $M = \partial P$ where $P$ is a compact smooth manifold and let $f : M \to N$ be a smooth map. Prove that if $f$ extends to a smooth map $F : P \to N$ then $\int_M f^*\omega = 0$ for every closed $n$-form $\omega$ in $N$, where $n = \dim N$.

**16** Assume $M$ is a compact smooth manifold of dimension $m$ and $f, g : M \to N$ are homotopic maps. Prove that

$$\int_M f^*\omega = \int_M g^*\omega$$

for every closed $m$-form $\omega$ in $N$.

**17** Prove that a 1-form $\omega$ on a smooth manifold $M$ has $\int_\gamma \omega = 0$ for every closed curve $\gamma$ in $M$ if and only if it is exact. (Hint: Show that $f(p) = \int_{p_0}^p \omega$ is well defined and satisfies $df = \omega$.)

**18** Prove that $H^1(M) = 0$ for a simply-connected smooth manifold $M$. (Hint: By approximation results, a smooth manifold is simply-connected if and only if every smooth closed curve is smoothly homotopic to a point.)

§ 4.5

**19** Let $f : S^1 \to S^1$ be a smooth map.

*a.* Prove that there exists a smooth map $g : \mathbf{R} \to \mathbf{R}$ such that $f(e^{it}) = e^{ig(t)}$ and $g(t + 2\pi) = g(t) + 2\pi d$ for all $t \in \mathbf{R}$, where $d$ is the degree of $f$ integer.

*b.* Use part (a) to show that if $f, g : S^1 \to S^1$ have the same degree then they are homotopic. Deduce that homotopy classes of smooth maps $S^1 \to S^1$ are classified by their degree.

**20** Let $f : M \to N$ be a smooth map between oriented manifolds of the same dimension where $N$ is connected. Assume $M$ is the boundary $\partial P$ of a compact oriented smooth manifold $P$, $M$ has the induced orientation, and $f$ extends to a smooth map $F : P \to N$. Prove that $\deg f = 0$.

**21 (Fundamental theorem of algebra)** Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$ be a complex polynomial.

*a.* Consider the extended complex plane $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and show that $z : \bar{\mathbf{C}} \setminus \{\infty\} \to \mathbf{C} \cong \mathbf{R}^2$, $\frac{1}{z} : \bar{\mathbf{C}} \setminus \{0\} \to \mathbf{C} \cong \mathbf{R}^2$ define a smooth atlas on $\bar{\mathbf{C}}$. (Hint: Use Proposition 1.2.10.) Use stereographic projection from the north and south poles to construct a diffeomorphism $S^2 \cong \bar{\mathbf{C}}$.

*b.* Extend $f$ to a map $\tilde{f} : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ by putting $\tilde{f}(\infty) = \infty$. Check that $\tilde{f}$ is smooth using the atlas constructed in (a).

*c.* Show that $\tilde{f}$ is smoothly homotopic to $g : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ where $g(z) = z^k$. What is the degree of $g$?

*d.* Deduce from (c) that $f$ is surjective. In particular, there exists $z_0 \in \mathbf{C}$ such that $f(z_0) = 0$.

**22** Define the *Hopf map* $\pi : S^3 \to S^2$ by $\pi(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$, where we view $S^3 \subset \mathbf{C}^2$ and $S^2 \subset \mathbf{C} \times \mathbf{R}$.

*a.* Show that the level sets of $\pi$ are circles of the form $\{e^{it} \cdot p \mid t \in \mathbf{R}\}$ for some $p \in S^3$.

*b.* Compute the linking number of $\pi^{-1}(0, 1)$ and $\pi^{-1}(0, -1)$.

**23** Let $M$ be a compact connected orientable surface (2-dimensional manifold) in $\mathbf{R}^3$. Consider the Riemannian metric obtained by restriction of the scalar product of $\mathbf{R}^3$ to the tangent spaces of $M$.

*a.* According to Exercise 1, there exists a smooth normal unit vector field along $M$ in $\mathbf{R}^3$. Use the canonical parallelism in $\mathbf{R}^3$ to view this vector field as a smooth map $g : M \to S^2$; this map is called the *Gauss map* of $M$; check that it is uniquely defined, up to sign.

*b.* For $p \in M$, the differential $dg_p : T_pM \to T_{g(p)}S^2$ where $T_pM$ and $T_{g(p)}S^2$ can again be identified under the canonical parallelism in $\mathbf{R}^3$. The *Gaussian curvature* $\kappa(p)$ of $M$ at $p$ is the determinant $\det(dg_p)$, and does not depend on the choice of sign in (a). Prove that

$$\kappa \operatorname{vol}_M = g^* \operatorname{vol}_{S^2}.$$

*c.* Use (b) and the Gauss-Bonnet theorem to conclude that the degree of the Gauss map is half the Euler characteristic of $M$:

$$\deg g = \frac{1}{2}\chi(M).$$

§ 4.6

**24** Use Problem 19(a) to show that an odd smooth map $f : S^1 \to S^1$ has odd degree.

**25** Prove that there exists no antipode-preserving smooth map $f : S^n \to S^{n-1}$.

**26** Let $f : S^n \to \mathbf{R}^n$ be a continuous map.

*a.* Use the Stone-Weierstrass theorem to show that for all $\epsilon > 0$ there exists a smooth map $g : S^n \to \mathbf{R}^n$ such that $||g(x) - f(x)|| < \epsilon$ for all $x \in S^n$.

*b.* Prove that there exists a pair of antipodal points $\pm p \in S^n$ such that $f(p) = f(-p)$.

*c.* Deduce form part (b) the following results:

(i) **(Ham sandwich theorem)** Let $A_1, \ldots, A_n$ be $n$ Lebesgue measurable sets in $\mathbf{R}^n$. Then there exists a hyperplane $\mathcal{H}$ simultaneously bisecting all sets into half their volumes, that is,

$$\mathrm{vol}(A_i \cap \mathcal{H}^+) = \mathrm{vol}(A_i \cap \mathcal{H}^-)$$

where $\mathcal{H}^\pm$ are the half-spaces defined by $\mathcal{H}$.

(ii) **(Lyusternik-Schnirel'man)** For any cover $\{F_1, \ldots, F_{n+1}\}$ of $S^n$ by closed sets, there exists at least one set containing a pair of antipodal points, that is, $F_i \cap (-F_i) \neq \varnothing$ for some $i = 1, \ldots, n+1$.

CHAPTER A

# Covering manifolds

In this appendix, we summarize some properties of covering spaces in the context of smooth manifolds.

## A.1 Topological coverings

Recall that a (topological) *covering* of a space $X$ is another space $\tilde{X}$ with a continuous map $\pi : \tilde{X} \to X$ such that $X$ is a union of evenly covered open set, where a connected open subset $U$ of $X$ is called *evenly covered* if

$$\pi^{-1}U = \cup_{i \in I} \tilde{U}_i \qquad \text{(A.1.1)}$$

is a disjoint union of open sets $\tilde{U}_i$ of $\tilde{X}$, each of which is mapped homeomorphically onto $U$ under $\pi$. In particular, the fibers of $\pi$ are discrete subsets of $\tilde{X}$. It also follows from the definition that $\tilde{X}$ has the Hausdorff property if $X$ does. Further it is usual, as we shall do, to require that $X$ and $\tilde{X}$ be connected, and then the index set $I$ can be taken the same for all evenly covered open sets.

**A.1.2 Examples** *(a)* $\pi : \mathbf{R} \to S^1$, $\pi(t) = e^{it}$ *is a covering.*

*(b)* $\pi : S^1 \to S^1$, $\pi(z) = z^n$ *is a covering for any nonzero integer* $n$.

*(c)* $\pi : (0, 3\pi) \to S^1$, $\pi(t) = e^{it}$ *is a local homemeomorphism which is not a covering, since* $1 \in S^1$ *does not admit evenly covered neighborhoods.*

## A.2 Fundamental groups

Covering spaces are closely tied with fundamental groups. The *fundamental group* $\pi_1(X, x_0)$ of a topological space $X$ with basepoint $x_0$ is defined as follows. As a set, it consists of the homotopy classes of continuous loops based at $x_0$. The concatenation of such loops is compatible with the equivalence relation given by homotopy, so it induces a group operation on $\pi_1(X, x_0)$ making it into a group. If $X$ is arcwise connected, the isomorphism class of the fundamental group is independent of the choice of basepoint (indeed for $x_0$, $x_1 \in X$ and $c$ a continuous path from $x_0$ to $x_1$, conjugation by $c^{-1}$ induces an isomorphism from $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$) and thus is sometimes denoted by $\pi_1(X)$. Finally, a continuous map $f : X \to Y$ between topological spaces with $f(x_0) = y_0$ induces a homomorphism $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ so that the assignment $(X, x_0) \to \pi_1(X, x_0)$ is functorial. Of course the fundamental group is trivial if and only if the space is simply-connected.

Being locally Euclidean, a smooth manifold is locally arcwise connected and locally simply-connected. A connected space $X$ with such local connectivity properties admits a simply-connected covering space, which is unique up to isomorphism; an isomorphism between coverings $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $\pi_2 \circ f = \pi_1$. More generally, there exists a bijective correspondence between classes of basepoint-preserving isomorphisms of coverings $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$ given by $(\tilde{X}, \tilde{x}_0) \mapsto \pi_\#(\pi_1(\tilde{X}, \tilde{x}_0))$; moreover, a change of basepoint in $\tilde{X}$ corresponds to passing to a conjugate subgroup $\pi_1(X, x_0)$.

**A.2.1 Lemma** *The fundamental group of a connected smooth manifold $M$ is a countable group.*

*Proof.* Here we strongly use the second-countability of $M$. It implies that we can find a countable covering $\{B_i\}$ of $M$ by open sets, each of which diffeomorphic to a ball in Euclidean space. For all $i$, $j$, $B_i \cap B_j$ has countably many path-components; fix a point in each one of them and denote the (countable) collection of points thus obtained by $\mathcal{P}$. Finally, for all $p, p' \in \mathcal{P}$ with $p, p' \in B_i$ for some $i$, fix a path $\gamma^i_{p,p'}$ joining $p$ to $p'$ inside $B_i$, and denote the (countable) collection of paths thus obtained by $\mathcal{C}$. Taking $p_0 \in \mathcal{P}$; using the simple-connectedness of the $B_i$, it is now more or less clear that every loop based at $p_0$ is homotopic to a loop at $p_0$ consisting of the product of finitely many elements of $\mathcal{C}$. Hence $\pi_1(M, p_0)$ is countable. □

## A.3 Smooth coverings

Suppose $\pi : \tilde{M} \to M$ is a covering where $M$ is a smooth manifold. Then there is a natural structure of smooth manifold on $\tilde{M}$ such that the projection $\pi$ is smooth. In fact, for every chart $(U, \pi)$ of $M$ where $U$ is evenly covered as in (A.1.1), take a chart $(\tilde{U}_i, \varphi \circ \pi|_{\tilde{U}_i})$ for $\tilde{M}$. This gives an atlas of $\tilde{M}$, which is smooth because for another chart $(V, \psi)$ of $M$, $V$ evenly covered by $\cup_{i \in I} \tilde{V}_i$ and $\tilde{U}_i \cap \tilde{V}_j \neq \varnothing$ for some $i, j \in I$, we have that the transition map

$$(\psi \circ \pi|_{\tilde{V}_j})(\varphi \circ \pi|_{\tilde{U}_i})^{-1} = \psi \circ \varphi^{-1}$$

is smooth. We already know that $\tilde{M}$ is a Hausdorff space. It is possible to choose a countable basis of connected open sets for $M$ which are evenly covered. The connected components of the preimages under $\pi$ of the elements of this basis form a basis of connected open sets for $\tilde{M}$, which is countable as long as the index set $I$ is countable, but this follows from the countability of the fundamental group $\pi_1(M)$. Now, around any point in $\tilde{M}$, $\pi$ admits a local representation as the identity, so it is a local diffeomorphism. Note that we have indeed proved more: $M$ can be covered by evenly covered neighborhoods $U$ such that the restriction of $\pi$ to a connected component of $\pi^{-1}U$ is a diffeomorphism onto $U$. This is the definition of a *smooth covering*. Note that a topological covering whose covering map is smooth need not be a smooth covering (e.g. $\pi : \mathbf{R} \to \mathbf{R}$, $\pi(x) = x^3$).

Next, we can formulate basic results in covering theory for a smooth covering $\pi : \tilde{M} \to M$ of a smooth manifold $M$. Fix basepoints $\tilde{p} \in \tilde{M}$, $p \in M$ such that $\pi(\tilde{p}) = p$. We say that a map $f : N \to M$ admits a lifting if there exists a map $\tilde{f} : N \to \tilde{M}$ such that $\pi \circ \tilde{f} = f$.

**A.3.1 Theorem (Lifting criterion)** *Let $q \in f^{-1}(p)$. A smooth map $f : N \to M$ admits a smooth lifting $\tilde{f} : N \to \tilde{M}$ with $\tilde{f}(q) = \tilde{p}$ if and only if $f_\#(\pi_1(N, q)) \subset \pi_\#(\pi_1(\tilde{M}, \tilde{p}))$. In that case, if $N$ is connected, the lifting is unique.*

Taking $f : N \to M$ to be the universal covering of $M$ in Theorem A.3.1 shows that the universal covering of $M$ covers any other covering of $M$ and hence justifies its name.

## A.4 Deck transformations

For a topological covering $\pi : \tilde{X} \to X$, a *deck transformation* or *covering transformation* is an isomorphism $\tilde{X} \to \tilde{X}$, namely, a homeomorphism $f : \tilde{X} \to \tilde{X}$ such that $\pi \circ f = \pi$. The deck transformations form a group under composition. It follows from uniqueness of liftings that a deck transformation is uniquely determined by its action on one point. In particular, the only deck transformation admitting fixed points is the identity. Since a smooth covering map $\pi : \tilde{M} \to M$ is a local diffeomorphism, in this case the equation $\pi \circ f = \pi$ implies that deck transformations are diffeomorphisms of $\tilde{M}$.

An action of a (discrete) group on a topological space (resp. smooth manifold) is a homomorphism from the group to the group of homeomorphisms (resp. diffeomorphisms) of the space (resp. manifold). For a smooth manifold $M$, we now recall the canonical action of $\pi_1(M, p)$ on its universal covering $\tilde{M}$ by deck transformations. First we remark that by the lifting criterion, given $q \in M$ and $\tilde{q}_1$, $\tilde{q}_2 \in \pi^{-1}(q)$, there is a unique deck transformation mapping $\tilde{q}_1$ to $\tilde{q}_2$. Now let $\gamma$ be a continuous loop in $M$ based at $p$ representing an element $[\gamma] \in \pi_1(M, p)$, and fix a point $\tilde{p} \in \pi^{-1}(p)$. By the remark, it suffices to describe the action of $[\gamma]$ on $\tilde{p}$, which goes as follows: lift $\gamma$ uniquely to a path $\tilde{\gamma}$ starting at $\tilde{p}$; then $[\gamma]\cdot\tilde{p}$ is by definition the endpoint of $\tilde{\gamma}$, which sits in the fiber $\pi^{-1}(p)$. The definition independs of the choice made, namely, if we change $\gamma$ to a homotopic curve, we get the same result. This follows from Theorem A.3.1 applied to the homotopy, as it is defined on a square and a square is simply-connected. Since $\pi:\tilde{M}\to M$ is the universal covering, every deck transformation is obtained in this way from an element of $\pi_1(M,p)$.

An action of a (discrete) group $\Gamma$ on a topological space $X$ is called *free* if no nontrivial element of $\Gamma$ has fixed points, and it is called *proper* if any two points $x$, $y\in X$ admit open neighborhoods $U\ni x$, $V\ni y$ such that $\{\,\gamma\in\Gamma\mid\gamma U\cap V\neq\varnothing\,\}$ is finite. The action of $\pi_1(M,p)$ on the universal covering $\tilde{M}$ by deck transformations has both properties. In fact, we have already remarked it is free. To check properness, let $\tilde{p}$, $\tilde{q}\in\tilde{M}$. If these points lie in the same orbit of $\pi_1(M,p)$ or, equivalently, the same fiber of $\pi$, the required neighborhoods are the connected components of $\pi^{-1}(U)$ containing $\tilde{p}$ and $\tilde{q}$, resp., where $U$ is an evenly covered neighborhood of $\pi(\tilde{p})=\pi(\tilde{q})$. On the other hand, if $\pi(\tilde{p})=:p\neq q:=\pi(\tilde{p})$, we use the Hausdorff property of $M$ to find disjoint evenly covered neighborhoods $U\ni p$, $V\ni q$ and then it is clear that the connected component of $\pi^{-1}(U)$ containing $\tilde{p}$ and the connected component of $\pi^{-1}(V)$ containing $\tilde{q}$ do the job.

Conversely, we have:

**A.4.1 Theorem** *If the group $\Gamma$ acts freely and properly on a smooth manifold $\tilde{M}$, then the quotient space $M=\Gamma\backslash\tilde{M}$ endowed with the quotient topology admits a unique structure of smooth manifold such that the projection $\pi:\tilde{M}\to M$ is a smooth covering.*

*Proof*. The action of $\Gamma$ on $\tilde{M}$ determines a partition into equivalence classes or *orbits*, namely $\tilde{p}\sim\tilde{q}$ if and only if $\tilde{q}=\gamma\tilde{p}$ for some $\gamma\in\Gamma$. The orbit through $\tilde{p}$ is denoted $\Gamma(\tilde{p})$. The quotient space $\Gamma\backslash\tilde{M}$ is also called *orbit space*.

The quotient topology is defined by the condition that $U\subset M$ is open if and only if $\pi^{-1}(U)$ is open in $\tilde{M}$. In particular, for an open set $\tilde{U}\subset\tilde{M}$ we have $\pi^{-1}(\pi(\tilde{U}))=\cup_{\gamma\in\Gamma}\gamma(\tilde{U})$, a union of open sets, showing that $\pi(\tilde{U})$ is open and proving that $\pi$ is an open map. In particular, $\pi$ maps a countable basis of open sets in $\tilde{M}$ to a countable basis of open sets in $M$.

The covering property follows from the fact that $\Gamma$ is proper. In fact, let $\tilde{p}\in\tilde{M}$. From the definition of proeprness, we can choose a neighborhood $\tilde{U}\ni\tilde{p}$ such that $\{\,\gamma\in\Gamma\mid\gamma\tilde{U}\cap\tilde{U}\neq\varnothing\,\}$ is finite. Using the Hausdorff property of $\tilde{M}$ and the freeness of $\Gamma$, we can shrink $\tilde{U}$ so that this set becomes empty. Now the map $\pi$ identifies all disjoint homeomorphic open sets $\gamma U$ for $\gamma\in\Gamma$ to a single open set $\pi(U)$ in $M$, which is then evenly covered.

The Hausdorff property of $M$ also follows from properness of $\Gamma$. Indeed, let $p$, $q \in M$, $p \neq q$. Choose $\tilde{p} \in \pi^{-1}(p)$, $\tilde{q} \in \pi^{-1}(q)$ and neighborhoods $\tilde{U} \ni \tilde{p}$, $\tilde{V} \ni \tilde{q}$ such that $\{ \gamma \in \Gamma \mid \gamma\tilde{U} \cap \tilde{V} \neq \varnothing \}$ is finite. Note that $\tilde{q} \notin \Gamma(\tilde{p})$, so by the Hausdorff property for $\tilde{M}$, we can shrink $\tilde{U}$ so that this set becomes empty. Since $\pi$ is open, $U := \pi(\tilde{U})$ and $V := \pi(\tilde{V})$ are now disjoint neighborhoods of $p$ and $q$, respectively.

Finally, we construct a smooth atlas for $M$. Let $p \in M$ and choose an evenly covered neighborhood $U \ni p$. Write $\pi^{-1}U = \cup_{i\in I}\tilde{U}_i$ as in (A.1.1). By shrinking $U$ we can ensure that $\tilde{U}_i$ is the domain of a local chart $(\tilde{U}_i, \tilde{\varphi}_i)$ of $\tilde{M}$. Now $\varphi_i := \tilde{\varphi}_i \circ (\pi|_{\tilde{U}_i})^{-1} : U \to \mathbf{R}^n$ defines a homeomorphism onto the open set $\tilde{\varphi}_i(\tilde{U}_i)$ and thus a local chart $(U, \varphi_i)$ of $M$. The domains of such charts cover $M$ and it remains only to check that the transition maps are smooth. So let $V$ be another evenly covered neighborhood of $p$ with $\pi^{-1}V = \cup_{j\in I}\tilde{V}_j$ and associated local chart $\psi_j := \tilde{\psi}_j \circ (\pi|_{\tilde{V}_j})^{-1} : U \to \mathbf{R}^n$ where $(\tilde{V}_j, \tilde{\psi}_j)$ is a local chart of $\tilde{M}$. Then

$$\psi_j \circ \varphi_i^{-1} = \tilde{\psi}_j \circ (\pi|_{\tilde{V}_j})^{-1} \circ \pi \circ \tilde{\varphi}_i^{-1} \tag{A.4.2}$$

However, $(\pi|_{\tilde{V}_j})^{-1} \circ \pi$ is realized by a unique element $\gamma \in \Gamma$ in a neighborhood of $\tilde{p}_i = \pi|_{\tilde{U}_i}^{-1}(p)$. Since $\Gamma$ acts by diffeomorphisms, this shows that the transtion map (A.4.2) is smooth and finishes the proof. □

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