UNIVERSITY OF SÃO PAULO

Notes on Smooth Manifolds

CLAUDIO GORODSKI

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Foreword

The concept of smooth manifold is ubiquitous in Mathematics. Indeed smooth manifolds appear as Riemannian manifolds in differential geometry, space-times in general relativity, phase spaces and energy levels in mechanics, domains of definition of ODE's in dynamical systems, Riemann surfaces in theory of complex analytic functions, Lie groups in algebra and geometry..., to name a few instances.

The notion took some time to evolve until it reached its present form in H. Whitney's celebrated Annals of Mathematics paper in 1936. Whitney's paper in fact represents a culmination of diverse historical developments which took place separately, each in a different domain, all striving to make the passage from the local to the global.

From the modern point of view, the initial goal of introducing smooth manifolds is to generalize the methods and results of differential and integral calculus, in special, the inverse and implicit function theorems, the theorem on existence, uniqueness and regularity of ODE's and Stokes' theorem. As usual in Mathematics, once introduced such objects start to atract interest on their own and new structure is uncovered. The subject of differential topology studies smooth manifolds *per se*. Many important results about the topology of smooth manifolds were obtained in the 1950's and 1960's in the high dimensional range. For instance, there exist topological manifolds admitting several non-diffeomorphic smooth structures (Milnor, 1956, in the case of S^7), and there exist topological manifolds admitting no smooth strucuture at all (Kervaire, 1961). Moreover the Poincaré conjecture in dimensions bigger than 4 was proved independently by Stallings and Smale in the 1960's. On the other hand, the topology of compact surfaces is a classical subject already tackled in the nineteenth century; the very important case of dimension 3 has seen tremendous development after the works of Thurston (late 1970's), Hamilton (1981) and Perelman (2003), and continues to attract a lot of attention; and the case of dimension 4, despite the breakthroughs of Donaldson and Freedman in the 1980's, is largely terra incognita.

The aim of these notes is much more modest. Their contents cover, with some looseness, the syllabus of the course "Differentiable manifolds and Lie groups" that I taught at the Graduate Program of the University of São Paulo in 2001, 2008, 2013 and 2015. Chapter 1 introduces the basic language of smooth manifolds, culminating with the Frobenius theorem. Chapter 2 introduces the basic language of tensors. The most important construction there is perhaps the exterior derivative of differential forms. Chapter 3 is a first encounter with Lie groups and their Lie algebras, in which also homogeneous manifolds are briefly discussed. Finally, Chapter 4 is about integration on manifolds and explains Stokes' theorem, de Rham cohomology and some rudiments of differential topology. Routine exercises are scattered throughout the text, which aim to help the reader digest the material. More elaborate problems can be found in the final section of each chapter. Needless to say, working arduously in problems is a necessary (but not sufficient) condition to advance one's comprehension of a mathematical theory.

I am indebted to the (dozens of) graduate students who took my courses and impelled me to write this set of notes. Special thanks go to Dr. Pedro Zühlke whose careful reading and suggestions has helped improve the text. Any remaining errors are of course my own fault.

São Paulo, December 2015

Contents

CHAPTER 1

Smooth manifolds

In order to motivate the definition of abstract smooth manifold, we first define submanifolds of Euclidean spaces. Recall from vector calculus and differential geometry the ideas of parametrizations and inverse images of regular values.

1.1 Submanifolds of Euclidean spaces

A smooth map $f : U \to \mathbf{R}^{n+k}$, where $U \subset \mathbf{R}^n$ is open, is called an *immersion at* p, where $p \in U$, if $df_p : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective. f is called simply an *immersion* if it is an immersion everywhere. An injective immersion will be called a *parametrization*.

A smooth map $F : W \to \mathbf{R}^k$, where $W \subset \mathbf{R}^{n+k}$ is open, is called a *submersion at* p, where $p \in W$, if $dF_p : \mathbf{R}^{n+k} \to \mathbf{R}^k$ is surjective. F is called simply a *submersion* if it is a submersion everywhere. For $z_0 \in \mathbf{R}^k$, if F is a submersion along the level set $F^{-1}(z_0)$, then z_0 is called a *regular value* of F (in particular, a point $z_0 \in \mathbf{R}^k$ *not* in the image of F is always a regular value!).

Images of parametrizations and inverse images of regular values are thus candidates to be submanifolds of Euclidean spaces. Next we would like to explain why the second class has stronger properties than the first one. The argument involves the implicit function theorem, and how it is proved to be a consequence of the inverse function theorem.

Assume then z_0 is a regular value of F as above and $F^{-1}(z_0)$ is non-empty; write M for this set and consider $p \in M$. Then dF_p is surjective and, up to relabeling the coordinates, we may assume that $(d_2F)_p$, which is the restriction of dF_p to $\{0\} \oplus \mathbf{R}^k \subset \mathbf{R}^{n+k}$, is an isomorphism onto $\mathbf{R}^k$. Write $p = (x_0, y_0)$ where $x_0 \in \mathbf{R}^n$, $y_0 \in \mathbf{R}^k$. Define a smooth map

$$\Phi : W \to \mathbf{R}^{n+k}, \qquad \Phi(x, y) = (x, F(x, y) - z_0)$$

Then $d\Phi_{(x_0,y_0)}$ is easily seen to be an isomorphism, so the inverse function theorem implies that there exist open neighborhoods U, V of x_0, y_0 in $\mathbf{R}^n$,

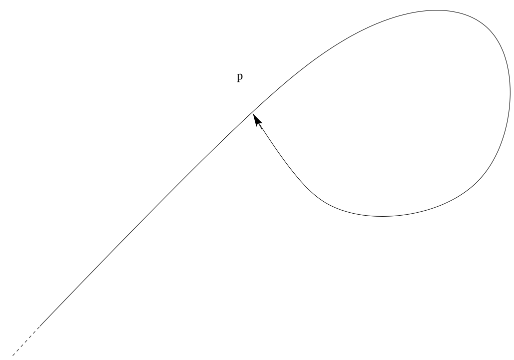


Figure 1.1: A non-embedded submanifold of $\mathbf{R}^2$.

$\mathbf{R}^k$, respectively, such that Φ is a diffeomorphism of $U \times V$ onto an open subset of $\mathbf{R}^{n+k}$, i.e. Φ is a smooth bijective map onto its image and the inverse map is also smooth. Now the fundamental fact is that

$$\Phi(M \cap (U \times V)) = (\mathbf{R}^n \times \{0\}) \cap \Phi(U \times V),$$

as it follows from the form of Φ; namely, Φ "rectifies" M.

Let $\varphi : M \cap (U \times V) \to \mathbf{R}^n$ be the restriction of Φ. Then φ^{-1} is the restriction of Φ to $\mathbf{R}^n$ and thus smooth. It also follows from the above calculation that $M \cap (U \times V)$ is exactly the graph of the smooth map $f : U \to V$, satisfying $f(x_0) = y_0$, given by $f = \mathrm{proj}_{\mathbf{R}^k} \circ \varphi^{-1}$. Another way to put it is that $M \cap (U \times V)$ is the image of a parametrization $\varphi^{-1} : \varphi(M \cap (U \times V)) \subset \mathbf{R}^n \to \mathbf{R}^{n+k}$ which is a homeomorphism onto its image, where the latter is equipped with the topology induced from $\mathbf{R}^{n+k}$.

1.1.1 Definition (i) A subset $M \subset \mathbf{R}^{n+k}$ will be called a *embedded submanifold of dimension n of* $\mathbf{R}^{n+k}$ if for every $p \in M$, there exists a diffeomorphism Φ from an open neighborhood U of p in $\mathbf{R}^{n+k}$ onto its image such that $\Phi(M \cap U) = (\mathbf{R}^n \times \{0\}) \cap \Phi(U)$. In this case we will say that (U, Φ) is a *local chart of* $\mathbf{R}^{n+k}$ *adapted to* M.

(ii) A *parametrized submanifold of dimension n of* $\mathbf{R}^{n+k}$ is a pair (U, f) where $U \subset \mathbf{R}^n$ is open and $f : U \to \mathbf{R}^{n+k}$ is an injective immersion.

1.1.2 Example Let $(\mathbf{R}, f)$ be a parametrized submanifold of dimension 1 of $\mathbf{R}^2$, where $f : \mathbf{R} \to \mathbf{R}^2$ has image M described in Figure 1.1. Then M is non-embedded. In fact no connected neighborhood of p can be homeomorphic to an interval of $\mathbf{R}$ (restrict such a homeomorphism to the complement of $\{p\}$ to get a contradiction). Note that f is not a homeomorphism onto its image.

1.1.3 Exercise Prove that the graph of a smooth map $f : U \to \mathbf{R}^k$, where $U \subset \mathbf{R}^n$ is open, is an embedded submanifold of dimension n of $\mathbf{R}^{n+k}$.

1.1.4 Exercise Let $f, g : (0, 2\pi) \to \mathbf{R}^2$ be defined by

$$f(t) = (\sin t, \sin t \cos t), \qquad g(t) = (\sin t, -\sin t \cos t).$$

a. Check that f, g are injective immersions with the same image.
b. Sketch a drawing of their image.
c. Write a formula for $g^{-1} \circ f : (0, 2\pi) \to (0, 2\pi)$.
d. Deduce that the identity map $\mathrm{id} : \mathrm{im}\, f \to \mathrm{im}\, g$ is not continuous, where $\mathrm{im}\, f$ and $\mathrm{im}\, g$ are equipped with the topology induced from $\mathbf{R}$ via f and g, respectively.

The algebra $C^\infty(M)$ of real smooth functions on M

Let M be an embedded submanifold of $\mathbf{R}^{n+k}$.

1.1.5 Definition A function $f : M \to \mathbf{R}$ is said to be *smooth* at $p \in M$ if $f \circ \Phi^{-1} : \Phi(U) \cap \mathbf{R}^n \to \mathbf{R}$ is a smooth function for some adapted local chart (U, Φ) around p.

1.1.6 Remark (i) The condition is independent of the choice of adapted local chart around p. Indeed if (V, Φ) is another one,

$$f \circ \Phi^{-1} = (f \circ \Psi^{-1}) \circ (\Psi \circ \Phi^{-1})$$

where $\Psi \circ \Phi^{-1} : \Phi(U \cap V) \to \Psi(U \cap V)$ is a diffeomorphism and the claim follows from the the chain rule for smooth maps between Euclidean spaces.

(ii) A smooth function on M is automatically continuous.

(iii) Let F be a smooth function defined on an open neighborhood of p in $\mathbf{R}^{n+k}$. The restriction of F to M is smooth at p.

1.2 Definition of abstract smooth manifold

Let M be a topological space. A *local chart* of M is a pair (U, φ), where U is an open subset of M and φ is a homeomorphism from U onto an open subset of $\mathbf{R}^n$. A local chart $\varphi : U \to \mathbf{R}^n$ introduces coordinates $(x_1, \ldots, x_n)$ on U, namely, the component functions of φ, and that is why (U, φ) is also called a *system of local coordinates* on M.

A *(topological) atlas* for M is a family $\{(U_\alpha, \varphi_\alpha)\}$ of local charts of M, where the dimension n of the Euclidean space is fixed, whose domains cover M, namely, $\bigcup U_\alpha = M$. If M admits an atlas, we say that M is *locally modeled on* $\mathbf{R}^n$ and M is a *topological manifold*.

A *smooth atlas* is an atlas whose local charts satisfy the additional *compatibility condition*:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{1.2.1}$$

is smooth, for all α, β. A smooth atlas $\mathcal{A}$ defines a notion of smooth function on M as above, namely, a function $f : M \to \mathbf{R}$ is *smooth* if $f \circ \varphi^{-1} : \varphi(U) \to \mathbf{R}$ is smooth for all $(U, \varphi) \in \mathcal{A}$. We say that two atlas $\mathcal{A}$, $\mathcal{B}$ for M are *equivalent* if the local charts of one are compatible with those of the other, namely, $\psi \circ \phi^{-1}$ is smooth for all $(U, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$. In this case, it is obvious that $\mathcal{A}$ and $\mathcal{B}$ define the same notion of smooth function on M.

A *smooth structure* on M is an equivalence class $[\mathcal{A}]$ of smooth atlases on M. Finally, a *smooth manifold* is a topological space M equipped with a smooth structure $[\mathcal{A}]$. In order to be able to do interesting analysis on M, we shall assume, as usual, that *the topology of M is Hausdorff and second countable*.

1.2.2 Remark (a) It follows from general results in topology that (smooth) manifolds are metrizable. Indeed, manifols are locally Euclidean and thus locally compact. A locally compact Hausdorff space is (completely) regular, and the Urysohn metrization theorem states that a second countable regular space is metrizable.

(b) The condition of second countability also rules out pathologies of the following kind. Consider $\mathbf{R}^2$ with the topology with basis of open sets $\{(a, b) \times \{c\} \mid a, b, c \in \mathbf{R}, a < b\}$. This topology is Hausdorff but not second countable, and it is compatible with a structure of smooth manifold of dimension 1 (a continuum of real lines)!

1.2.3 Exercise Let M be a topological space. Prove that two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if their union $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Deduce that every equivalence class of smooth atlases for M contains a unique representative which is *maximal* (i.e. not properly contained in any other smooth atlas in the same equivalence class).

Let M, N be smooth manifolds. A map $f : M \to N$ is called *smooth* if for every $p \in M$, there exist local charts (U, φ), (V, ψ) of M, N around p, $f(p)$, resp., such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

1.2.4 Remark (i) The definition is independent of the choice of local charts.

(ii) The definition is local in the sense that $f : M \to N$ is smooth if and only if its restriction to an open subset U of M is smooth (cf. Example 1.2.7(vi)).

(iii) A smooth map $M \to N$ is automatically continuous.

We have completed the definition of the category **DIFF**, whose objects are the smooth manifolds and whose morphisms are the smooth maps. An isomorphism in this category is usually called a *diffeomorphism*.

1.2.5 Exercise Let M be a smooth manifold with smooth atlas $\mathcal{A}$. Prove that any local chart $(U, \varphi) \in \mathcal{A}$ is a diffeomorphism onto its image. Conversely, prove any map $\tau : W \to \mathbf{R}^n$, where $n = \dim M$ and $W \subset M$ is open, which is a diffeomorphism onto its image belongs to a smooth atlas equivalent to $\mathcal{A}$; in particular, $(W, \tau) \in \mathcal{A}$ if $\mathcal{A}$ is maximal.

1.2.6 Remark In practice, explicitly written down atlases are finite (compare Problem 1 and Example 1.2.9). However, in view of the last assertion in Exercise 1.2.5, it is often convenient to implicitly represent a smooth structure by a maximal atlas, and we shall be doing that.

1.2.7 Examples (i) $\mathbf{R}^n$ has a canonical atlas consisting only of one local chart, namely, the identity map, which in fact is a *global* chart. This is the standard smooth structure on $\mathbf{R}^n$ with respect to which all definitions coincide with the usual ones. Unless explicit mention, we will always consider $\mathbf{R}^n$ with this smooth structure.

(ii) Any finite dimensional real vector space V has a canonical structure of smooth manifold. In fact a linear isomorphism $V \cong \mathbf{R}^n$ defines a global chart and thus an atlas, and two such atlases are always equivalent since the transition map between their global charts is a linear isomorphism of $\mathbf{R}^n$ and hence smooth.

(iii) Submanifolds of Euclidean spaces (Definition 1.1.1(i)) are smooth manifolds. Namely, atlases are construted by using restrictions of adapted charts. Note that the compatibility condition (1.2.1) is automatically satisfied.

(iv) Graphs of smooth maps defined on open subsets of $\mathbf{R}^n$ with values on $\mathbf{R}^{n+k}$ are smooth manifolds (cf. Exercise 1.1.3 and (iii)). More generally, a subset M of $\mathbf{R}^{n+k}$ with the property that every one of its points admits an open neighborhood in M which is a graph as above is a smooth manifold.

(v) It follows from (iv) that the *n-sphere*

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is a smooth manifold.

(vi) If $\mathcal{A}$ is an atlas for M and $V \subset M$ is open then $\mathcal{A}|_V := \{(V \cap U, \varphi|_{V\cap U}) : (U, \varphi) \in \mathcal{A}\}$ is an atlas for V. It follows that any open subset of a smooth manifold is a smooth manifold.

(vii) If M, N are smooth manifolds with atlases $\mathcal{A}$, $\mathcal{B}$, resp., then $\mathcal{A} \times \mathcal{B}$ is an atlas for the Cartesian product $M \times N$ with the product topology, and hence $M \times N$ is canonically a smooth manifold of dimension $\dim M + \dim N$.

(viii) It follows from (iv) and (vi) that the *n-torus*

$$T^n = S^1 \times \cdots \times S^1 \qquad (n \text{ factors})$$

is a smooth manifold.

(ix) The *general linear group* $\mathbf{GL}(n,\mathbf{R})$ is the set of all $n\times n$ non-singular real matrices. Since the set of $n\times n$ real matrices can be identified with a $\mathbf{R}^{n^2}$ and as such the determinant becomes a continuous function, $\mathbf{GL}(n,\mathbf{R})$ can be viewed as the open subset of $\mathbf{R}^{n^2}$ where the determinant does not vanish and hence acquires the structure of a smooth manifold of dimension n^2.

The following two examples deserve a separate discussion.

1.2.8 Example The map $f:\mathbf{R}\to\mathbf{R}$ given by $f(x)=x^3$ is a homeomorphism, so it defines a local chart around any point of $\mathbf{R}$ and we can use it to define an atlas $\{f\}$ for $\mathbf{R}$; denote the resulting smooth manifold by $\tilde{\mathbf{R}}$. We claim that $\tilde{\mathbf{R}}\neq\mathbf{R}$ as smooth manifolds, because $C^\infty(\tilde{\mathbf{R}})\neq C^\infty(\mathbf{R})$. In fact, $\mathrm{id}:\mathbf{R}\to\mathbf{R}$ is obviously smooth, but $\mathrm{id}:\tilde{\mathbf{R}}\to\mathbf{R}$ is not, because $\mathrm{id}\circ f^{-1}:\mathbf{R}\to\mathbf{R}$ maps x to $\sqrt[3]{x}$ so it is not differentiable at 0. On the other hand, $\tilde{\mathbf{R}}$ is diffeomorphic to $\mathbf{R}$. Indeed $f:\tilde{\mathbf{R}}\to\mathbf{R}$ defines a diffeomorphism since its local representation $\mathrm{id}\circ f\circ f^{-1}$ is the identity.

1.2.9 Example The *real projective space*, denoted $\mathbf{R}P^n$, as a set consists of all one-dimensional subspaces of $\mathbf{R}^{n+1}$. We introduce a structure of smooth manifold of dimension n on $\mathbf{R}P^n$. Each subspace is spanned by a non-zero vector $v\in\mathbf{R}^{n+1}$. Let U_i be the subset of $\mathbf{R}P^n$ specified by the condition that the i-th coordinate of v is not zero. Then $\{U_i\}_{i=1}^{n+1}$ covers $\mathbf{R}P^n$. Each line in U_i meets the hyperplane $x_i=1$ in exactly one point, so there is a bijective map $\varphi_i:U_i\to\mathbf{R}^n\subset\mathbf{R}^{n+1}$. For $i\neq j$, $\varphi_i(U_i\cap U_j)\subset\mathbf{R}^n\subset\mathbf{R}^{n+1}$ is precisely the open subset of the hyperplane $x_i=1$ defined by $x_j\neq 0$, and

$$\varphi_j\circ\varphi_i^{-1}:\{x\in\mathbf{R}^{n+1}:x_i=1,\ x_j\neq 0\}\to\{x\in\mathbf{R}^{n+1}:x_j=1,\ x_i\neq 0\}$$

is the map

$$v\mapsto\frac{1}{x_j}v,$$

thus smooth. So far there is no topology in $\mathbf{R}P^n$, and we introduce one by declaring

$$\cup_{i=1}^{n+1}\{\varphi_i^{-1}(W):W\subset\varphi_i(U_i)=\mathbf{R}^n\text{ is open}\}$$

to be a basis of open sets. It is clear that $\varnothing$ and M are open sets (since each U_i is open) and we have only to check that finite intersections of open sets are open. Let $W_i\subset\varphi_i(U_i)$ and $W_j\subset\varphi_i(U_j)$ be open. Then

$$\varphi_i^{-1}(W_i)\cap\varphi_j^{-1}(W_j)=\varphi_j^{-1}\left(\varphi_j\varphi_i^{-1}(W_i\cap\varphi_i(U_i\cap U_j))\cap W_j\right).$$

Since $\varphi_j\varphi_i^{-1}$ is a homeomorphism, a necessary and sufficient condition for the left hand side to describe an open set for all i, j, is that $\varphi_i(U_i\cap U_j)$ be open for all i, j, and this does occur in this example. Now the topology is well defined, second countable, and the φ_i are homeomorhisms onto their images. It is also clear that for $\ell \in \mathbf{R}P^n$ the sets

$$\{\ell' \in \mathbf{R}P : \angle(\ell, \ell') < \epsilon\}$$

for $\epsilon > 0$ are open neighborhoods of ℓ. It follows that the topology is Hausdorff.

The argument in Example 1.2.9 is immediately generalized to prove the following proposition.

1.2.10 Proposition *Let M be a set and let n be a non-negative integer. A countable collection $\{(U_\alpha, \varphi_\alpha)\}$ of injective maps $\varphi : U_\alpha \to \mathbf{R}^n$ whose domains cover M satisfying*

a. $\varphi_\alpha(U_\alpha)$ is open for all α;
b. $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open for all α, β;
c. $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth for all α, β;

defines a second countable topology and smooth structure on M (the Hausdorff condition is not automatic and must be checked in each case).

1.3 Tangent space

As a motivation, we first discuss the case of an embedded submanifold M of $\mathbf{R}^{n+k}$. Fix $p \in M$ and take an adapted local chart (U, Φ) around p. Recall that we get a parametrization of M around p by setting $\varphi := \mathrm{proj}_{\mathbf{R}^n} \circ \Phi|_{M\cap U}$ and taking

$$\varphi^{-1} : \mathbf{R}^n \cap \Phi(U) \to \mathbf{R}^{n+k}.$$

It is then natural to define the *tangent space* of M at p to be the image of the differential of the parametrization, namely,

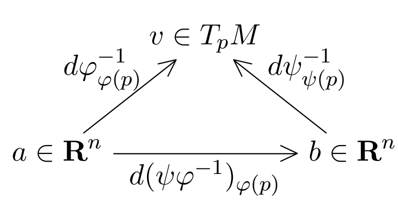
$$T_pM := d(\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n).$$

If (V, Ψ) is another adapted local chart around p, $\psi := \mathrm{proj}_{\mathbf{R}^n} \circ \Psi|_{M\cap V}$ and $\psi^{-1} : \mathbf{R}^n \cap \Psi(V) \to \mathbf{R}^{n+k}$ is the associated parametrization, then

$$\begin{aligned} d(\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n) &= d(\psi^{-1})_{\psi(p)} d(\psi\varphi^{-1})_{\varphi(p)}(\mathbf{R}^n) \\ &= d(\psi^{-1})_{\varphi(p)}(\mathbf{R}^n) \end{aligned}$$

since $d(\psi\varphi^{-1})_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. It follows that T_pM is well defined as a subspace of dimension n of $\mathbf{R}^{n+k}$.

Note that we have the following situation:



Namely, the tangent vector $v \in T_pM$ is represented by two different vectors a, $b \in \mathbf{R}^n$ which are related by the differential of the transition map. We can use this idea to generalize the construction of the tangent space to an abstract smooth manifold.

Let M be a smooth manifold of dimension n, and fix $p \in M$. Suppose that $\mathcal{A}$ is an atlas defining the smooth structure of M. The *tangent space* of M at p is the set T_pM of all pairs (a, φ) — where $a \in \mathbf{R}^n$ and $(U, \varphi) \in \mathcal{A}$ is a local chart around p — quotiented by the equivalence relation

$$(a, \varphi) \sim (b, \psi) \quad \text{if and only if} \quad d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

It follows from the chain rule in $\mathbf{R}^n$ that this is indeed an equivalence relation, and we denote the equivalence class of (a, φ) by $[a, \varphi]$. Each such equivalence class is called a *tangent vector* at p. For a fixed local chart (U, φ) around p, the map

$$a \in \mathbf{R}^n \mapsto [a, \varphi] \in T_pM$$

is a bijection, and it follows from the linearity of $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ that we can use it to transfer the vector space structure of $\mathbf{R}^n$ to T_pM. Note that $\dim T_pM = \dim M$.

1.3.1 Exercise Let M be a smooth manifold and let $V \subset M$ be an open subset. Prove that there is a canonical isomorphism $T_pV \cong T_pM$ for all $p \in V$.

Let $(U, \varphi = (x_1, \ldots, x_n))$ be a local chart of M, and denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$. The *coordinate vectors* at p are with respect to this chart are defined to be

$$\left.\frac{\partial}{\partial x_i}\right|_p = [e_i, \varphi].$$

Note that

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \tag{1.3.2}$$

is a basis of T_pM.

In the case of $\mathbf{R}^n$, for each $p \in \mathbf{R}^n$ there is a canonical isomorphism $\mathbf{R}^n \to T_p\mathbf{R}^n$ given by

$$a \mapsto [a, \mathrm{id}], \tag{1.3.3}$$

where id is the identity map of $\mathbf{R}^n$. Usually we will make this identification without further comment. In particular, $T_p\mathbf{R}^n$ and $T_q\mathbf{R}^n$ are canonically isomorphic for every p, $q \in \mathbf{R}^n$. In the case of a general smooth manifold M, obviously there are no such canonical isomorphisms. Occasionally we shall denote by $(r_1, \ldots, r_n)$ the coordinates on $\mathbf{R}^n$ corresponding to id.

Tangent vectors as directional derivatives

Let M be a smooth manifold, and fix a point $p \in M$. For each tangent vector $v \in T_pM$ of the form $v = [a, \varphi]$, where $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M, and for each $f \in C^\infty(U)$, we define the *directional derivative of f in the direction of v* to be the real number

$$\begin{aligned} v(f) &= \frac{d}{dt}\Big|_{t=0}(f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= d(f \circ \varphi^{-1})(a). \end{aligned}$$

It is a simple consequence of the chain rule that this definition does not depend on the choice of representative of v.

In the case of $\mathbf{R}^n$, $\frac{\partial}{\partial r_i}\big|_p f$ is simply the partial derivative in the direction e_i, the ith vector in the canonical basis of $\mathbf{R}^n$. In general, if $\varphi = (x_1, \ldots, x_n)$, then $x_i \circ \varphi^{-1} = r_i$, so

$$v(x_i) = d(r_i)_{\varphi(p)}(a) = a_i,$$

where $a = \sum_{i=1}^n a_i e_i$. Since $v = [a, \varphi] = \sum_{i=1}^n a_i [e_i, \varphi]$, it follows that

$$v = \sum_{i=1}^n v(x_i) \frac{\partial}{\partial x_i}\Big|_p. \tag{1.3.4}$$

If v is a coordinate vector $\frac{\partial}{\partial x_i}$ and $f \in C^\infty(U)$, we also write

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial f}{\partial x_i}\Big|_p.$$

As a particular case of (1.3.4), take now v to be a coordinate vector of another local chart $(V, \psi = (y_1, \ldots, y_n))$ around p. Then

$$\frac{\partial}{\partial y_j}\Big|_p = \sum_{i=1}^n \frac{\partial x_i}{\partial y_j}\Big|_p \frac{\partial}{\partial x_i}\Big|_p.$$

Note that the preceding formula shows that even if $x_1 = y_1$ we do not need to have $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}$.

The differential

Let $f : M \to N$ be a smooth map between smooth manifolds. Fix a point $p \in M$, and local charts (U, φ) of M around p, and (V, ψ) of N around $q = f(p)$. The *differential* or *tangent map* of f at p is the linear map

$$df_p : T_pM \to T_qN$$

given by

$$[a, \varphi] \mapsto [d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(a), \psi].$$

It is easy to check that this definition does not depend on the choices of local charts. Using the identification (1.3.3), one checks that $d\varphi_p : T_pM \to \mathbf{R}^n$ and $d\psi_q : T_pM \to \mathbf{R}^n$ are linear isomorphisms and

$$df_p = (d\psi_q)^{-1} \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} \circ d\varphi_p.$$

1.3.5 Proposition (Chain rule) *Let M, N, P be smooth manifolds. If $f : M \to N$ and $g : N \to P$ are smooth maps, then $g \circ f : M \to P$ is a smooth map and*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

for $p \in M$.

1.3.6 Exercise Prove Proposition 1.3.5.

If $f \in C^\infty(M, N)$, $g \in C^\infty(N)$ and $v \in T_pM$, then it is a simple matter of unravelling the definitions to check that

$$df_p(v)(g) = v(g \circ f).$$

Now (1.3.4) together with this equation gives that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right) &= \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right)(y_i)\frac{\partial}{\partial y_i}\Big|_{f(p)} \\ &= \sum_{i=1}^n \frac{\partial(y_i \circ f)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_i}\Big|_{f(p)}. \end{aligned}$$

The matrix

$$\left(\frac{\partial(y_i \circ f)}{\partial x_j}\Big|_p\right)$$

is called the *Jacobian matrix of f at p* relative to the given coordinate systems. Observe that the chain rule (Proposition 1.3.5) is equivalent to saying that the Jacobian matrix of $g \circ f$ at a point is the product of the Jacobian matrices of g and f at the appropriate points.

Consider now the case in which $N = \mathbf{R}$ and $f \in C^\infty(M)$. Then $df_p : T_pM \to T_{f(p)}\mathbf{R}$, and upon the identification between $T_{f(p)}\mathbf{R}$ and $\mathbf{R}$, we easily see that $df_p(v) = v(f)$. Applying this to $f = x_i$, where $(U, \varphi = (x_1, \ldots, x_n))$ is a local chart around p, and using again (1.3.4) shows that

(1.3.7) $$\{dx_1|_p, \ldots, dx_n|_p\}$$

is the basis of T_pM^* dual of the basis (1.3.2), and hence

$$df_p = \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) dx_i|_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i|_p.$$

Finally, we discuss smooth curves on M. A *smooth curve* in M is simply a smooth map $\gamma : (a,b) \to M$ where (a,b) is an interval of $\mathbf{R}$. One can also consider smooth curves γ in M defined on a closed interval $[a,b]$. This simply means that γ admits a smooth extension to an open interval $(a-\epsilon, b+\epsilon)$ for some $\epsilon > 0$.

If $\gamma : (a,b) \to M$ is a smooth curve, the *tangent vector* to γ at $t \in (a,b)$ is

$$\dot\gamma(t) = d\gamma_t\left(\frac{\partial}{\partial r}\Big|_t\right) \in T_{\gamma(t)}M,$$

where r is the canonical coordinate of $\mathbf{R}$. Note that an arbitrary vector $v \in T_pM$ can be considered to be the tangent vector at 0 to the curve $\gamma(t) = \varphi^{-1}(ta)$, where (U,φ) is a local chart around p with $\varphi(p) = 0$ and $d\varphi_p(v) = a$.

In the case in which $M = \mathbf{R}^n$, upon identifying $T_{\gamma(t)}\mathbf{R}^n$ and $\mathbf{R}^n$, it is easily seen that

$$\dot\gamma(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

The inverse function theorem

It is now straightforward to state and prove the inverse function theorem for smooth manifolds.

1.3.8 Theorem (Inverse function theorem) *Let $f : M \to N$ be a smooth map between two smooth manifolds M, N, and let $p \in M$ and $q = f(p)$. If $df_p : T_pM \to T_qN$ is an isomorphism, then there exists an open neighborhood W of p such that $f(W)$ is an open neighborhood of q and f restricts to a diffeomorphism from W onto $f(W)$.*

Proof. The proof is really a transposition of the inverse function theorem for smooth mappings between Euclidean spaces to manifolds using local charts. Note that M and N have the same dimension, say, n. Take local charts (U,φ) of M around p and (V,ψ) of N around q such that $f(U) \subset V$. Set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. By the inverse function theorem for smooth mappings of $\mathbf{R}^n$, there exists an open subset $\tilde W \subset \varphi(U)$ with $\varphi(p) \in \tilde W$ such that $\alpha(\tilde W)$ is an open neighborhood of $\psi(q)$ and α restricts to a diffeomorphism from $\tilde W$ onto $\alpha(\tilde W)$. It follows that $f = \psi^{-1} \circ \alpha \circ \varphi$ is a diffeomorphism from the open neighborhood $W = \varphi^{-1}(\tilde W)$ of p onto the open neighborhood $\psi^{-1}(\alpha(\tilde W))$ of q. $\square$

A smooth map $f : M \to N$ satisfying the conclusion of Theorem 1.3.8 at a point $p \in M$ is called a *local diffeomorphism* at p. It follows from the above and the chain rule that f is a local diffeomorphism at p if and only if $df_p : T_pM \to T_qN$ is an isomorphism. In this case, there exist local charts

(U, φ) of M around p and (V, ψ) of N around $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ of f is the identity, owing to Problem 1.2.5, after enlarging the atlas of M, if necessary.

1.3.9 Exercise Let $f : M \to N$ be a smooth bijective map that is a local diffeomorphism everywhere. Show that f is a diffeomorphism.

1.4 Submanifolds of smooth manifolds

Similar to the situation of submanifolds of Euclidean spaces, some manifolds are contained in other manifolds in a natural way (compare Definition 1.1.1). Let N be a smooth manifold of dimension $n + k$. A subset M of N is called an *embedded submanifold* of N of dimension n if, for every $p \in M$, there exists a local chart (V, ψ) of N such that $p \in V$ and $\psi(V \cap M) = \psi(V) \cap \mathbf{R}^n$, where we identify $\mathbf{R}^n$ with $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^n \times \mathbf{R}^k = \mathbf{R}^{n+k}$. We say that (V, ψ) is a local chart of N *adapted* to M. An embedded submanifold M of N is a smooth manifold in its own right, with respect to the relative topology, in a canonical way. In fact an atlas of M is furnished by the restrictions to M of those local charts of N that are adapted to M. Namely, if $\{(V_\alpha, \psi_\alpha)\}$ is an atlas of N consisting of adapted charts, then $\{(V_\alpha \cap M, \psi_\alpha|_{V_\alpha \cap M})\}$ becomes an atlas of M. Note that the compatibility condition for the local charts of M follows automatically from the compatibility condition for N.

1.4.1 Exercise Let N be a smooth manifold and let M be an embedded submanifold of N. Prove that T_pM is canonically isomorphic to a subspace of T_pN for every $p \in M$.

Immersions and embeddings

Another class of submanifolds can be introduced as follows. Let $f : M \to N$ be a smooth map between smooth manifolds. The map f is called an *immersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is injective. If f is an immersion everywhere it is simply called an *immersion*. Now call the pair (M, f) an *immersed submanifold* or simply a *submanifold* of N if $f : M \to N$ is an injective immersion.

Let M be an embedded submanifold of N and consider the inclusion $\iota : M \to N$. The existence of adapted local charts implies that ι can be locally represented around any point of M by the standard inclusion $x \mapsto (x, 0)$, $\mathbf{R}^n \to \mathbf{R}^{n+k}$. Since this map is an immersion, also ι is an immersion. It follows that (M, ι) is an immersed submanifold of N. This shows that every embedded submanifold of a smooth manifold is an immersed submanifold, but the converse is not true.

1.4.2 Example Let N be the 2-torus $T^2 = S^1 \times S^1$ viewed as an embedded submanifold of $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$ and consider the smooth map

$$F : \mathbf{R} \to \mathbf{R}^4, \qquad F(t) = (\cos at, \sin at, \cos bt, \sin bt),$$

where a, b are non-zero real numbers. Note that the image of F lies in T^2. Denote by (r_1, r_2, r_3, r_4) the coordinates on $\mathbf{R}^4$. Choosing r_i, r_j where $i \in \{1, 2\}$ and $j \in \{3, 4\}$ gives a system of coordinates defined on an open subset of T^2, and in this way we obtain atlas for T^2. It follows that the induced map $f : \mathbf{R} \to T^2$ is smooth. Since N is an embedded submanifold of $\mathbf{R}^4$, we can consider $T_{f(t)}N$ to be a subspace of $\mathbf{R}^4$, and the tangent vector $f'(t) \in T_{f(t)}N$ is the usual derivative $F'(t)$. Since $f'(t)$ never vanishes, f is an immersion. Note that if b/a is an irrational number, then f is an injective map, so $(\mathbf{R}, f)$ is an immersed submanifold which we claim is *not* an embedded submanifold of T^2. In fact, the assumption on b/a implies that M is a dense subset of T^2, but an embedded submanifold of another manifold is always locally closed.

We would like to further investigate the gap between immersed submanifolds and embedded submanifolds.

1.4.3 Lemma (Local form of an immersion) *Let M and N be smooth manifolds of dimensions n and $n + k$, respectively, and suppose that $f : M \to N$ is an immersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, such that $f(U) \subset V$, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective, so, up to rearranging indices, we can assume that $d(\pi_1 \circ \alpha)_{\varphi(p)} = \pi_1 \circ d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\pi_1 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$ is the projection onto the first factor. By the inverse function theorem, by shrinking U, we can assume that $\pi_1 \circ \alpha$ is a diffeomorphism from $U_0 = \varphi(U)$ onto its image V_0; let $\beta : V_0 \to U_0$ be its smooth inverse. Now we can describe $\alpha(U_0)$ as being the graph of the smooth map $\gamma = \pi_2 \circ \alpha \circ \beta : V_0 \subset \mathbf{R}^n \to \mathbf{R}^k$, where $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ is the projection onto the second factor. By Exercise 1.1.3, $\alpha(U_0)$ is a submanifold of $\mathbf{R}^{n+k}$ and the map $\tau : V_0 \times \mathbf{R}^k \to V_0 \times \mathbf{R}^k$ given by $\tau(x, y) = (x, y - \gamma(x))$ is a diffeomorphism such that $\tau(\alpha(U_0)) = V_0 \times \{0\}$. Finally, we put $\tilde{\varphi} = \pi_1 \circ \alpha \circ \varphi$ and $\tilde{\psi} = \tau \circ \psi$, shrinking V if necessary. Then $(U, \tilde{\varphi})$ and $(V, \tilde{\psi})$ are local charts, and for $x \in \tilde{\varphi}(U) = V_0$ we have that

$$\begin{aligned} \tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}(x) &= \tau \circ \psi \circ f \circ \varphi^{-1} \circ \beta(x) = \tau \circ \alpha \circ \beta(x) \\ &= \tau(x, \gamma(x)) = (x, 0). \end{aligned}$$

□

1.4.4 Proposition *If $f : M \to N$ is an immersion at $p \in M$, then there exists an open neighborhood U of p in M such that $f|_U$ is injective and $f(U)$ is an embedded submanifold of N.*

Proof. The local injectivity of f at p is an immediate consequence of the fact that some local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$, hence, injective. Moreover, in the course of proof of Lemma 1.4.3, we have produced a local chart $(V, \tilde{\psi})$ of N adapted to $f(U)$. □

A smooth map $f : M \to N$ is called an *embedding* if it is an immersion and a homeomorphism from M onto $f(M)$ with the induced topology.

1.4.5 Proposition *Let N be a smooth manifold. A subset $P \subset N$ is an embedded submanifold of N if and only if it is the image of an embedding.*

Proof. Let $f : M \to N$ be an embedding with $P = f(M)$. To prove that P is an embedded submanifold of N, it suffices to check that it can be covered by open sets in the relative topology each of which is an embedded submanifold of N. Owing to Proposition 1.4.4, any point of P lies in a set of the form $f(U)$, where U is an open subset of M and $f(U)$ is an embedded submanifold of N. Since f is an open map into P with the relative topology, $f(U)$ is open in the relative topology and we are done. Conversely, if P is an embedded submanifold of N, it has the relative topology and thus the inclusion $\iota : P \to N$ is a homeomorphism onto its image. Moreover, we have seen above that ι is an immersion, whence it is an embedding. □

Recall that a continuous map between locally compact, Hausdorff topological spaces is called *proper* if the inverse image of a compact subset of the target space is a compact subset of the domain. It is known that proper maps are closed. Also, it is clear that if the domain is compact, then every continuous map is automatically proper. An embedded submanifold M of a smooth manifold N is called *properly embedded* if the inclusion map is proper.

1.4.6 Proposition *If $f : M \to N$ is an injective immersion which is also a proper map, then the image $f(M)$ is a properly embedded submanifold of N.*

Proof. Let $P = f(M)$ have the relative topology. A proper map is closed. Since f viewed as a map $M \to P$ is bijective and closed, it is an open map and thus a homeomorphism. Due to Proposition 1.4.5, P is an embedded submanifold of N. The properness of the inclusion $P \to N$ clearly follows from that of f. □

1.4.7 Exercise Give an example of an embedded submanifold of a smooth manifold which is not properly embedded.

1.4.8 Exercise Decide whether a closed embedded submanifold of a smooth manifold is necessarily properly embedded.

Exercise 1.1.4 dealt with a situation in which a smooth map $f : M \to N$ factors through an immersed submanifold (P, g) of N (namely, $f(M) \subset g(P)$) and the induced map $f_0 : M \to P$ (namely, $g \circ f_0 = f$) is discontinuous.

1.4.9 Proposition *Suppose that $f : M \to N$ is smooth and (P, g) is an immersed submanifold of N such that $f(M) \subset g(P)$. Consider the induced map $f_0 : M \to P$ that satisfies $g \circ f_0 = f$.*

a. If g is an embedding, then f_0 is continuous.

b. If f_0 is continuous, then it is smooth.

Proof. (a) In this case g is a homeomorphism onto $g(P)$ with the relative topology. If $V \subset P$ is open, then $g(V) = W \cap g(P)$ for some open subset $W \subset N$. By continuity of f, we have that $f_0^{-1}(V) = f_0^{-1}(g^{-1}(W)) = f^{-1}(W)$ is open in M, hence also f_0 is continuous.

(b) Let $p \in M$ and $q = f_0(p) \in P$. By Proposition 1.4.4, there exists a neighborhood U of q and a local chart (V, ψ) of N^n adapted to $g(U)$, with $g(U) \subset V$. In particular, there exists a projection π from $\mathbf{R}^n$ onto a subspace obtained by setting some coordinates equal to 0 such that $\tau = \pi \circ \psi \circ g$ is a local chart of P around q. Note that $f_0^{-1}(U)$ is a neighborhood of p in M. Now

$$\tau \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ g \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ f|_{f_0^{-1}(U)},$$

and the latter is smooth. □

An immersed submanifold (P, g) of N with the property that $f_0 : M \to P$ is smooth for every smooth map $f : M \to N$ with $f(M) \subset g(P)$ will be called an *initial submanifold*.

1.4.10 Exercise Use Exercise 1.3.9 and Propositions 1.4.5 and 1.4.9 to deduce that an embedding $f : M \to N$ induces a diffeomorphism from M onto a submanifold of N.

1.4.11 Exercise For an immersed submanifold (M, f) of N, show that there is a natural structure of smooth manifold on $f(M)$ and that $(f(M), \iota)$ is an immersed submanifold of N, where $\iota : f(M) \to N$ denotes the inclusion.

Submersions

A smooth map $f : M \to N$ is called a *submersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is surjective. If f is a submersion everywhere it is simply called a *submersion*. A point $q \in N$ is called a *regular value* of f if f is a submersion at all points in $f^{-1}(q)$; otherwise q is called a *singular value* of f.

1.4.12 Lemma (Local form of a submersion) *Let M an N be smooth manifolds of dimensions $n+k$ and k, respectively, and suppose that $f : M \to N$ is a submersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard projection of $\mathbf{R}^{n+k}$ onto $\mathbf{R}^k$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^{n+k} \to \mathbf{R}^k$ is surjective, so, up to rearranging indices, we can assume that $d(\alpha \circ \iota_2)_{\varphi(p)} = d\alpha_{\varphi(p)} \circ \iota_2 : \mathbf{R}^k \to \mathbf{R}^k$ is an isomorphism, where $\iota_2 : \mathbf{R}^k \to \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k$ is the standard inclusion. Define $\tilde{\alpha} : \varphi(U) \subset \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k$ by $\tilde{\alpha}(x, y) = (x, \alpha(x, y))$. Since $d\alpha_{\varphi(p)} \circ \iota_2$ is an isomorphism, it is clear that $d\tilde{\alpha}_{\varphi(p)} : \mathbf{R}^n \oplus \mathbf{R}^k \to \mathbf{R}^n \oplus \mathbf{R}^k$ is an isomorphism. By the inverse function theorem, there exists an open neighborhood U_0 of $\varphi(p)$ contained in $\varphi(U)$ such that $\tilde{\alpha}$ is a diffeomorphism from U_0 onto its image V_0; let $\tilde{\beta} : V_0 \to U_0$ be its smooth inverse. We put $\tilde{\varphi} = \tilde{\alpha} \circ \varphi$. Then $(\varphi^{-1}(U_0), \tilde{\varphi})$ is a local chart of M around p and

$$\begin{aligned} \psi \circ f \circ \tilde{\varphi}^{-1}(x, y) &= \psi \circ f \circ \varphi^{-1} \circ \tilde{\beta}(x, y) = \alpha \circ \tilde{\beta}(x, y) \\ &= \pi_2 \circ \tilde{\alpha} \circ \tilde{\beta}(x, y) = y. \end{aligned}$$

□

1.4.13 Proposition *Let $f : M \to N$ be a smooth map, and let $q \in N$ be a regular value of f such that $f^{-1}(q) \neq \varnothing$. Then $P = f^{-1}(q)$ is an embedded submanifold of M of dimension $\dim M - \dim N$. Moreover, for $p \in P$ we have $T_pP = \ker df_p$.*

Proof. It is enough to construct local charts of M that are adapted to P and whose domains cover P. So suppose $\dim M = n + k$, $\dim N = k$, let $p \in P$ and consider local charts $(W := \varphi^{-1}(U_0), \tilde{\varphi})$ and (V, ψ) as in Theorem 1.4.12 such that $p \in U$ and $q \in V$. We can assume that $\psi(q) = 0$. Now

$$\pi_2 \circ \tilde{\varphi}(W \cap P) = \alpha \circ \varphi(W \cap P) = \psi \circ f(W \cap P) = \{0\},$$

so $\tilde{\varphi}(W \cap P) = \tilde{\varphi}(W) \cap \mathbf{R}^n$ and thus φ is an adapted chart around p. Finally, the local representation of f at p is the projection $\mathbf{R}^{n+k} \to \mathbf{R}^k$. This is a linear map with kernel $\mathbf{R}^n$. It follows that $\ker df_p = (d\tilde{\varphi}^{-1})_{\varphi(p)}(\mathbf{R}^n) = T_pP$. □

1.4.14 Examples (a) Let A be a non-singular real symmetric matrix of order $n + 1$ and define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(p) = \langle Ap, p \rangle$ where $\langle , \rangle$ is the standard Euclidean inner product. Then $df_p : \mathbf{R}^{n+1} \to \mathbf{R}$ is given by $df_p(v) = 2\langle Ap, v \rangle$, so it is surjective if $p \neq 0$. It follows that f is a submersion on $\mathbf{R}^{n+1} \setminus \{0\}$, and then $f^{-1}(r)$ for $r \in \mathbf{R}$ is an embedded submanifold of $\mathbf{R}^{n+1}$ of dimension n if it is nonempty. In particular, by taking A to be the identity matrix we get a manifold structure for S^n which coincides with the one previously constructed.

(b) Denote by $Sym(n, \mathbf{R})$ the vector space of real symmetric matrices of order n, and define $f : M(n, \mathbf{R}) \to Sym(n, \mathbf{R})$ by $f(A) = AA^t$. This is map between vector spaces whose local representations components are quadratic polynomials. It follows that f is smooth and that df_A can be viewed as a map $M(n, \mathbf{R}) \to Sym(n, \mathbf{R})$ for all $A \in M(n, \mathbf{R})$. We claim that I is a regular value of f. For the purpose of checking that, we first compute for $A \in f^{-1}(I)$ and $B \in M(n, \mathbf{R})$ that

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \frac{(A+hB)(A+hB)^t - I}{h} \\ &= \lim_{h\to 0} \frac{h(AB^t + BA^t) + h^2 BB^t}{h} \\ &= AB^t + BA^t. \end{aligned}$$

Now given $C \in Sym(n, \mathbf{R})$, we have $df_A(\frac{1}{2}CA) = C$, and this proves that f is a submersion at A, as desired. Hence $f^{-1}(I) = \{ A \in ML(n, \mathbf{R}) \mid AA^t = I \}$ is an embedded submanifold of $M(n, \mathbf{R})$ of dimension

$$\dim M(n, \mathbf{R}) - \dim V = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Note that $f^{-1}(I)$ is a group with respect to the multiplication of matrices; it is called the *orthogonal group* of order n and is usually denoted by $O(n)$. It is obvious that $O(n) \subset GL(n, \mathbf{R})$.

We close this section by mentioning a generalization of Proposition 1.4.13. Let $f : M \to N$ be a smooth map and let Q be an embedded submanifold of N. We say that f is *transverse* to Q, in symbols $f \pitchfork Q$, if

$$df_p(T_pM) + T_{f(p)}Q = T_{f(p)}N$$

for every $p \in f^{-1}(Q)$.

1.4.15 Exercise Let $f : M \to N$ be a smooth map and let $q \in N$. Prove that $f \pitchfork \{q\}$ if and only if q is a regular value of f.

For an immersed submanifold (M, f) of a smooth manifold N, its *codimension* is the number $\dim N - \dim M$.

1.4.16 Proposition *If $f : M \to N$ is a smooth map which is transverse to an embedded submanifold Q of N of codimension k and $P = f^{-1}(Q)$ is non-empty, then P is an embedded submanifold of M of codimension k. Moreover $T_pP = (df_p)^{-1}(T_{f(p)}Q)$ for every $p \in P$.*

Proof. For the first assertion, it suffices to check that P is an embedded submanifold of M in a neighborhod of a point $p \in P$. Let (V, ψ) be a local chart of N adapted to Q around $q := f(p)$. Then $\psi : V \to \mathbf{R}^{n+k}$ and

$\psi(V \cap Q) = \psi(V) \cap \mathbf{R}^n$, where $n = \dim Q$. Let $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ be the standard projection and put $g = \pi_2 \circ \psi$. Then $g : V \to \mathbf{R}^k$ is a submersion and $g^{-1}(0) = V \cap Q$. Moreover

$$\begin{aligned} d(g \circ f)_p(T_pM) &= dg_q \circ df_p(T_pM) \\ &= dg_q(T_qN) \\ &= \mathbf{R}^k \end{aligned}$$

where, in view of $\ker dg_q = T_qQ$, the second equality follows from the assumption $f \pitchfork Q$. Now $h := g \circ f : f^{-1}(V) \to \mathbf{R}^k$ is a submersion at p and $h^{-1}(0) = f^{-1}(V \cap Q) = f^{-1}(V) \cap P$ and $f^{-1}(V)$ is an open neighborhood of p in M, so we can apply Proposition 1.4.13. All the assertions follow. □

As a most important special case, two embedded submanifolds M, P of N are called *transverse*, denoted $M \pitchfork P$, if the inclusion map $\iota : M \to N$ is transverse to P. It is easy to see that this is a symmetric relation.

1.4.17 Corollary *If M and P are transverse embedded submanifolds of N then $M \cap P$ is an embedded submanifold of N and*

$$\operatorname{codim}(M \cap P) = \operatorname{codim}(M) + \operatorname{codim}(P).$$

1.5 Partitions of unity

Many important constructions for smooth manifolds rely on the existence of smooth partitions of unity. This technique allows for a much greater flexibility of smooth manifolds as compared, for instance, with real analytic or complex manifolds.

Bump functions

We start with the remark that the function

$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is smooth everywhere. Therefore the function

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

is smooth, flat and equal to 0 on $(-\infty, 0]$, and flat and equal to 1 on $[1, +\infty)$. Finally,

$$h(t) = g(t+2)g(2-t)$$

is smooth, flat and equal to 1 on $[-1,1]$ and its support lies in $(-2,2)$; h is called a *bump function*. We can also make an n-dimensional version of a bump function by setting

$$k(x_1,\dots,x_n) = h(x_1)\cdots h(x_n),$$

and we can rescale k by precomposing with $x \mapsto r^{-1}x$ to have a smooth function on $\mathbf{R}^n$ which is flat and equal to 1 on a closed ball of radius r and with support contained in an open ball of radius $2r$.

Bump functions are very useful. As one application, note that for a given smooth manifold M so far we do not know whether the algebra $C^\infty(M)$ of smooth functions on M contains functions other than the constants (of course, the components of local charts are smooth, but these are not *globally* defined on M). We claim that $C^\infty(M)$ is indeed in general huge. In fact, let (U,φ) be a local chart of M and take a bump function $k : \mathbf{R}^n \to \mathbf{R}$ whose support lies in $\varphi(U)$. Then

$$f(x) := \begin{cases} k\circ\varphi(x) & \text{if} \in U, \\ 0 & \text{if } x \in M\setminus U \end{cases}$$

is a smooth function on M: this is clear for a point $p \in U$; if $p \notin U$, then we can find a neighborhood V of p which does not meet the compact set $\varphi^{-1}(\mathrm{supp}(k))$, so $f|_V = 0$ and thus f is smooth at p.

Partitions of unity

Let M be a smooth manifold. A *partition of unity* on M is a collection $\{\rho_i\}_{i\in I}$ of smooth functions on M, where I is an index set, such that:

(i) $\rho_i(p) \geq 0$ for all $p \in M$ and all $i \in I$;

(ii) the collection of supports $\{\mathrm{supp}(\rho)\}_{i\in I}$ is locally finite (i.e. every point of M admits a neighborhood meeting $\mathrm{supp}(\rho_i)$ for only finitely many indices i);

(iii) $\sum_{i\in I}\rho_i(p) = 1$ for all $p \in M$ (the sum is finite in view of (ii)).

Let $\{U_\alpha\}_{\alpha\in A}$ be a cover of M by open sets. We say that a partition of unity $\{\rho_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in A}$ if for every $i \in I$ there is some $\alpha \in A$ such that $\mathrm{supp}(\rho_i) \subset U_\alpha$; and we say $\{\rho_i\}_{i\in I}$ is *strictly subordinate* to $\{U_\alpha\}_{\alpha\in A}$ if $I = A$ and $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ for every $\alpha \in A$.

Partitions of unity are used to piece together global objects out of local ones, and conversely to decompose global objects as locally finite sums of locally defined ones. For instance, suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M and $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity strictly subordinate to $\{U_\alpha\}$. If we are given $f_\alpha \in C^\infty(U_\alpha)$ for all $\alpha \in A$, then $f = \sum_{\alpha\in A}\rho_\alpha f_\alpha$ is a smooth function on M. Indeed for $p \in M$ and $\alpha \in A$, it is true that either $p \in U_\alpha$ and then f_α is defined at p, or $p \notin U_\alpha$ and then $\rho_\alpha(p) = 0$. Moreover, since the sum is locally finite, f is locally the sum of finitely many smooth functions and hence smooth. Conversely, if we start with $f \in C^\infty(M)$ then $f = \sum_{\alpha \in A} f_\alpha$ for smooth functions f_α with $\mathrm{supp}(f_\alpha) \subset U_\alpha$, namely, $f_\alpha := \rho_\alpha f$.

1.5.1 Exercise Let C be closed in M and let U be open in M with $C \subset U$. Prove that there exists a smooth function $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda|_C = 1$ and $\mathrm{supp}\,\lambda \subset U$.

If M is compact, it is a lot easier to prove the existence of a partition of unity subordinate to any given open cover $\{U_\alpha\}$ of M. In fact for each $x \in U_\alpha$ we construct as above a bump function λ_x which is flat and equal to 1 on a neighborhood V_x of x and whose (compact) support lies in U_α. Owing to compactness of M, we can extract a finite subcover of $\{V_x\}$ and thus we get non-negative smooth functions $\lambda_i := \lambda_{x_i}$ for $i = 1, \ldots, n$ such that λ_i is 1 on V_{x_i}. In particular, their sum is positive, so

$$\rho_i := \frac{\lambda_i}{\sum_{i=1}^n \lambda_i}$$

for $i = 1, \ldots, n$ yields the desired partition of unity.

1.5.2 Theorem (Easy Whitney embedding theorem) *Let M be a compact smooth manifold. Then there exists an embedding of M into $\mathbf{R}^m$ for m suffciently big.*

Proof. Since M is compact, there exists an open covering $\{V_i\}_{i=1}^a$ such that for each i, $\bar{V}_i \subset U_i$ where (U_i, φ_i) is a local chart of M. For each i, we can find $\rho_i \in C^\infty(M)$ such that $0 \leq \rho_i \leq 1$, $\rho_i|_{\bar{V}_i} = 1$ and $\mathrm{supp}\,\rho_i \subset U_i$. Put

$$f_i(x) = \begin{cases} \rho_i(x)\varphi_i(x), & \text{if } x \in U_i, \\ 0, & \text{if } x \in M \setminus U_i. \end{cases}$$

Then $f_i : M \to \mathbf{R}^n$ is smooth, where $n = \dim M$. Define also smooth functions

$$g_i = (f_i, \rho_i) : M \to \mathbf{R}^{n+1} \quad \text{and} \quad g = (g_1, \ldots, g_a) : M \to \mathbf{R}^{a(n+1)}.$$

It is enough to check that g is an injective immersion. In fact, on the open set V_i, we have that $g_i = (\varphi_i, 1)$ is an immersion, so g is an immersion. Further, if $g(x) = g(y)$ for x, $y \in M$, then $\rho_i(x) = \rho_i(y)$ and $f_i(x) = f_i(y)$ for all i. Take an index j such that $\rho_j(x) = \rho_j(y) \neq 0$. Then x, $y \in U_j$ and $\varphi_j(x) = \varphi_j(y)$. Due to the injectivity of φ_j, we must have $x = y$. Hence g is injective. □

1.5.3 Remark In the noncompact case, one can still construct partitions of unity and modify the proof of Theorem 1.5.2 to prove that M properly embedds into $\mathbf{R}^m$ for some m. Then a standard trick involving Sard's theorem and projections into lower dimensional subspaces of $\mathbf{R}^m$ allows to find the bound $m \leq 2n + 1$, where $n = \dim M$. A more difficult result, the *strong Whitney embedding theorem* asserts that in fact $m \leq 2n$.

In general, a reasonable substitute for compactness is paracompactness. A topological space is called *paracompact* if every open covering admits an open locally finite refinement. It turns out that every locally compact, second countable, Hausdorff space is paracompact. Hence manifolds are paracompact. Now the above argument can be extended to give the following theorem, for whose proof we refer the reader to [War83].

1.5.4 Theorem (Existence of partitions of unity) *Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then there exists a countable partition of unity $\{\rho_i : i = 1,\ 2,\ 3, \ldots\}$ subordinate to $\{U_\alpha\}$ with $\mathrm{supp}(\rho_i)$ compact for each i. If one does not require compact supports, then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ strictly subordinate to $\{U_\alpha\}$ with at most countably many of the ρ_α not zero.*

1.6 Vector fields

Let M be a smooth manifold. A *vector field* on M is an assigment of a tangent vector $X(p)$ in T_pM for all $p \in M$. Sometimes, we also write X_p instead of $X(p)$. So a vector field is a map $X : M \to TM$ where $TM = \dot{\cup}_{p\in M} T_pM$ (disjoint union), and

$$\pi \circ X = \mathrm{id} \tag{1.6.1}$$

where $\pi : TM \to M$ is the natural projection $\pi(v) = p$ if $v \in T_pM$. In account of property (1.6.1), we say that X is a *section* of TM.

We shall need to talk about continuity and differentiability of vector fields, so we next explain that TM carries a canonical manifold structure induced from that of M.

The tangent bundle

Let M be a smooth manifold and consider the disjoint union

$$TM = \dot{\bigcup}_{p\in M} T_pM.$$

We can view the elements of TM as equivalence classes of triples (p, a, φ), where $p \in M$, $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M such that $p \in U$, and

$$(p, a, \varphi) \sim (q, b, \psi) \quad \text{if and only if } p = q \text{ and } d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

There is a natural projection $\pi : TM \to M$ given by $\pi[p, a, \varphi] = p$, and then $\pi^{-1}(p) = T_pM$.

Suppose $\dim M = n$. Note that we have n degrees of freedom for a point p in M and n degrees of freedom for a vector $v \in T_pM$, so we expect

TM to be $2n$-dimensional. We will use Proposition 1.2.10 to simultaneously introduce a topology and smooth structure on TM. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for M with countably many elements (recall that every second countable space is Lindelöf). For each α, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a diffeomorphism and, for each $p \in U_\alpha$, $d(\varphi_\alpha)_p : T_pU_\alpha = T_pM \to \mathbf{R}^n$ is the isomorphism mapping $[p, a, \varphi]$ to a. Set

$$\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbf{R}^n, \qquad [p, a, \varphi] \to (\varphi_\alpha(p), a).$$

Then $\tilde{\varphi}_\alpha$ is a bijection and $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbf{R}^{2n}$. Moreover, the maps

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

are defined on open subsets of $\mathbf{R}^{2n}$ and are given by

$$(x, a) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(x)\,,\, d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)).$$

Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth diffeomorphism, we have that $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x$ is a linear isomorphism and $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)$ is also smooth on x. It follows that $\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}$ defines a topology and a smooth atlas for M and we need only to check the Hausdorff condition. Namely, let v, $w \in TM$ with $v \neq w$. Note that π is an open map. If v, $w \in TM$ and $\pi(v) \neq \pi(w)$, we can use the Hausdorff property of M to separate v and w from each other with open sets of TM. On the other hand, if v, $w \in T_pM$, they lie in the domain of the same local chart of TM and the result also follows.

Note that, in particular, we have shown that every system of local coordinates $(x_1, \ldots, x_n)$ on an open subset U of M induces a system of local coordinates $(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ on $TM|_U$.

If $f \in C^\infty(M, N)$, then we define the *differential of* f to be the map

$$df : TM \to TN$$

that restricts to $df_p : T_pM \to T_{f(p)}N$ for each $p \in M$. Using the above atlases for TM and TN, we immediately see that $df \in C^\infty(TM, TN)$.

1.6.2 Remark The mapping that associates to each manifold M its tangent bundle TM and associates to each smooth map $f : M \to N$ its tangent map $df : TM \to TN$ can be thought of a functor $\mathbf{DIFF} \to \mathbf{VB}$ from the category of smooth manifolds to the category of smooth vector bundles. In fact, $d(\mathrm{id}_M) = \mathrm{id}_{TM}$, and $d(g \circ f) = dg \circ df$ for a sequence of smooth maps $M \xrightarrow{f} N \xrightarrow{g} P$.

Smooth vector fields

A vector field X on M is called *smooth* (resp. *continuous*) if the map $X : M \to TM$ is smooth (resp. continuous).

More generally, let $f : M \to N$ be a smooth mapping. Then a (smooth, continuous) *vector field along* f is a (smooth, continuous) map $X : M \to TN$ such that $X(p) \in T_{f(p)}N$ for $p \in M$. The most important case is that in which f is a smooth curve $\gamma : [a, b] \to N$. A vector field along γ is a map $X : [a, b] \to TN$ such that $X(t) \in T_{\gamma(t)}N$ for $t \in [a, b]$. A typical example is the tangent vector field $\dot{\gamma}$.

For practical purposes, we reformulate the notion of smoothness as follows. Let X be a vector field on M. Given a smooth function $f \in C^\infty(U)$ where U is an open subset of M, the directional derivative $X(f) : U \to \mathbf{R}$ is defined to be the function $p \in U \mapsto X_p(f)$. Further, if $(x_1, \ldots, x_n)$ is a coordinate system on U, we have already seen that $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of T_pM for $p \in U$. It follows that there are functions $a_i : U \to \mathbf{R}$ such that

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}. \tag{1.6.3}$$

1.6.4 Proposition *Let X be a vector field on M. Then the following assertions are equivalent:*

a. X is smooth.

b. For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, the functions a_i defined by (1.6.3) are smooth.

c. For every open set V of M and $f \in C^\infty(V)$, the function $X(f) \in C^\infty(V)$.

Proof. Suppose X is smooth and let $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ be a coordinate system on U. Then $X|_U$ is smooth and $a_i = dx_i \circ X|_U$ is also smooth.

Next, assume (b) and let $f \in C^\infty(V)$. Take a coordinate system

$$(U, (x_1, \ldots, x_n))$$

with $U \subset V$. Then, by using (b) and the fact that $\frac{\partial f}{\partial x_i}$ is smooth,

$$X(f)|_U = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} \in C^\infty(U).$$

Since V can be covered by such U, this proves (c).

Finally, assume (c). For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, we have a corresponding coordinate system $(\pi^{-1}(U), x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$ of TM. Then

$$(x_i \circ \pi) \circ X|_U = x_i \quad \text{and} \quad dx_i \circ X|_U = X(x_i)$$

are smooth. This proves that X is smooth. □

In particular, the proposition shows that the coordinate vector fields $\frac{\partial}{\partial x_i}$ associated to a local chart are smooth. Since $a_i = X(x_i)$ in (1.6.3), we have

1.6.5 Scholium *If X is a smooth vector field on M and $X(f) = 0$ for every smooth function, then $X = 0$.*

1.6.6 Remark Part (c) of Proposition 1.6.4 in fact says that every smooth vector field on M defines a *derivation* of the algebra $C^\infty(M)$, namely, a differential operator that maps constants to zero and satisfies the Leibniz identity $X(fg) = X(f)g + fX(g)$.

Flow of a vector field

We have now come to the integration of vector fields. Let $\varphi_t : M \to M$ be a diffeomorphism such that the curve $t \mapsto \varphi_t(p)$ is smooth for each p. Then $X_p := \frac{d}{dt}\big|_{t=0}\varphi_t(p)$ defines a vector field on M. Conversely, one can integrate smooth vector fields to obtain (local) diffeomorphisms. Actually, this is the extension of ODE theory to smooth manifolds that we discuss below.

An *integral curve* of X is a smooth curve $\gamma : I \to M$, where I is an open interval, such that

$$\dot\gamma(t) = X(\gamma(t))$$

for all $t \in I$. We write this equation in local coordinates. Suppose X has the form (1.6.3), $\gamma_i = x_i \circ \gamma$ and $\tilde a_i = a_i \circ \varphi^{-1}$. Then γ is an integral curve of X in $\gamma^{-1}(U)$ if and only if

$$\frac{d\gamma_i}{dr}\Big|_t = \tilde a_i(\gamma_1(t), \ldots, \gamma_n(t)) \tag{1.6.7}$$

for $i = 1, \ldots, n$ and $t \in \gamma^{-1}(U)$. Equation (1.6.7) is a system of first order ordinary differential equations for which existence and uniqueness theorems are known. These, translated into manifold terminology yield local existence and uniqueness of integral curves for smooth vector fields. Moreover, one can cover M by domains of local charts and, using uniqueness, piece together the locally defined integral curves of X to obtain, for any given point $p \in M$, a *maximal* integral curve γ_p of X through p defined on a possibly infinite interval $(a(p), b(p))$.

Even more interesting is to reverse the rôles of p and t by setting

$$\varphi_t(p) := \gamma_p(t)$$

for all p such that $t \in (a(p), b(p))$. The smooth dependence of solutions of ODE on the initial conditions implies that for every $p \in M$, there exists an open neighborhood V of p and $\epsilon > 0$ such that the map

$$(-\epsilon, \epsilon) \times V \to M, \quad (t, q) \mapsto \varphi_t(q) \tag{1.6.8}$$

is well defined and smooth. The same theorem also shows that, for fixed $t > 0$, the domain of φ_t is an open subset $\mathcal{D}_t$ of M.

The uniqueness of solutions of ODE with given initial conditions implies that

$$\varphi_{s+t} = \varphi_s \circ \varphi_t \tag{1.6.9}$$

whenever both hand sides are defined. In fact, for each t, the curve $s \mapsto \varphi_{s+t}(p)$ is an integral curve of X passing through the point $\varphi_t(p)$ at $s = 0$, so it must locally coincide with $\varphi_s(\varphi_t(p))$.

Obviously φ_0 is the identity, so φ_t is a diffeomorphism $\mathcal{D}_t \to \mathcal{D}_{-t}$ with inverse φ_{-t}. The collection $\{\varphi_t\}$ is called the *flow* of X. Owing to property (1.6.9), the flow of X is also called the *one-parameter local group* of locally defined diffeomorphisms generated by X, and X is called the *infinitesimal generator* of $\{\varphi_t\}$. If φ_t is defined for all $t \in \mathbf{R}$, the vector field X is called *complete*. This is equivalent to requiring that the maximal integral curves of X be defined on the entire $\mathbf{R}$, or yet, that the domain of each φ_t be M. In this case we refer to $\{\varphi_t\}$ as the *one-parameter group* of diffeomorphisms of M generated by X.

1.6.10 Proposition *Every smooth vector field X defined on a compact smooth manifold M is complete.*

Proof. If M is compact, we can find a finite open covering $\{V_i\}$ of it and $\epsilon_i > 0$ such that $(-\epsilon_i, \epsilon_i) \times V_i \to M$, $(t, p) \mapsto \varphi_t(p)$ is well defined and smooth for all i, as in (1.6.8). Let $\epsilon = \min_i\{\epsilon_i\}$. Now this map is defined on $(-\epsilon, \epsilon) \times M \to M$. This means that any integral curve of X starting at any point of M is defined at least on the interval $(-\epsilon, \epsilon)$. The argument using the uniqueness of solutions of ODE as in (1.6.9) and piecing together integral curves of X shows that any integral curve of X is defined on $(-k\epsilon, k\epsilon)$ for all positive integer k, hence it is defined on $\mathbf{R}$. □

1.6.11 Examples (a) Take $M = \mathbf{R}^2$ and $X = \frac{\partial}{\partial x_1}$. Then X is complete and $\varphi_t(x_1, x_2) = (x_1 + t, x_2)$ for $(x_1, x_2) \in \mathbf{R}^2$. Note that if we replace $\mathbf{R}^2$ by the punctured plane $\mathbf{R}^2 \setminus \{(0, 0)\}$, the domains of φ_t become proper subsets of M.

(b) Consider the smooth vector field on $\mathbf{R}^{2n}$ defined by

$$X(x_1, \ldots, x_{2n}) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \cdots - x_{2n}\frac{\partial}{\partial x_{2n-1}} + x_{2n-1}\frac{\partial}{\partial x_{2n}}.$$

The flow of X is given the linear map

$$\varphi_t\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} R_t & & \\ & \ddots & \\ & & R_t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix}$$

where R_t is the 2×2 block

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

It is clear that X restricts to a smooth vector field $\bar{X}$ on S^{2n-1}. The flow of $\bar{X}$ is of course the restriction of φ_t to S^{2n-1}. X and $\bar{X}$ are complete vector fields.

(c) Take $M = \mathbf{R}$ and $X(x) = x^2 \frac{\partial}{\partial x}$. Solving the ODE we find $\varphi_t(x) = \frac{x}{1-tx}$. It follows that the domain of φ_t is $(-\infty, \frac{1}{t})$ if $t > 0$ and $(\frac{1}{t}, +\infty)$ if $t < 0$.

Lie bracket

If X is a smooth vector field on M and $f : M \to \mathbf{R}$ is a smooth function, the directional derivative $X(f) : M \to \mathbf{R}$ is also smooth and so it makes sense to derivate it again as in $Y(X(f))$ where Y is another smooth vector field on M. For instance, in a local chart $(U, \varphi = (x_1, \ldots, x_n))$, we have the first order partial derivative

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial f}{\partial x_i}\Big|_p$$

and the second order partial derivative

$$\left(\frac{\partial}{\partial x_j}\right)_p \left(\frac{\partial}{\partial x_i}(f)\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}\Big|_p$$

and it follows from Schwarz theorem on the commutativity of mixed partial derivatives of smooth functions on $\mathbf{R}^n$ that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}\Big|_p = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial r_j \partial r_i}\Big|_p = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial r_i \partial r_j}\Big|_p = \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p, \tag{1.6.12}$$

where $\mathrm{id} = (r_1, \ldots, r_n)$ denote the canonical coordinates on $\mathbf{R}^n$.

On the other hand, for general smooth vector fields X, Y on M the second derivative depends on the order of the vector fields and the failure of the commutativity is measured by the *commutator* or *Lie bracket*

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{1.6.13}$$

for every smooth function $f : M \to \mathbf{R}$. We say that X, Y *commute* if $[X, Y] = 0$. It turns out that formula (1.6.13) defines a smooth vector field on M! Indeed, Scholium 1.6.5 says that such a vector field is unique, if it exists. In order to prove existence, consider a coordinate system $(U, (x_1, \ldots, x_n))$. Then we can write

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y|_U = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$$

for $a_i, b_j \in C^\infty(U)$. If $[X,Y]$ exists, we must have

$$[X,Y]|_U = \sum_{i,j=1}^{n} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}, \tag{1.6.14}$$

because the coefficients of $[X,Y]|_U$ in the local frame $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ must be given by $[X,Y](x_j) = X(Y(x_j)) - Y(X(x_j))$. We can use formula (1.6.14) as the definition of a vector field on U; note that such a vector field is smooth and satisfies property (1.6.13) for functions in $C^\infty(U)$. We finally define $[X,Y]$ globally by covering M with domains of local charts: on the overlap of two charts, the different definitions coming from the two charts must agree by the above uniqueness result; it follows that $[X,Y]$ is well defined.

1.6.15 Examples (a) Schwarz theorem (1.6.12) now means $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for coordinate vector fields associated to a local chart.

(b) Let $X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial z}$ be smooth vector fields on $\mathbf{R}^3$. Then $[X,Y] = Z$, $[Z,X] = [Z,Y] = 0$.

1.6.16 Proposition *Let X, Y and Z be smooth vector fields on M. Then*

a. $[Y,X] = -[X,Y]$.

b. If $f, g \in C^\infty(M)$, then

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

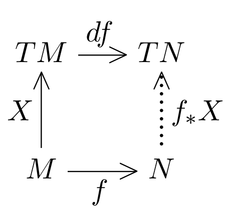
c. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. *(Jacobi identity)*

1.6.17 Exercise Prove Proposition 1.6.16. (Hint: Use (1.6.13).)

Let $f: M \to N$ be a diffeomorphism. For every smooth vector field X on M, the formula $df \circ X \circ f^{-1}$ defines a smooth vector field on N, called the *push-forward* of X under f, which we denote by f_*X. If the flow of X is $\{\varphi_t\}$, then the flow of f_*X is $f \circ \varphi_t \circ f^{-1}$, as

$$\frac{d}{dt} f(\varphi_t(f^{-1}(p))) = df\left(\frac{d}{dt}\varphi_t(f^{-1}(p))\right) = df(X_{f^{-1}(p)}).$$

More generally, if $f: M \to N$ is a smooth map which needs not be a diffeomorphism, smooth vector fields X on M and Y on N are called *f-related* if $df \circ X = Y \circ f$.



1.6.18 Proposition *Let $f : M \to M'$ be smooth. Let X, Y be smooth vector fields on M, and let X', Y' be smooth vector fields on M'. If X and X' are f-related and Y and Y' are f-related, then also $[X, Y]$ and $[X', Y']$ are f-related.*

Proof. Let $h \in C^\infty(M')$ and $q \in M$. Note first that

$$\begin{aligned} X_q(h \circ f) &= d(h \circ f)(X_q) \\ &= dh(df(X_q)) \\ &= (df \circ X)_q(h) \\ &= X'_{f(q)}(h), \end{aligned}$$

namely,

$$X(h \circ f) = X'(h) \circ f. \tag{1.6.19}$$

Similarly, $Y(h \circ f) = Y'(h) \circ f$.

We now prove $df \circ [X, Y] = [X', Y'] \circ f$. Let $g \in C^\infty(M')$ and $p \in M$. Use (1.6.13) and the above identities:

$$\begin{aligned} df([X, Y]_p)(g) &= [X, Y]_p(g \circ f) \\ &= X_p(Y(g \circ f)) - Y_p(X(g \circ f)) \\ &= X_p(Y'(g) \circ f) - Y_p(X'(g) \circ f) \\ &= X'_{f(p)}(Y'(g)) - Y'_{f(p)}(X'(g)) \\ &= [X', Y']_{f(p)}(g), \end{aligned}$$

as we wished. □

What is the relation between flows and Lie brackets? In order to discuss that, let X, Y be smooth vector fields on M. Denote the flow of X by $\{\varphi_t\}$ and let f be a smooth function on M. Then

$$\frac{d}{dt} f(\varphi_t) = X(f),$$

and

$$((\varphi_{-t})_* Y)(f \circ \varphi_t) = Y(f) \circ \varphi_t \tag{1.6.20}$$

as $(\varphi_{-t})_* Y$ and Y are φ_t-related (cf. (1.6.19)).

1.6.21 Exercise Let Z_t be a smooth curve in T_pM and let $h_t(x) = H(t, x)$, where $H \in C^\infty(\mathbf{R} \times M)$. Prove that

$$\left.\frac{d}{dt}\right|_{t=0} Z_t(h_t) = \left(\left.\frac{d}{dt}\right|_{t=0} Z_t\right)(h_0) + Z_0 \left(\left.\frac{d}{dt}\right|_{t=0} h_t\right).$$

(Hint: Here $\frac{d}{dt}|_{t=0} h_t(x)$ means $\frac{\partial H}{\partial t}(0, x)$. Consider $\Gamma \in C^\infty(\mathbf{R} \times \mathbf{R})$ such that $\Gamma(t, 0) = p$ and $\frac{\partial}{\partial s}|_{s=0} \Gamma(t, s) = Z_t$ for all $t \in \mathbf{R}$, and use the chain rule.)

Differentiate identity (1.6.20) at $t=0$ to get

$$\frac{d}{dt}\Big|_{t=0}((\varphi_{-t})_*Y)(f) + Y(X(f)) = X(Y(f)).$$

Note that $t \mapsto ((\varphi_{-t})_*Y)_p$ is a smooth curve in T_pM. Its tangent vector at $t=0$ is called the *Lie derivative* of Y with respect to X at p, denoted by $(L_XY)_p$, and this defines the Lie derivative L_XY as a smooth vector field on M. The above calculation shows that

$$L_XY = [X,Y]. \tag{1.6.22}$$

1.6.23 Proposition *X and Y commute if and only if their corresponding flows $\{\varphi_t\}$, $\{\psi_s\}$ commute.*

Proof. $[X,Y]=0$ if and only if $0 = \frac{d}{dt}\big|_{t=0}(\varphi_{-t})_*Y$. Since $\{\varphi_t\}$ is a one-parameter group,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0}(\varphi_{-t})_*Y &= \frac{d}{dh}\Big|_{h=0}(\varphi_{-(t_0+h)})_*Y \\ &= d(\varphi_{-t_0})\left(\frac{d}{dh}\Big|_{h=0}(\varphi_{-h})_*Y\circ\varphi_{t_0}\right),\end{aligned}$$

this is equivalent to $(\varphi_{-t})_*Y = Y$ for all t. However the flow of $(\varphi_{-t})_*Y$ is $\{\varphi_{-t}\psi_s\varphi_t\}$, so this means $\varphi_{-t}\psi_s\varphi_t = \psi_s$. □

We know that, for a local chart (U,φ), the set of coordinate vector fields $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$ is linearly independent at every point of U and the $\frac{\partial}{\partial x_i}$ pairwise commute. It turns out these two conditions locally characterize coordinate vector fields. Namely, we call a set $\{X_1,\ldots,X_k\}$ of smooth vector fields defined on an open set V of M a *local k-frame* if it is linearly independent at every point of V; if $k=\dim M$, we simply say *local frame*.

1.6.24 Proposition *Let $\{X_1,\ldots,X_k\}$ be a local k-frame on V such that $[X_i,X_j]=0$ for all i, $j=1,\ldots,k$. Then for every $p\in V$ there exists an open neighborhood U of p in V and a local chart (U,φ) whose first k coordinate vector fields are exactly the X_i.*

Proof. Complete $\{X_1,\ldots,X_k\}$ to a local frame $\{X_1,\ldots,X_n\}$ in smaller neighborhood $\tilde{V}\subset V$ of p. (One can do that by first completing

$$\{X_1(p),\ldots,X_k(p)\}$$

to a basis

$$\{X_1(p),\ldots,X_k(p),v_{k+1},\ldots,v_n\}$$

of T_pM and then declaring $X_{k+1},\ldots,X_n$ to be the vector fields defined on the domain of a system of local coordinates $(W,y_1,\ldots,y_n)$ around p, $W\subset$

V, with constant coefficients in $\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ that extend $v_{k+1}, \ldots, v_n$. By continuity, $\{X_1, \ldots, X_k\}$ will be a local frame in a neighborhood $\tilde{V} \subset W$ of p.) Let $\{\varphi_t^i\}$ be the flow of X_i and put $F(t_1, \ldots, t_n) := \varphi_{t_1}^1 \circ \cdots \circ \varphi_{t_n}^n(p)$, smooth map defined on a neighborhood of 0 in $\mathbf{R}^n$. Then $dF_0(e_i) = X_i(p)$ for all i, so F is a local diffeomorphism at 0 by the inverse function theorem. The local inverse F^{-1} defines a local chart $(U, x_1, \ldots, x_n)$ around p. Finally, for $q = F(t_1, \ldots, t_n)$,

$$\begin{aligned} \frac{\partial}{\partial x_i}\Big|_q &= dF_{F^{-1}(q)}(e_i) \\ &= \frac{d}{dh}\Big|_{t=0} \varphi_{t_i+h}^i \varphi_{t_1}^1 \cdots \hat{\varphi_{t_i}^i} \cdots \varphi_{t_n}^n(p) \\ &= X_i\left(\varphi_{t_i}^i \varphi_{t_1}^1 \cdots \hat{\varphi_{t_i}^i} \cdots \varphi_{t_n}^n(p)\right) \\ &= X_i\left(\varphi_{t_1}^1 \cdots \varphi_{t_n}^n(p)\right) \\ &= X_i(q), \end{aligned}$$

where we have used Proposition 1.6.23 twice. □

1.7 Distributions and foliations

We seek to generalize the theory of the previous section to higher dimensions, so let us rephrase it in the following terms. Let X be a smooth vector field on M *which is nowhere zero*. On one hand, the $\mathbf{R}$-span of X_p defines a family $\mathcal{D}$ of one-dimensional subspaces $\mathcal{D}_p$ of T_pM for each $p \in M$. On the other hand, the maximal integral curves of X define a partition $\mathcal{F}$ of M into regular parametrized curves, or 1-dimensional immersed submanifolds of M. The relation between $\mathcal{D}$ and $\mathcal{F}$ is that $T_pL = \mathcal{D}_p$ for every $L \in \mathcal{F}$ and every $p \in L$.

In view of the above, we give the following definition. Suppose $\dim M = n$. A *rank k (smooth) distribution* $\mathcal{D}$ on M, $0 \leq k \leq n$, is an assignment of a k-dimensional subspace $\mathcal{D}_p$ of T_pM to each $p \in M$, where any $p \in M$ admits an open neighborhood U with the property that there exist smooth vector fields $X_1, \ldots, X_k$ on U such that the span of $X_1(q), \ldots, X_k(q)$ coincides with $\mathcal{D}_q$ for all $q \in U$.

Before continuing, we recall a consequence of Proposition 1.6.24, namely, that the flow of a non-vanishing vector field can be locally "rectified" in the following sense.

1.7.1 Proposition *Let X be a smooth vector field on M such that $X_p \neq 0$ for some $p \in M$. Then there exists a system of local coordinates $(U, (x_1, \ldots, x_n))$ around p such that $X|_U = \frac{\partial}{\partial x_1}$. Equivalently, the integral curves of X in U are of the form $x_2 = c_2, \ldots, x_n = c_n$ for some $c_2, \ldots, c_n \in \mathbf{R}$.*

Based on Proposition 1.7.1, we make the following definition. A k-*dimensional foliation* of M, $0 \leq k \leq n$, is a partition $\mathcal{F}$ of M into piecewise smooth arc-connected subsets, where any $p \in M$ admits a coordinate neighborhood $(U, (x_1, \ldots, x_n))$ such that U is connected and, for every $L \in \mathcal{F}$, the piecewise smooth arc-connected components of $L \cap U$ are coincide with the "slices"

$$x_{k+1} = c_{k+1}, \ \ldots, \ x_n = c_n$$

for some $c_{k+1}, \ldots, c_n \in \mathbf{R}$. The elements of $\mathcal{F}$ are called *leaves*. A coordinate system $(U, (x_1, \ldots, x_n))$ as above will be called *distinguished*. If $L \in \mathcal{F}$, the piecewise smooth arc-components of $L \cap U$ are called *plaques*.

1.7.2 Examples (i) The levels sets of a submersion $M \to N$ form a foliation of rank $\dim M - \dim N$, by the local form of a submersion, where the leaves are embedded submanifolds. Indeed, this is the local model of a general foliation, by definition.

(ii) Recall the skew-line in the torus in Example 1.4.2. The traces of the immersions

$$F_s : \mathbf{R} \to \mathbf{R}^4, \qquad F(t) = (\cos at, \sin at, \cos(bt + 2\pi s), \sin(bt + 2\pi s)),$$

where a, b are non-zero real numbers, for $s \in [0, 1]$, form a foliation of rank 1 of T^2. If b/a is an irrational number, the leaves are dense in T^2.

Each leaf $L \in \mathcal{F}$ has a canonical structure of immersed submanifold of M of dimension k. In fact, we can use Proposition 1.2.10. For any distinguished chart (U, φ), $\varphi|_P$ is a bijective map from a plaque (arc component) P of $L \cap U$ onto an open subset of $\mathbf{R}^k$. In this way, if we start with a countable collection $\{(U_m, \varphi_m)\}_{m \in \mathbb{N}}$ of distinguished charts of M whose domains cover L, we construct a collection $\{(P_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, where P_α is a plaque of $L \cap U_m$ for some m and φ_α is the restriction of φ_m to P_α. It is clear that this collection satisfies conditions (a), (b) and (c) of Proposition 1.2.10, but it remains to be checked that the index set A is countable. For that purpose, it suffices to see that $U_m \cap L$ has countably many arc components, for every m. Fix a plaque P_0 of L in $\{U_m\}$. Since L is arc connected, for any other plaque P there exists a sequence $P_1, \ldots, P_\ell = P$ of plaques such that $P_{i-1} \cap P_i \neq \varnothing$ for all $i = 1, \ldots, \ell$. So any plaque of L in $\{U_m\}$ can be reached by a finite path of plaques that originates at P_0. It suffices to show that the collection of such paths is countable. In order to do that, it is enough to prove that a given plaque P' of L in $\{U_m\}$ can meet only countably many other plaques of L in $\{U_m\}$. For any m, $P' \cap (L \cap U_m) = P' \cap U_m$ is an open subset of the locally Euclidean space P' and thus has countably many components, each such component being contained in a plaque of $L \cap U_m$. It follows that P' can meet at most countably many components of $L \cap U_m$, as we wished.

In this way, we have a structure of smooth manifold of L such that each plaque of L is an open submanifold of L. The underlying topology in L can be much finer than the induced topology. In any case, the Hausdorff condition follows because the inclusion map $L \to M$ is continuous and M is Hausdorff. In addition (recall Proposition 1.4.9):

1.7.3 Proposition *Every leaf L of a foliation of N is an initial submanifold.*

Proof. Let $f : M \to N$ be a smooth map such that $f(M) \subset L$ and consider the induced map $f_0 : M \to L$ such that $\iota \circ f_0 = f$, where $\iota : L \to N$ is the inclusion. We need to show that f_0 is continuous. We will prove that $f_0^{-1}(U)$ is open in M for any given open subset U of L. We may assume $f_0^{-1}(U) \neq \varnothing$, so let $p \in f_0^{-1}(U)$ and $q = f_0(p) \in U$. It suffices to show that p is an interior point of $f_0^{-1}(U)$. Let $(V, y_1, \ldots, y_n)$ be a distinguished chart of N around q, so that the plaques of L in V are of the form

$$y_i = \text{constant} \qquad \text{for } i = k+1, \ldots, n \tag{1.7.4}$$

and the plaque containing q is

$$y_{k+1} = \cdots = y_n = 0 \tag{1.7.5}$$

By shrinking V, we may assume that (1.7.5) is an open set $\tilde{U} \subset U$. Note that $f^{-1}(V)$ an open neighborhood of p in M; let W be its connected component containing p. Of course, W is open. It is enough to show that $f_0(W) \subset \tilde{U}$, or what amounts to the same, $f(W)$ is contained in (1.7.5). Since $f(W)$ is connected, it is contained in a plaque of of $V \cap L$; since $f(W)$ meets q, it must be (1.7.5). □

The Frobenius theorem

Let M be a smooth manifold. It is clear that every foliation of M gives rise to a distribution simply by taking the tangent spaces to the leaves at each point; locally, for a distinguished chart $(U, (x_1, \ldots, x_n))$, the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ span the distribution on U. What about the converse? If we start with a distribution, can we produce an "integral" foliation? Well, in case $k = 1$, locally we can find a smooth vector field X that spans the line distribution and we have seen how to construct a local foliation by integral curves of X; in fact, the global problem can also be solved by passing to a double covering of M. It turns out that in case $k = 1$ there are no obstructions to the integrability of distributions, and this is in line with the fact that there are no obstructions to the integrability of ordinary differential equations. On the other hand, the situation is different when we pass to distributions of rank $k > 1$, what amounts to consider certain kinds of partial differential equations.

Let $\mathcal{D}$ be a distribution on M. We say that $\mathcal{D}$ is *integrable* if there exists a foliation $\mathcal{F}$ such that $T_pL_p = \mathcal{D}_p$ for every $p \in M$, where $L_p \in \mathcal{F}$ denotes the leaf thorugh p. Such an $\mathcal{F}$ is called an *integral foliation* of $\mathcal{D}$.

1.7.6 Proposition *If $\mathcal{D}$ is an integrable foliation on M then the integral foliation $\mathcal{F}$ is unique.*

Proof. Define an equivalence relation on M by declaring two points equivalent if and only if they can be joined by a piecewise smooth curve whose smooth arcs are tangent to $\mathcal{D}$. For $p \in M$, denote by L_p the leaf of $\mathcal{F}$ through p. Since L_p is arc connected, it is a union of equivalence classes. Now the existence of distinguished charts implies that each such equivalence class is open in L_p, so L_p coincides with the equivalence class of p. This already characterizes the leaves of $\mathcal{F}$ as subsets of M. Each leaf is an initial submanifold of M, so the structure of smooth manifold on the leaf is unique up to equivalence, as in Problem 19(d). □

More generally, an *integral manifold* of a distribution $\mathcal{D}$ on M is a submanifold (L, f) of M such that $df_p(T_pL) = \mathcal{D}_{f(p)}$ for every $p \in L$. A *maximal integral manifold* of $\mathcal{D}$ is a connected integral manifold whose image in M is not a proper subset of another connected integral manifold of $\mathcal{D}$, that is, there is no connected integral manifold (L', f') such that $f(L)$ is a proper subset of $f'(L')$.

1.7.7 Exercise Let L_1, L_2 be two integral manifolds of a distribution $\mathcal{D}$ on M. Use adapted charts to show that either L_1 and L_2 are disjoint or $L_1 \cap L_2$ is open in both L_1 and L_2. Deduce that, if $\mathcal{D}$ is integrable, then the leaves of the integral foliation are the maximal integral manifolds of $\mathcal{D}$.

We say that a vector field X on M *lies* in $\mathcal{D}$ if $X(p) \in \mathcal{D}_p$ for all $p \in M$; in this case, we write $X \in \mathcal{D}$. We say that $\mathcal{D}$ is *involutive* if X, $Y \in \mathcal{D}$ implies $[X, Y] \in \mathcal{D}$, namely, if $\mathcal{D}$ is closed under Lie brackets. Involutivity is a necessary condition for a distribution to be integrable.

1.7.8 Proposition *Every integrable distribution is involutive.*

Proof. Let $\mathcal{D}$ be an integrable distribution on a smooth manifold M. Given smooth vector fields X, $Y \in \mathcal{D}$ and $p \in M$, we need to show that $[X, Y]_p \in \mathcal{D}_p$. By assumption, there exists a distinguished coordinate system $(U, (x_1, \ldots, x_n))$ around p such that the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ span the distribution $\mathcal{D}$ on U. Now $X|_U$, $Y|_U$ are linear combinations of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ with $C^\infty(U)$-coefficients, and so is their bracket, as we wished. □

It so happens that involutivity is also a sufficient condition for a distribution to be integrable. This is the contents of the celebrated Frobenius theorem.

Despite being named after Frobenius, the theorem seems to be proved first by Clebsch and Deahna. The merit of Frobenius in his 1875 Crelle's paper was to apply the theorem to Pfaffian systems, or systems of partial differential equations that are usefully formulated, from the point of view of their underlying geometric and algebraic structure, in terms of a system of differential forms of degree one. The proof below is accredited to Lundell [Lun92] who found inspiration in Chern and Wolfson.

We first prove an elementary, general lemma.

1.7.9 Lemma *Let $\mathcal{D}$ be any rank k distribution on a smooth manifold M. Then there exists a system of local coordinates $(U, x_1, \ldots, x_n)$ around any given point p in M such that $\mathcal{D}$ is spanned by the k vector fields*

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=k+1}^{n} a_{ij} \frac{\partial}{\partial x_i} \qquad \text{for } j = 1, \ldots, k$$

at all points in U, where $a_{ij} \in C^\infty(U)$.

Proof. Let $(V, x_1, \ldots, x_n)$ be any system of local coordinates around p. Let $Y_1, \ldots, Y_k$ be arbitrary smooth vector fields spanning $\mathcal{D}$ on an open set $\tilde{U} \subset V$. Then $Y_j = \sum_{i=1}^n b_{ij} \frac{\partial}{\partial x_i}$ for $j = 1, \ldots, k$ and $b_{ij} \in C^\infty(\tilde{U})$. Since $Y_1, \ldots, Y_k$ is linearly independent at every point of $\tilde{U}$, the matrix $B(q) = (b_{ij}(q))$ has rank k for all $q \in \tilde{U}$. By relabeling the x_i, we may assume that the $1 \leq i, j \leq k$-block B' is non-singular in an open neighborhood $U \subset \tilde{U}$ of p. Now the $1 \leq i, j \leq k$-block of $B(B')^{-1}$ is the identity, namely, $X_j = \sum_{i=1}^k \hat{b}_{ij} Y_i$ has the desired form, where $(B')^{-1} = (\hat{b}_{ij})$. ☐

1.7.10 Theorem *Every involutive distribution is integrable.*

Proof. Let $\mathcal{D}$ be an involutive distribution on a smooth manifold M. We first prove the local integrability, namely, the existence around any given point $p \in M$ of a system of local coordinates $(V, y_1, \ldots, y_n)$ such that $\mathcal{D}_q$ is spanned by $\frac{\partial}{\partial y_1}|_1, \ldots, \frac{\partial}{\partial y_k}|_q$ for every $q \in V$. Indeed let $(U, x_1, \ldots, x_n)$ and $X_1, \ldots, X_k$ be as in Lemma 1.7.9. Note that

$$[X_i, X_j] \in \operatorname{span}\left\{ \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n} \right\},$$

so the involutivity of $\mathcal{D}$ implies that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, k$. The desired result follows from Proposition 1.6.24.

Finally, we construct the integral foliation. According to Proposition 1.7.6, the leaf L_p through a given point $p \in M$ must be the set of points $q \in M$ that can be reached from p by a piecewise smooth curve whose smooth arcs are tangent to $\mathcal{D}$. This defines a partition $\mathcal{F}$ of M into piecewise smooth arc connected subsets. Given $q \in L_p$, let $(V, y_1, \ldots, y_n)$ be a system of local coordinates around q such that $\mathcal{D}$ is spanned by $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_k}$ at all points in V. It is clear that the arc connected components of $L_p \cap V$ are

$$y_{k+1} = \text{constant}, \ \ldots, \ y_n = \text{constant}.$$

This proves that $\mathcal{F}$ is a foliation. □

1.8 Problems

§ 1.2

1 a. Use stereographic projection $\varphi_N : U_N = S^2 \setminus \{(0,0,1)\} \to \mathbf{R}^2$ to define a local chart on S^2 and write a formula for φ_N in terms of the coordinates of $\mathbf{R}^3$. Do the same for $\varphi_S : U_S = S^2 \setminus \{(0,0,-1)\} \to \mathbf{R}^2$.
b. Show that $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a smooth atlas for S^2. Compare the smooth structure defined by this atlas with that defined in example 1.2.7 (viewing S^2 as a union of graphs of smooth maps).

2 Let M be the set of all (affine) lines in $\mathbf{R}^2$. Construct a natural structure of smooth manifold in M. What is the dimension of M? (Hint: Parametrize lines in terms of their equations.)

3 Let M, N, P be smooth manifolds and denote by $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ the canonical projections. Define maps $\iota_1 : M \to M \times N$, $\iota_2 : N \to M \times N$, where $\iota_1(x) = (x, q)$, $\iota_2(y) = (p, y)$ and $p \in M$, $q \in N$.
a. Show that π_1, π_2, ι_1, ι_2 are smooth maps.
b. Show that $f : P \to M \times N$ is smooth if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are smooth.

4 Let $f : M \to N$ be a map. Prove that $f \in C^\infty(M, N)$ if and only if $g \circ f \in C^\infty(M)$ for all $g \in C^\infty(N)$.

5 Let $\pi : \tilde{M} \to M$ be a topological covering of a smooth manifold M. Check that $\tilde{M}$ is necessarily Hausdorff, second-countable (here you need to know that the fundamental group $\pi(M)$ is at most countable) and locally Euclidean. Prove also that there exists a unique smooth structure on $\tilde{M}$ which makes π smooth and a local diffeomorphism (compare Appendix A).

§ 1.4

6 a. Prove that the composition and the product of immersions are immersions.
b. In case $\dim M = \dim N$, check that the immersions $M \to N$ coincide with the local diffeomorphisms.

7 Prove that every submersion is an open map.

8 a. Prove that if M is compact and N is connected then every submersion $M \to N$ is surjective.

b. Show that there are no submersions of compact manifolds into Euclidean spaces.

9 Show that every smooth real function on a compact manifold has at least two critical points.

10 Let M be a compact manifold of dimension n and let $f : M \to \mathbf{R}^n$ be smooth. Prove that f has at least one critical point.

11 Let $p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a polynomial with complex coefficients and consider the associated polynomial map $\mathbf{C} \to \mathbf{C}$. Show that this map is a submersion out of finitely many points.

12 (*Generalized inverse function theorem.*) Let $f : M \to N$ be a smooth map which is injective on a compact submanifold P of M. Assume that $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for every $p \in P$.

a. Prove that $f(P)$ is a submanifold of N and that f restricts to a diffeomorphism $P \to f(P)$.

b. Prove that indeed f maps some open neighborhood of P in M diffeomorphically onto an open neighborhood of $f(P)$ in N. (Hint: It suffices to show that f is injective on some neighborhood of P; if this is not the case, there exist sequences $\{p_i\}$, $\{q_i\}$ in M both converging to a point $p \in P$, with $p_i \neq q_i$ but $f(p_i) = f(q_i)$ for all i, and this contradicts the non-singularity of df_p.)

13 Let p be a homogeneous polynomial of degree m in n variables $t_1, \ldots, t_n$. Show that $p^{-1}(a)$ is a submanifold of codimension one of $\mathbf{R}^n$ if $a \neq 0$. Show that the submanifolds obtained with $a > 0$ are all diffeomorphic, as well as those with $a < 0$. (Hint: Use Euler's identity

$$\sum_{i=1}^{n} t_i \frac{\partial p}{\partial t_i} = mp.)$$

14 The $n \times n$ real matrices with determinar 1 form a group denoted $SL(n, \mathbf{R})$. Prove that $SL(n, \mathbf{R})$ is a submanifold of $GL(n, \mathbf{R})$. (Hint: Use Problem 13.)

15 Consider the submanifolds $GL(n, \mathbf{R})$, $O(n)$ and $SL(n, \mathbf{R})$ of the vector space $M(n, \mathbf{R})$ (see Examples 1.2.7(ix) and 1.4.14(b), and Problem 14, respectively).

a. Check that the tangent space of $GL(n, \mathbf{R})$ at the identity is canonically isomorphic to $M(n, \mathbf{R})$.

b. Check that the tangent space of $SL(n, \mathbf{R})$ at the identity is canonically isomorphic to the subspace of $M(n, \mathbf{R})$ consisting of matrices of trace zero.

c. Check that the tangent space of $O(n)$ at the identity is canonically isomorphic to the subspace of $M(n, \mathbf{R})$ consisting of the skew-symmetric matrices.

16 Denote by $M(m \times n, \mathbf{R})$ the vector space of real $m \times n$ matrices.

a. Show that the subset of $M(m \times n, \mathbf{R})$ consisting of matrices of rank at least k $(0 \leq k \leq \min\{m, n\})$ is a smooth manifold.

b. Show that the subset of $M(m \times n, \mathbf{R})$ consisting of matrices of rank equal to k $(0 \leq k \leq \min\{m, n\})$ is a smooth manifold. What is its dimension? (Hint: We may work in a neighborhood of a matrix

$$g = \begin{array}{c} \scriptstyle k \\ \scriptstyle m-k \end{array} \overset{\qquad k \quad\ n-k}{\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)}$$

where A is nonsingular and right multiply by

$$\left(\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right)$$

to check that g has rank k if and only if $D - CA^{-1}B = 0$.)

17 Let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of smooth maps between smooth manifolds. Assume that $g \pitchfork Q$ for a submanifold Q of P. Prove that $f \pitchfork g^{-1}(Q)$ if and only if $g \circ f \pitchfork Q$.

18 Let $G \subset \mathbf{R}^2$ be the graph of $g : \mathbf{R} \to \mathbf{R}$, $g(x) = |x|^{1/3}$. Show that G admits a smooth structure so that the inclusion $G \to \mathbf{R}^2$ is smooth. Is it an immersion? (Hint: consider the map $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(t) = \begin{cases} te^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ te^{1/t} & \text{if } t < 0.) \end{cases}$$

19 Define submanifolds (M_1, f_1), (M_2, f_2) of N to be *equivalent* if there exists a diffeomorphism $g : M_1 \to M_2$ such that $f_2 \circ g = f_1$.

a. Show that this is indeed an equivalence relation.

b. Show that each equivalence class of submanifolds of N contains a unique representative of the form (M, ι), where M is a subset of N with a manifold structure such that $\iota : M \to N$ is a smooth immersion.

c. Let N be a smooth manifold, and let M be a subset of N equipped with a given topology. Prove that there exists at most one smooth structure on M, up to equivalence, which makes (M, ι) an immersed submanifold of N, where $\iota : M \to N$ is the inclusion. (Hint: Use Proposition 1.4.9.)

d. Let N be a smooth manifold, and let M be a subset of N. Prove that there exists at most one structure of smooth manifold on M, up to equivalence, which makes (M, ι) an initial submanifold of N, where $\iota : M \to N$ is the inclusion. (Hint: Use Proposition 1.4.9.)

20 Let N be a smooth manifold of dimension $n+k$. For a point $q \in N$ and a subset $A \subset N$, denote by $C_q(A)$ the set of all points of A that can be joined to q by a smooth curve in M whose image lies in A.

a. Prove that if (P, g) is an initial submanifold of dimension n of N then for every $p \in P$ there exists a local chart (V, ψ) of N around $g(p)$ such that

$$\psi(C_{g(p)}(V \cap g(P))) = \psi(V) \cap (\mathbf{R}^n \times \{0\}).$$

(Hint: Use Proposition 1.4.5.)

b. Conversely, assume P is a subset of N with the property that around any point $p \in P$ there exists a local chart (V, ψ) of N around p such that

$$\psi(C_p(V \cap P)) = \psi(V) \cap (\mathbf{R}^n \times \{0\}).$$

Prove that there exists a topology on P that makes each connected component of P into an initial submanifold of dimension n of N with respect to the inclusion. (Hint: Apply Proposition 1.2.10 to the restrictions $\psi|_{C_p(V\cap P)}$. Proving second-countability requires the following facts: for locally Euclidean Hausdorff spaces, paracompactness is equivalent to the property that each connected component is second-countable; every metric space is paracompact; the topology on P is metrizable since it is compatible with the Riemannian distance for the Riemannian metric induced from a given Riemannian metric on N; Riemannian metrics can be constructed on N using partitions of unity.)

21 Show that the product of any number of spheres can be embedded in some Euclidean space with codimension one.

§ 1.5

22 Let M be a closed submanifold of N. Prove that the restriction map $C^\infty(N) \to C^\infty(M)$ is well defined and surjective. Show that the result ceases to be true if: (i) M is not closed; or (ii) $M \subset N$ is closed but merely assumed to be an immersed submanifold.

23 Let M be a smooth manifold of dimension n. Given $p \in M$, construct a local chart (U, φ) of M around p such that φ is the restriction of a smooth map $M \to \mathbf{R}^n$.

24 Prove that on any smooth manifold M there exists a proper smooth map $f : M \to \mathbf{R}$. (Hint: Use σ-compactness of manifolds and partitions of unity.)

§ 1.6

25 Determine the vector field on $\mathbf{R}^2$ with flow $\varphi_t(x, y) = (xe^{2t}, ye^{-3t})$.

26 Determine the flow of the vector field X on $\mathbf{R}^2$ when:
a. $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.
b. $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

27 Given the following vector fields in $\mathbf{R}^3$,

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

compute their Lie brackets.

28 Show that the restriction of the vector field defined on $\mathbf{R}^{2n}$

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \cdots - x_{2n}\frac{\partial}{\partial x_{2n-1}} + x_{2n-1}\frac{\partial}{\partial x_{2n}}$$

to the unit sphere S^{2n-1} defines a nowhere vanishing smooth vector field.

29 Let X and Y be smooth vector fields on M and N with flows $\{\varphi_t\}$ and $\{\psi_t\}$, respectively, and let $f : M \to N$ be smooth. Show that X and Y are f-related if and only if $f \circ \varphi_t = \psi_t \circ f$ for all t.

30 Let M be a properly embedded submanifold of N. Prove that every smooth vector field on M can be smoothly extended to a vector field on N.

31 Construct a natural diffeomorphism $TS^1 \approx S^1 \times \mathbf{R}$ which restricts to a linear isomorphism $T_pS^1 \to \{p\} \times \mathbf{R}$ for every $p \in S^1$ (we say that such a diffeomorphism maps fibers to fibers and is linear on the fibers).

32 Construct a natural diffeomorphism $T(M \times N) \approx TM \times TN$ that maps fibers to fibers and is linear on the fibers.

33 Construct a natural diffeomorphism $T\mathbf{R}^n \approx \mathbf{R}^n \times \mathbf{R}^n$ that maps fibers to fibers and is linear on the fibers.

34 Show that $TS^n \times \mathbf{R}$ is diffeomorphic to $S^n \times \mathbf{R}^{n+1}$. (Hint: There are natural isomorphisms $T_pS^n \oplus \mathbf{R} \cong \mathbf{R}^{n+1}$.)

35 A smooth manifold M of dimension n is called *parallelizable* if $TM \approx M \times \mathbf{R}^n$ by a diffeomorphism that maps fibers to fibers and is linear on the fibers. Prove that M is parallelizable if and only if there exists a globally defined n-frame $\{X_1, \ldots, X_n\}$ on M.

§ 1.7

36 Is there a non-constant smooth function f defined on an open subset of $\mathbf{R}^3$ such that

$$\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z} = 0?$$

(Hint: Consider a regular level set of f.)

37 Consider the first order system of partial differential equations

$$\frac{\partial z}{\partial x} = \alpha(x, y, z), \quad \frac{\partial z}{\partial y} = \beta(x, y, z)$$

where α, β are smooth functions defined on an open subset of $\mathbf{R}^3$.

a. Show that if f is a solution, then the smooth vector fields $X = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial z}$ e $Y = \frac{\partial}{\partial y} + \beta\frac{\partial}{\partial z}$ span the tangent space to the graph of f at all points.

b. Prove that the system admits local solutions if and only if

$$\frac{\partial \beta}{\partial x} + \alpha\frac{\partial \beta}{\partial z} = \frac{\partial \alpha}{\partial y} + \beta\frac{\partial \alpha}{\partial z}.$$

38 Prove that there exists a smooth function f defined on a neighborhood of $(0, 0)$ in $\mathbf{R}^2$ such that $f(0, 0) = 0$ and $\frac{\partial f}{\partial x} = ye^{-(x+y)} - f$, $\frac{\partial f}{\partial y} = xe^{-(x+y)} - f$.