CHAPTER 3

Lie groups

Lie groups are amongst the most important examples of smooth manifolds. At the same time, almost all usually encountered examples of smooth manifolds are related to Lie groups, in a way or another. A Lie group is a smooth manifold with an additional, compatible structure of group. Here compatibility refers to the fact that the group operations are smooth (another point of view is to regard a Lie group as a group with an additional structure of manifold...). The reader can keep in mind the matrix group $\mathbf{GL}(n,\mathbf{R})$ of non-singular real $n \times n$ matrices (Examples 1.2.7) in which the n^2 matrix coefficients form a global coordinate system. The conjuction of the smooth and the group structures allows one to give a more explicit description of the differential invariants attached to a manifold. For this reason, Lie groups form a class of manifolds suitable for testing general hypotheses and conjectures. The same remarks apply to homogeneous spaces, which are certain quotients of Lie groups.

3.1 Basic definitions and examples

A *Lie group* G is a smooth manifold endowed with a group structure such that the group operations are smooth. More concretely, the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are required to be smooth.

3.1.1 Examples (a) The Euclidean space $\mathbf{R}^n$ with its additive vector space structure is a Lie group. Since the multiplication is commutative, this is an example of a *Abelian* (or *commutative*) Lie group.

(b) The multiplicative group of nonzero complex numbers $\mathbf{C}^\times$. The subgroup of unit complex numbers is also a Lie group, and as a smooth manifold it is diffeomorphic to the circle S^1. This is also an Abelian Lie group.

(c) If G and H are Lie groups, the direct product group structure turns the product manifold $G \times H$ into a Lie group.

(d) It follows from (b) and (c) that the *n-torus* $T^n = S^1 \times \cdots \times S^1$ (n times) is a Lie group. Of course, T^n is a compact connected Abelian Lie group. Conversely, we will see in Theorem 3.5.3 that every compact connected Abelian Lie group is an n-torus.

(e) If G is a Lie group, the connected component of the identity of G, denoted by G°, is also a Lie group. Indeed, G° is open in G, so it inherits a smooth structure from G just by restricting the local charts. Since $\mu(G^\circ \times G^\circ)$ is connected and $\mu(1,1) = 1$, we must have $\mu(G^\circ \times G^\circ) \subset G^\circ$. Similarly, $\iota(G^\circ) \subset G^\circ$. Since $G^\circ \subset G$ is an open submanifold, it follows that the group operations restricted to G° are smooth.

(f) Any finite or countable group endowed with the discrete topology becomes a 0-dimensional Lie group. Such examples are called *discrete Lie groups*.

(g) We now turn to some of the classical matrix groups. The general linear group $\mathbf{GL}(n, \mathbf{R})$ is a Lie group since the entries of the product of two matrices is a quadratic polynomial on the entries of the two matrices, and the entries of inverse of a non-singular matrix is a rational function on the entries of the matrix.

Similarly, one defines the *complex general linear group of order* n, which is denoted by $\mathbf{GL}(n, \mathbf{C})$, as the group consisting of all nonsingular $n \times n$ complex matrices, and checks that it is a Lie group. Note that $\dim \mathbf{GL}(n, \mathbf{C}) = 2n^2$ and $\mathbf{GL}(1, \mathbf{C}) = \mathbf{C}^\times$.

We have already encountered the orthogonal group $\mathbf{O}(n)$ as a closed embedded submanifold of $\mathbf{GL}(n, \mathbf{R})$ in 1.4.14. Since $\mathbf{O}(n)$ is an embedded submanifold, it follows from Theorem 1.4.9 that the group operations of $\mathbf{O}(n)$ are smooth, and hence $\mathbf{O}(n)$ is a Lie group.

Similarly to $\mathbf{O}(n)$, one checks that the

$$\begin{aligned}
\mathbf{SL}(n, \mathbf{R}) &= \{A \in \mathbf{GL}(n, \mathbf{R}) \mid \det(A) = 1\} \quad (\textit{real special linear group}) \\
\mathbf{SL}(n, \mathbf{C}) &= \{A \in \mathbf{GL}(n, \mathbf{C}) \mid \det(A) = 1\} \quad (\textit{complex special linear group}) \\
\mathbf{U}(n) &= \{A \in \mathbf{GL}(n, \mathbf{C}) \mid AA^* = I\} \quad (\textit{unitary group}) \\
\mathbf{SO}(n) &= \{A \in \mathbf{O}(n) \mid \det(A) = 1\} \quad (\textit{special orthogonal group}) \\
\mathbf{SU}(n) &= \{A \in \mathbf{U}(n) \mid \det(A) = 1\} \quad (\textit{special unitary group})
\end{aligned}$$

are Lie groups, where A^* denotes the complex conjugate transpose matrix of A. Note that $\mathbf{U}(1) = S^1$.

Lie algebras

For an arbitrary smooth manifold M, the space $\mathfrak{X}(M)$ of smooth vector fields on M is an infinite-dimensional vector space over $\mathbf{R}$. In addition, we have already encountered the Lie bracket, a bilinear map $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying:

$a.$ $[Y, X] = -[X, Y]$;

b. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity);
for every $X, Y \in \mathfrak{X}(M)$. In general, a vector space with a bilinear operation satisfying (a) and (b) above is called a *Lie algebra*. So $\mathfrak{X}(M)$ is Lie algebra over $\mathbf{R}$.

It turns out in case of a Lie group G, we can single out a finite dimensional subalgebra of $\mathfrak{X}(M)$. For that purpose, let us first introduce translations in G. The *left translation* defined by $g \in G$ is the map $L_g : G \to G$, $L_g(x) = gx$. It is a diffeomorphism of G, its inverse being given by $L_{g^{-1}}$. Similarly, the *right translation* defined by $g \in G$ is the map $R_g : G \to G$, $R_g(x) = xg$. It is also a diffeomorphism of G, and its inverse is given by $R_{g^{-1}}$.

The translations in G define canonical identifications between the tangent spaces to G at different points. For instance, $dL_g : T_hG \to T_{gh}G$ is an isomorphism for every g, $h \in G$. This allows us to consider invariant tensors, the most important case being that of vector fields. A vector field X on G is called *left-invariant* if $d(L_g)_x(X_x) = X_{gx}$ for every g, $x \in X$. This condition is simply $dL_g \circ X = X \circ L_g$ for every $g \in G$; equivalently, X is L_g-related to itself, or yet $L_{g*}X = X$ (since L_g is a diffeomorphism), for all $g \in G$. We can similarly define *right-invariant* vector fields, but most often we will be considering the left-invariant variety. Note that left-invariance and right-invariance are the same property in case of an Abelian group.

3.1.2 Lemma *Every left invariant vector field X in G is smooth.*

Proof. Let f be a smooth function defined on a neighborhood of 1 in G, and let $\gamma : (-\epsilon, \epsilon) \to G$ be a smooth curve with $\gamma(0) = 1$ and $\gamma'(0) = X_1$. Then the value of X on f is given by

$$X_g(f) = dL_g(X_1)(f) = X_1(f \circ L_g) = \frac{d}{dt}\Big|_{t=0} f(g\gamma(t)) = \frac{d}{dt}\Big|_{t=0} f \circ \mu(g, \gamma(t)),$$

and hence, it is a smooth function of g. $\square$

Let $\mathfrak{g}$ denote the set of left invariant vector fields on G. It follows that $\mathfrak{g}$ is a vector subspace of $\mathfrak{X}(M)$. Further, $\mathfrak{g}$ is a subalgebra of $\mathfrak{X}(M)$, for given $X, Y \in \mathfrak{g}$, we have by Proposition 1.6.18 that

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y],$$

for every $g \in G$. Finally, we explain why $\mathfrak{g}$ is finite-dimensional: the map $X \in \mathfrak{g} \mapsto X_1$ defines a linear isomorphism between $\mathfrak{g}$ and the tangent space to G at the identity T_1G, since any left invariant vector field is completely defined by its value at the identity.

The discussion above shows that to any Lie group G is naturally associated a (real) finite-dimensional Lie algebra $\mathfrak{g}$ of the same dimension as G, consisting of the left invariant vector fields on G. This Lie algebra is the infinitesimal object associated to G and, as we shall see, completely determines its local structure.

3.1.3 Examples (*The Lie algebras of some known Lie groups*)

(i) The left-invariant vector fields on $\mathbf{R}^n$ are precisely the constant vector fields, namely, the linear combinations of coordinate vector fields (in the canonical coordinate system) with constant coefficients. The bracket of two constant vector fields on $\mathbf{R}^n$ is zero. It follows that the Lie algebra of $\mathbf{R}^n$ is $\mathbf{R}^n$ itself with the null bracket. In general, a vector space equipped with the null bracket is called an *Abelian* Lie algebra.

(ii) The Lie algebra of the direct product $G \times H$ is the direct sum of Lie algebras $\mathfrak{g} \oplus \mathfrak{h}$, where the bracket is taken componentwise.

(iii) Owing to the skew-symmetry of the Lie bracket, every one-dimensional Lie algebra is Abelian. In particular, the Lie algebra of S^1 is Abelian. It follows from (ii) that also the Lie algebra of T^n is Abelian.

(iv) G and G° have the same Lie algebra.

(v) The Lie algebra of a discrete group is $\{0\}$.

3.1.4 Examples (*Some abstract Lie algebras*)

(i) Let A be any real associative algebra and set $[a,b] = ab - ba$ for a, $b \in A$. It is easy to see that A becomes a Lie algebra.

(ii) The cross-product $\times$ on $\mathbf{R}^3$ is easily seen to define a Lie algebra structure.

(iii) If V is a two-dimensional vector space and X, $Y \in V$ are linearly independent, the conditions $[X,X] = [Y,Y] = 0$, $[X,Y] = X$ define a Lie algebra structure on V.

(iv) If V is a three-dimensional vector space spanned by X, Y, Z, the conditions $[X,Y] = Z$, $[Z,X] = [Z,Y] = 0$ define a Lie algebra structure on V, called the *(3-dimensional) Heisenberg algebra*. It can be realized as a Lie algebra of smooth vector fields on $\mathbf{R}^3$ as in Example 1.6.15(b).

3.1.5 Exercise Check the assertions of Examples 3.1.3 and 3.1.4.

3.2 The exponential map

For a Lie group G, we have constructed its most basic invariant, its Lie algebra $\mathfrak{g}$. Our next step will be to present the fundamental map that relates G and $\mathfrak{g}$, namely, the exponential map $\exp : \mathfrak{g} \to G$.

Matrix exponential

Recall that the exponential of a matrix $A \in \mathbf{M}(n,\mathbf{R})$ (or $\mathbf{M}(n,\mathbf{C})$) is given by the formula:

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}A^n. \end{aligned}$$

Since $||\sum_{k=0}^{n}\frac{1}{k!}A^k|| \leq e^{||A||}$ for all $n \geq 0$, the series is absolutely convergent on the entire $\mathbf{M}(n,\mathbf{R})$; here $||\cdot||$ denotes the usual Euclidean norm in $\mathbf{M}(n,\mathbf{R}) = \mathbf{R}^{n^2}$. In case $n=1$, we recover the usual exponential map on the line. In general, note that:

a. $e^0 = I$;

b. $e^{A+B} = e^A e^B$ if A and B commute.

Indeed, to check (b) notice that one can compute the product of e^A and e^B by multiplying the individual terms and rearranging, by absolute convergence. In particular:

c. $e^{(s+t)A} = e^{sA}e^{tB}$ for all $s,t \in \mathbf{R}$;

d. e^A is invertible and $(e^A)^{-1} = e^{-A}$.

View $t \in \mathbf{R} \mapsto e^{tA}$ as a curve in $\mathbf{M}(n,\mathbf{R})$. The last property worth mentioning is

e. $\frac{d}{dt}\Big|_{t=t_0} e^{tA} = A\,e^{t_0A} = e^{t_0A}\,A$.

Flow of left-invariant vector fields

Let G be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra.

3.2.1 Proposition *Every left-invariant vector field is complete.*

Proof. Given $X \in \mathfrak{g}$, there exists a maximal integral curve $\gamma_X : (a,b) \to G$ of X with $0 \in (a,b)$ (a, $b \in [-\infty,\infty]$) and $\gamma_X(0) = 1$; namely, $\gamma_X'(t) = X_{\gamma_X(t)}$. Since

$$\frac{d}{dt}\Big|_{t=t_0} L_g(\gamma_X(t)) = d(L_g)(X_{\gamma_X(t_0)}) = X_{L_g(\gamma_X(t_0))},$$

we have that $L_g \circ \gamma_X$ is an integral curve of X starting at g. In particular, if $b < \infty$, by taking $g = \gamma(s)$ with s very close to b, this shows that γ_X can be extended beyond b, leading to a contradiction. Similarly, one sees that $a = -\infty$. Hence X is complete. $\square$

Now the integral curve γ_X of any $X \in \mathfrak{g}$ starting at the identity is defined on $\mathbf{R}$. The *exponential map* of G is the map $\exp : \mathfrak{g} \to G$ defined by $\exp X = \gamma_X(1)$.

Note that $\frac{d}{ds}\Big|_{s=s_0}\gamma_X(ts) = t\gamma_X'(ts_0) = tX(\gamma_X(ts_0))$. This implies $\gamma_X(ts) = \gamma_{tX}(s)$ for all $s,t \in \mathbf{R}$ and therefore

$$\begin{aligned} \gamma_X(t) &= \gamma_{tX}(1) \\ &= \exp(tX), \end{aligned} \tag{3.2.2}$$

namely, every integral curve of a left-invariant vector field through the identity factors through the exponential map.

3.2.3 Exercise Check that the flow $\{\varphi_t\}$ of a left-invariant vector field X is given by $\varphi_t = R_{\exp tX}$ (recall that R_g denotes a right-translation). What is the corresponding result for right-invariant vector fields?

Moreover, we state:

3.2.4 Proposition *The exponential map* $\exp : \mathfrak{g} \to G$ *is smooth and it is a local diffeomorphism at* 0*.*

Proof. Smoothness follows from general properties of flows, namely, smooth dependence on parameters of solutions of ODE's. Moreover, $d\exp_0 : T_0\mathfrak{g} \cong \mathfrak{g} \to T_1G \cong \mathfrak{g}$ is the identity, since

$$d\exp_0(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX) = \varphi_X'(0) = X.$$

Thus, $\exp$ is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of 1 in G by the Inverse Function Theorem (1.3.8). □

Recall that the identity component G° is an open subgroup of G.

3.2.5 Proposition G° *is generated as a group by any neighborhood* U *of* 1 *in* G°*, namely,*

$$G^\circ = \bigcup_{n\geq 1} U^n,$$

where U^n *denotes the set of* n*-fold products of elements in* U*. In particular,* G° *is generated by* $\exp[\mathfrak{g}]$*.*

Proof. By replacing U by $U \cap U^{-1}$, if necessary, we may assume that $U = U^{-1}$. Define $V = \cup_{n\geq 0} U^n$ and consider the relation in G° given by $g \sim g'$ if and only if $g^{-1}g' \in V$. Note that this is an equivalence relation, and equivalence classes are open as $g' \sim g$ implies $g'U \sim g$, where $g'U$ is an open neighborhood of g'. Hence $V = G^\circ$. □

The case of $\mathbf{GL}(n, \mathbf{R})$

Recall that $G = \mathbf{GL}(n, \mathbf{R})$ inherits its manifold structure as an open subset of the Euclidean space $\mathbf{M}(n, \mathbf{R})$. In particular, the tangent space at the identity $T_IG = \mathbf{M}(n, \mathbf{R})$. Let $A \in \mathbf{M}(n, \mathbf{R})$ and denote by $\tilde{A} \in \mathfrak{g}$ the corresponding left-invariant vector field on G. For any $g \in G$, we have $\tilde{A}_g = (dL_g)(A) = gA$ (matrix multiplication on the right hand side).

Using property (e) of the matrix exponential,

$$\frac{d}{dt}\Big|_{t=t_0} e^{tA} = e^{t_0A}A = \tilde{A}_{e^{t_0A}}$$

shows that $t \mapsto e^{tA}$ is the integral curve of $\tilde{A}$ through the identity, namely

$$\exp \tilde{A} = e^A$$

for all $A \in \mathbf{M}(n, \mathbf{R})$.

Finally, to determine the Lie bracket in $\mathfrak{g}$, we resort to (1.6.22). Let A, $B \in M(n, \mathbf{R})$, denote by $\tilde{A}$, $\tilde{B}$ the corresponding left-invariant vector fields on G, let $\{\varphi_t = R_{e^{tA}}\}$ be the flow of $\tilde{A}$ (cf. Exercise 3.2.3):

$$\begin{aligned}
[A, B] &= [\tilde{A}, \tilde{B}]_I \\
&= (L_{\tilde{A}}\tilde{B})_I \\
&= \frac{d}{dt}\Big|_{t=0} d\varphi_{-t}(\tilde{B}_{\varphi_t(I)}) \\
&= \frac{d}{dt}\Big|_{t=0} e^{tA}Be^{-tA} \\
&= AB - BA.
\end{aligned}$$

Note that the Lie algebra structure in $\mathbf{M}(n, \mathbf{R})$ is induced from its associative algebra structure as in Example 3.1.4(i). The space $\mathbf{M}(n, \mathbf{R})$ with this Lie algebra structure will be denoted by $\mathfrak{gl}(n, \mathbf{R})$.

The case of $\mathbf{GL}(n, \mathbf{C})$ is completely analogous.

3.3 Homomorphisms and Lie subgroups

A *(Lie group) homomorphism* between Lie groups G and H is map $\varphi : G \to H$ which is both a group homomorphism and a smooth map. φ is called an *isomorphism* if, in addition, it is a diffeomorphism. An *automorphism* of a Lie group is an isomorphism of the Lie group with itself. A *(Lie algebra) homomorphism* between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ which preserves brackets. Φ is called an *isomorphism* if, in addition, it is bijective. An *automorphism* of a Lie algebra is an isomorphism of the Lie algebra with itself.

3.3.1 Exercise For a homomorphism $\varphi : G \to H$, check that $L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$ for all $g \in G$.

A homomorphism $\varphi : G \to H$ between Lie groups induces a linear map $d\varphi_1 : T_1G \to T_1H$ and hence a linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Indeed, if X is a left invariant vector field on G, let X' be the unique left invariant vector field on H such that $X'_1 = d\varphi_1(X_1)$ and put $d\varphi(X) = X'$.

3.3.2 Proposition *If $\varphi : G \to H$ is a homomorphism between Lie groups then $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism between the corresponding Lie algebras.*

Proof. Let $X \in \mathfrak{g}$. We first claim that X and $X' := d\varphi(X)$ are φ-related. In fact,

$$X'_{\varphi(g)} = d(L_{\varphi(g)})_1(X'_1) = d(L_{\varphi(g)} \circ \varphi)_1(X_1) = d(\varphi \circ L_g)_1(X_1) = d\varphi_g(X_g),$$

proving the claim. Now, if $Y \in \mathfrak{g}$, then Y and $\varphi(Y)$ are φ-related. Therefore $[X,Y]$ and $[d\varphi(X), d\varphi(Y)]$ are φ-related and thus

$$d\varphi([X,Y]_1) = [d\varphi(X), d\varphi(Y)]_{\varphi(1)},$$

or

$$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)].$$

This shows that $d\varphi$ is a Lie algebra homomorphism. □

Let G be a Lie group. A *Lie subgroup* of G is an immersed submanifold (H,φ) of G such that H is a Lie group and $\varphi : H \to G$ is a homomorphism.

3.3.3 Remark Similarly as in the case of immersed submanifolds (Problem 19 in Chapter 1), we consider two Lie subgroups (H_1,φ_1) and (H_2,φ_2) of G equivalent if there exists a Lie group isomorphism $\alpha : H_1 \to H_2$ such that $\varphi_1 = \varphi_2 \circ \alpha$. This is an equivalence relation in the class of Lie subgroups of G and each equivalence class contains a unique representative of the form (A,ι), where A is a subset of G (an actual subgroup) and $\iota : A \to G$ is the inclusion. So we lose no generality in assuming that a Lie subgroup of G is an abstract subgroup H of G which is an immersed submanifold of G and a Lie group with respect to the operations induced from G; namely, the multiplication and inversion in G must restrict to smooth maps $H \times H \to H$ and $H \to H$, respectively.

3.3.4 Example The skew-line $(\mathbf{R}, f)$ in T^2 (Example 1.4.2) is an example of a Lie subgroup of T^2 which is not closed.

If $\mathfrak{g}$ is a Lie algebra, a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Lie subalgebra* if $\mathfrak{h}$ is closed under the bracket of $\mathfrak{g}$.

Let H be a Lie subgroup of G, say, $\iota : H \to G$ is the inclusion map. Since ι is an immersion, $d\iota : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras, and we may and will view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$. Conversely, as our most important application of Frobenius' theorem, we have:

3.3.5 Theorem (Lie) *Let G be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup H of G such that the Lie algebra of H is $\mathfrak{h}$.*

Proof. We have that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and so defines a subspace $\mathfrak{h}(1) := \{X(1) \mid X \in \mathfrak{h}\}$ of T_1G. Let $\mathcal{D}$ be the left-invariant distribution on G defined by $\mathfrak{h}$, namely, $\mathcal{D}_g = dL_g(\mathfrak{h}(1))$ for all $g \in G$. Then $\mathcal{D}$ is a smooth distribution, as it is globally generated by left-invariant vector fields $X_1, \ldots, X_k$ in $\mathfrak{h}$. The fact that $\mathcal{D}$ is involutive follows from (and is equivalent to) $\mathfrak{h}$ being a

Lie subalgebra of $\mathfrak{g}$. In fact, suppose X and Y lie in $\mathcal{D}$ over the open subset U of G. Write $X = \sum_i a_i X_i$, $Y = \sum_j b_j X_j$ for $a_i, b_j \in C^\infty(U)$. Then

$$[X, Y] = \sum_{i,j} a_i b_j [X_i, Y_j] + a_i X_i(b_j) X_j - b_j X_j(a_i) X_i$$

also lies in $\mathcal{D}$, as $[X_i, Y_j] \in \mathfrak{h}$.

By Frobenius theorem (1.7.10), there exists a unique maximal integral manifold of $\mathcal{D}$ passing through 1, which we call H. Since $\mathcal{D}$ is left-invariant, for every $h \in H$, $L_{h^{-1}}(H) = h^{-1}H$ is also a maximal integral manifold of $\mathcal{D}$, and it passes through through $h^{-1}h = 1$. This implies $h^{-1}H = H$, by uniqueness. It follows that H is a subgroup of G. The operations induced by G on H are smooth because H is an initial submanifold, due to Proposition 1.7.3. This proves that H is a Lie group. Its Lie algebra is $\mathfrak{h}$ because $\mathfrak{h}$ consists precisely of the elements of $\mathfrak{g}$ whose value at 1 lies in $\mathcal{D}_1 = T_1 H$, and these are exacly the elements of the Lie algebra of H.

Suppose now H' is another Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then H' must also be an integral manifold of $\mathcal{D}$ through 1. By the maximality of H, we have $H' \subset H$, and the inclusion map is smooth by Proposition 1.7.3 and thus an immersion. Now H' is an open submanifold of H and contains a neighborhood of 1 in H. Owing to Propostion 3.2.5, $H' = H$. □

3.3.6 Corollary *There is a bijective correspondence between connected Lie subgroups of a Lie group and subalgebras of its Lie algebra.*

3.3.7 Example Let G be a Lie group. A subgroup H of G which is an embedded submanifold of G is a Lie subgroup of G by Proposition 1.4.9. It follows from Example 1.4.14(b) that $\mathbf{O}(n)$ is a closed Lie subgroup of $\mathbf{GL}(n, \mathbf{R})$. Similarly, the other matrix groups listed in Examples 3.1.1(g) are closed Lie subgroups of $\mathbf{GL}(n, \mathbf{R})$, except that $\mathbf{SL}(n, \mathbf{C})$ is a closed Lie subgroup of $\mathbf{GL}(n, \mathbf{C})$. In particular, the Lie bracket in those subgroups is given by $[A, B] = AB - BA$.

3.3.8 Exercise Show that the Lie algebras of the matrix groups listed in Examples 3.1.1(g) are respectively as follows:

$$\begin{aligned}
\mathfrak{o}(n) &= \{A \in \mathfrak{gl}(n, \mathbf{R}) \mid A + A^t = 0\} \\
\mathfrak{sl}(n, \mathbf{R}) &= \{A \in \mathfrak{gl}(n, \mathbf{R}) \mid \operatorname{trace}(A) = 0\} \\
\mathfrak{sl}(n, \mathbf{C}) &= \{A \in \mathfrak{gl}(n, \mathbf{C}) \mid \operatorname{trace}(A) = 0\} \\
\mathfrak{u}(n) &= \{A \in \mathfrak{gl}(n, \mathbf{C}) \mid A + A^* = 0\} \\
\mathfrak{so}(n) &= \mathfrak{o}(n) \\
\mathfrak{su}(n) &= \{A \in \mathfrak{u}(n) \mid \operatorname{trace}(A) = 0\}
\end{aligned}$$

A Lie group homomorphism $\varphi : \mathbf{R} \to G$ is called a *(smooth) one-parameter subgroup*. Note that such a φ is the integral curve of $X := d\varphi(1) \in \mathfrak{g}$, and we have seen in (3.2.2) that $\varphi(t) = \exp(tX)$ for all $t \in \mathbf{R}$.

More generally, let $\varphi : G \to H$ be a homomorphism between Lie groups. Then, for a left invariant vector field X on G, $t \mapsto \varphi(\exp^G(tX))$ is a one-parameter subgroup of H with $\frac{d}{dt}\big|_{t=0}\varphi(\exp^G tX) = d\varphi(X_1)$. In view of the above,

$$\varphi \circ \exp^G X = \exp^H \circ d\varphi(X), \tag{3.3.9}$$

for every $X \in \mathfrak{g}$. In particular, if K is a Lie subgroup of G, then the inclusion map $\iota : K \to G$ is a Lie group homomorphism, so that the exponential map of G restricts to the exponential map of K, and the connected component of K is generated by $\exp^G[\mathfrak{k}]$, where $\mathfrak{k}$ is the Lie algebra of K. It follows also that

$$\mathfrak{k} = \{X \in \mathfrak{g} : \exp^G(tX) \in K, \text{ for all } t \in \mathbf{R}\}. \tag{3.3.10}$$

Indeed, let $X \in \mathfrak{g}$ with $\exp^G(tX) \in K$ for all $t \in \mathbf{R}$. Since K is an integral manifold of an involutive distribution (compare Theorem 3.3.5), $t \mapsto \exp^G(tX)$ defines a smooth map $\mathbf{R} \to K$ and thus a one-parameter subgroup of K. Therefore $\exp^G(tX) = i \circ \exp^K(tX')$ for some $X' \in \mathfrak{k}$, and hence $X = di(X')$.

3.4 Covering Lie groups

Let G be a connected Lie group. Consider the universal covering $\pi : \tilde{G} \to G$. By Problem 5 in Chapter 1 or the results in Appendix A, $\tilde{G}$ has a unique smooth structure for which π is a local diffeomorphism.

3.4.1 Theorem *Every connected Lie group G has a simply-connected covering $\pi : \tilde{G} \to G$ such that $\tilde{G}$ is a Lie group and π is a Lie group homomorphism.*

Proof. Consider the smooth map $\alpha : \tilde{G} \times \tilde{G} \to G$ given by $\alpha(\tilde{g}, \tilde{h}) = \pi(\tilde{g})\pi(\tilde{h})^{-1}$. Choose $\tilde{1} \in \pi^{-1}(1)$. As $\tilde{G}$ is simply-connected, so is $\tilde{G} \times \tilde{G}$. By the lifting criterion, there exists a unique map smooth $\tilde{\alpha} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\pi \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$. Put

$$\tilde{g}^{-1} := \tilde{\alpha}(\tilde{1}, \tilde{g}), \qquad \tilde{g}\tilde{h} := \tilde{\alpha}(\tilde{g}, \tilde{h}^{-1})$$

for $\tilde{g}$, $\tilde{h} \in \tilde{G}$. These operations are shown to make $\tilde{G}$ into a group by use of the uniqueness part in the lifting criterion. As an example,

$$\pi(\tilde{g}\tilde{1}) = \pi\tilde{\alpha}(\tilde{g}, \tilde{1}^{-1}) = \alpha(\tilde{g}, \tilde{1}^{-1}) = \pi(\tilde{g})\pi(\tilde{1}^{-1})^{-1} = \pi(\tilde{g}) \tag{3.4.2}$$

since $\tilde{1}^{-1} = \tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$ and $\pi(\tilde{1}) = 1$. Identity (3.4.2) shows that $\tilde{g} \mapsto \tilde{g}\tilde{1}$ is a lifting of $\tilde{g} \mapsto \pi(\tilde{g})$, $\tilde{G} \to G$, to a map $\tilde{G} \to \tilde{G}$ which takes $\tilde{1}$ to $\tilde{1} \cdot \tilde{1} =$

$\tilde{\alpha}(\tilde{1}, \tilde{1}^{-1}) = \tilde{\alpha}(\tilde{1}, \tilde{1}) = \tilde{1}$. However, the identity map of $\tilde{G}$ is also a lifting of $\tilde{g} \mapsto \pi(\tilde{G})$ which takes $\tilde{1}$ to $\tilde{1}$. By uniqueness, both liftings coincide and $\tilde{g}\tilde{1} = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$.

Now $\tilde{G}$ is a group. Since $\tilde{\alpha}$ is smooth, $\tilde{G}$ is a Lie group. Finally,

$$\pi(\tilde{g}^{-1}) = \pi\tilde{\alpha}(\tilde{1}, \tilde{g}) = \alpha(\tilde{1}, \tilde{g}) = \pi(\tilde{1})\pi(\tilde{g})^{-1} = \pi(\tilde{g})^{-1}$$

and

$$\pi(\tilde{g}\tilde{h}) = \pi\tilde{\alpha}(\tilde{g}, \tilde{h}^{-1}) = \alpha(\tilde{g}, \tilde{h}^{-1}) = \pi(\tilde{g})\pi(\tilde{h}^{-1})^{-1} = \pi(\tilde{g})\pi(\tilde{h}).$$

Hence, $\pi : \tilde{G} \to G$ is a Lie group homomorphism. □

3.4.3 Remark It follows from Lemma 3.4.4(c) and Theorem 3.7.7 that the structure of Lie group on the universal covering $\tilde{G}$ of G is unique, up to isomorphism.

3.4.4 Lemma *Let $\varphi : G \to H$ be a homomorphism between Lie groups. Consider the induced homomorphism between the corresponding Lie algebras $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Then:*

a. $d\varphi$ is injective if and only if the kernel of φ is discrete.
b. $d\varphi$ is surjective if and only if $\varphi(G^\circ) = H^\circ$.
c. $d\varphi$ is bijective if and only if φ is a topological covering (here we assume G and H connected).

Proof. (a) If $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is injective, then φ is an immersion at 1 and thus everywhere by Exercise 3.3.1. Therefore φ is locally injective and hence $\ker\varphi$ is discrete. Conversely, if $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is not injective, $\ker d\varphi_g$ is positive-dimensional for all $g \in G$ and thus defines a smooth distribution $\mathcal{D}$. Note that X lies in $\mathcal{D}$ if and only if X is φ-related to the null vector field on H. It follows that $\mathcal{D}$ is involutive. The maximal integral manifold of $\mathcal{D}$ through the identity is collapsed to a point under φ implying that $\ker\varphi$ is not discrete.

(b) Since $\varphi \circ \exp = \exp \circ d\varphi$ and G° is generated by $\exp[\mathfrak{g}]$, we have that $\varphi(G^\circ)$ is the subgroup of H° generated by $\exp[d\varphi(\mathfrak{g})]$, thus $\varphi(G^\circ) = H^\circ$ if $d\varphi$ is surjective. On the other hand, if $d\varphi$ is not surjective, $d\varphi(\mathfrak{g})$ is a proper subalgebra of $\mathfrak{h}$ to which there corresponds a connected, proper subgroup K of H°, and $\exp[d\varphi(\mathfrak{g})]$ generates K.

(c) Assume G, H connected. If φ is a covering then $\ker d\varphi$ is discrete and φ is surjective, so $d\varphi$ is an isomorphism by (a) and (b). Conversely, suppose that $d\varphi : \mathfrak{g} \to \mathfrak{h}$ an isomorphism. Then φ is surjective by (b). Let U be a neighborhood of 1 in G such that $\varphi : U \to \varphi(U) := V$ is a diffeomorphism. We can choose U so that $U \cap \ker d\varphi = \{1\}$ by (a). Then $\varphi^{-1}(V) = \cup_{n \in \ker\varphi} nU$, and this is a (disjoint union) for $ng = n'g'$ with n, $n' \in \ker\varphi$ and g, $g' \in U$ implies $gg'^{-1} = n^{-1}n' \in \ker\varphi$ and so $\varphi(g) = \varphi(g')$ and then $g = g'$. Since $\varphi \circ L_n = \varphi$ for $n \in \ker\varphi$, we also have that $\varphi|nU$ is a diffeomorphism onto

V. This shows that V is an evenly covered neighborhood of 1. Now hV is an evenly covered neighborhood of any given $h \in H$, and thus φ is a covering. □

3.4.5 Theorem *Let G_1, G_2 be Lie groups, and assume that G_1 is connected and simply-connected. Then, given a homomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between the Lie algebras, there exists a unique homomorphism $\varphi : G_1 \to G_2$ such that $d\varphi = \Phi$.*

Proof. The graph of Φ, $\mathfrak{h} = \{(X, \Phi(X)) : X \in \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let H be the subgroup of $G_1 \times G_2$ defined by $\mathfrak{h}$ (Theorem 3.3.5). Consider the projections

$$\Phi_i : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_i, \qquad \varphi_i : G_1 \times G_2 \to G_i,$$

for $i = 1$, 2. Since $\Phi_1|\mathfrak{h} : \mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism, we have that $\Phi = \Phi_2 \circ (\Phi_1|\mathfrak{h})^{-1}$ and $\varphi_1 : H \to G_1$ is a covering. Since G_1 is simply-connected, $\varphi_1|H : H \to G_1$ is an isomorphism of Lie groups, and we can thus define $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$. This proves the existence part. The uniqueness part comes from the fact that $d\varphi = \Phi$ specifies φ in a neighborhood of 1 (by using the exponential map as in (3.3.9)), and G_1 is generated by this neighborhood. □

3.5 The adjoint representation

Let G be a Lie group, and denote its Lie algebra by $\mathfrak{g}$. The noncommutativity of G is organized by the adjoint representation. In order to introduce it, let $g \in G$, and define a map $\mathrm{Inn}_g : G \to G$ by $\mathrm{Inn}_g(x) = gxg^{-1}$. Then Inn_g is an automorphism of G, which is called the *inner automorphism defined by* g. The differential $d(\mathrm{Inn}_g) : \mathfrak{g} \to \mathfrak{g}$ defines an automorphism of $\mathfrak{g}$, which we denote by Ad_g. Then

$$\mathrm{Ad}_g X = \frac{d}{dt}\Big|_{t=0} \mathrm{Inn}(g)(\exp tX) = \frac{d}{dt}\Big|_{t=0} g \exp tX g^{-1}.$$

3.5.1 Example *In case $G = GL(n, \mathbf{R})$, $\mathrm{Inn_g}$ is the restriction of the linear map $\mathbf{M}(n, \mathbf{R}) \to \mathbf{M}(n, \mathbf{R})$, $X \mapsto gXg^{-1}$, so $\mathrm{Ad}_g X = d(\mathrm{Inn}_g)_1(X) = gXg^{-1}$.*

This defines a homomorphism

$$\mathrm{Ad} : g \in G \to \mathrm{Ad}_g \in \mathbf{GL}(\mathfrak{g}),$$

which is called the *adjoint representation* of G on $\mathfrak{g}$.

We have

$$\begin{aligned}(\mathrm{Ad}_g X)_1 &= (dL_g)(dR_{g^{-1}})X_1 \\ &= (dR_{g^{-1}})(dL_g)X_1 \\ &= (dR_{g^{-1}})(X_g) \\ &= (dR_g^{-1}\circ X\circ R_g)(1) \\ &= \big((R_{g^{-1}})_*X\big)_1 .\end{aligned}$$

Recall that $\mathbf{GL}(\mathfrak{g})$ is itself a Lie group isomorphic to $\mathbf{GL}(n,\mathbf{R})$, where $n=\dim\mathfrak{g}$. Its Lie algebra consists of all linear endomorphisms of $\mathfrak{g}$ with the bracket $[A,B]=AB-BA$ and it is denoted by $\mathfrak{gl}(\mathfrak{g})$. Note that $\mathrm{Ad}_g = D_2F(g,1)$, where $F:G\times G\to G$ is the smooth function $F(g,x)=gxg^{-1}$, so the linear endomorphism Ad_g of $\mathfrak{g}$ depends smoothly on g. Now $\mathrm{Ad}: g\in G\to \mathrm{Ad}_g\in\mathbf{GL}(\mathfrak{g})$ is homomorphism of Lie groups and its differential $d(\mathrm{Ad})$ defines the *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad}: X\in\mathfrak{g}\to\mathrm{ad}_X = \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}_{\exp tX}\in\mathfrak{gl}(\mathfrak{g}).$$

Since $\varphi_t = R_{\exp tX}$ is the flow of X, we get

$$\mathrm{ad}_X Y = \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}_{\exp tX}Y = \frac{d}{dt}\Big|_{t=0}\big((R_{\exp(-tX)})_*Y\big)_1 = (L_XY)_1 = [X,Y].$$

As an important special case of (3.3.9), we have

$$\begin{aligned}\mathrm{Ad}_{\exp X} &= e^{\mathrm{ad}_X} \\ &= I+\mathrm{ad}_X+\frac{1}{2}\mathrm{ad}_X^2+\frac{1}{3!}\mathrm{ad}_X^3+\cdots\end{aligned}$$

for all $X\in\mathfrak{g}$.

3.5.2 Lemma *For given X, $Y\in\mathfrak{g}$, we have that $[X,Y]=0$ if and only if $\exp X\exp Y=\exp Y\exp X$. In that case, $\exp(t(X+Y))=\exp tX\exp tY$ for all $t\in\mathbf{R}$. It follows that a connected Lie group is Abelian if and only if its Lie algebra is Abelian.*

Proof. The first assertion is a special case of Proposition 1.6.23 using that $\varphi_t=R_{\exp tX}$ is the flow of X and $\psi_s = R_{\exp sY}$ is the flow of Y. The second one follows from noting that both $t\mapsto\exp(t(X+Y))$ and $t\mapsto\exp tX\exp tY$ are one-parameter groups with initial speed $X+Y$. Finally, we have seen that $\mathfrak{g}$ is Abelian if and only if $\exp[\mathfrak{g}]$ is Abelian, but the latter generates G°. □

3.5.3 Theorem *Every connected Abelian Lie group G is isomorphic to $\mathbf{R}^{n-k}\times T^k$. In particular, a simply-connected Abelian Lie group is isomorphic to $\mathbf{R}^n$ and a compact connected Abelian Lie group is isomorphic to T^n.*

Proof. It follows from Lemma 3.5.2 that $\mathfrak{g}$ is Abelian and $\exp : \mathfrak{g} \to G$ is a homomorphism, where $\mathfrak{g} \cong \mathbf{R}^n$ as a Lie group, thus $\exp$ is a smooth covering by Lemma 3.4.4(c). Hence G is isomorphic to the quotient of $\mathbf{R}^n$ by the discrete group $\ker \exp$. □

3.6 Homogeneous manifolds

Let G be a Lie group and let H be a closed subgroup. Consider the set G/H of left cosets of H in G equipped with the quotient topology with respect to the projection $\pi : G \to G/H$. Note also that left multiplication in G induces a map $\lambda : G \times G/H \to G/H$, namely, $\lambda(g, xH) = (gx)H$, and that

$$\pi \circ L_g = \lambda_g \circ \pi \tag{3.6.1}$$

for all $g \in G$, where $\lambda_g(p) = \lambda(g, p)$ for $p \in G/H$.

3.6.2 Lemma *A closed Lie subgroup H of a Lie group G must have the induced topology.*

Proof. We need to prove that the inclusion map $\iota : H \to G$ is an embedding. Since ι commutes with left translations, it suffices to find an open subset V of H such that the restriction $\iota|_V$ is an embedding into G. By the proof of Theorem 3.3.5, there exists a distinguished chart $(U, \varphi = (x_1, \ldots, x_n))$ of G around 1 such that $H \cap U$ consists of at most countably many plaques, each plaque being a slice of the form

$$x_{k+1} = c_{k+1}, \ \ldots, \ x_n = c_n$$

for some $c_{k+1}, \ldots, c_n \in \mathbf{R}$, where $k = \dim H$. Denote by $\tau : \mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ the projection. Let $\tilde{U}$ be a compact neighborhood of 1 contained in U. Now $H \cap \tilde{U}$ is compact, so $\tau(H \cap \tilde{U})$ is a non-empty closed countable subset of $\mathbf{R}^{n-k}$ which by the Baire category theorem must have an isolated point. This point specifies a isolated plaque V of H in U along which ι is an open mapping and hence a homeomorphism onto its image, as desired. □

3.6.3 Theorem *If G is a Lie group and H is a closed subgroup of G, then there is a unique smooth structure on the topological quotient G/H such that $\lambda : G \times G/H \to G/H$ is smooth. Moreover, $\pi : G \to G/H$ is a surjective submersion and* $\dim G/H = \dim G - \dim H$.

Proof. For an open set V of G/H we have that $\pi^{-1}(\pi(V)) = \bigcup_{g \in G} gV$ is a union of open sets and thus open. This shows that π is an open map and hence the projection of a countable basis of open sets of G yields a countable basis of open sets of G/H. To prove that G/H is Hausdorff, we use closedness of H. Indeed it implies that the equivalence relation $\mathcal{R} \subset$

$G \times G$, defined by specifying that $g \sim g'$ if and only if $g^{-1}g' \in H$, is a closed subset of $G \times G$. Now if $gH \neq g'H$ in G/H then $(g, g') \notin \mathcal{R}$ and there exist open neighborhoods W of g and W' of g' in G such that $(W \times W') \cap \mathcal{R} = \varnothing$. It follows that $\pi(W)$ and $\pi(W')$ are disjoint neighborhoods of g and g' in G/H, respectively.

We first construct a local chart of G/H around $p_0 = \pi(1) = 1H$. Recall from Proposition 3.2.4 and (3.3.9) that the exponential map $\exp = \exp^G$ gives a parametrization of G around the identity element and restricts to the exponential map of $\mathfrak{h}$. Denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, resp., and choose a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. We can choose a product neighborhood of 0 in $\mathfrak{g}$ of the form $U_0 \times V_0$, where U_0 is a neighborhood of 0 in $\mathfrak{h}$, V_0 is a neighborhood of 0 in $\mathfrak{m}$ such that the map

$$f : V_0 \times U_0 \to G, \qquad f(X, Y) = \exp X \exp Y$$

is a diffeomorphism from onto its image (apply the Inverse Function Theorem 1.3.8 to f). Owing to Lemma 3.6.2, H has the topology induced from G, so we may choose a neighborhood W of 1 in G such that $W \cap H = \exp(U_0)$. We also shrink V_0 so that $(\exp V_0)^{-1} \exp V_0 \subset W$. Now we claim that $\pi \circ \exp|_{V_0}$ is injective. Indeed, if $\pi(\exp X) = \pi(\exp X')$ for some $X, X' \in V_0$, then $(\exp X)^{-1} \exp X' \in H \cap W = \exp(U_0)$, so $\exp X' = \exp X \exp Y$ for some $Y \in U_0$. Since f is injective on $U_0 \times V_0$, this implies that $X' = X$ and $Y = 0$ and proves the claim. Note $\exp V_0 \exp U_0$ is open in G, so the image $\pi(\exp V_0) = \pi(\exp V_0 \exp U_0)$ is open in G/H. We have shown that $\pi \circ \exp$ defines a homeomorphism from V_0 onto the open neighborhood $V = \pi(\exp V_0)$ of p in G/H, whose inverse can then be used to define a local chart (V, ψ) of G/H around p_0.

Now the collection $\{(V^g, \psi^g)\}_{g \in G}$ defines an atlas of G/H, where $V^g = gV$ and $\psi^g = \psi \circ L_{g^{-1}}$, and we need to check the its smoothness. Suppose g, $g' \in G$ are such that $V^g \cap V^{g'} \neq \varnothing$, and that $p = (g \exp X)H = (g' \exp X')H$ is an element there, namely, $\psi^g(p) = X$ and $\psi^{g'}(p) = X'$. Then $\exp X' = (g')^{-1} g \exp X h \in \exp V_0$ for some $h \in H$, so there exists a neighborhood $\tilde{V_0}$ of X in V_0 such that $(g')^{-1} g(\exp \tilde{V_0})h \subset V_0$, and thus $\psi^{g'} \circ (\psi^g)^{-1}|_{\tilde{V_0}}$ can be written as the composite map

$$\tau \circ \log \circ R_h \circ L_{(g')^{-1}g} \circ \exp,$$

where $\log$ denotes the inverse map of $\exp : U_0 \times V_0 \to \exp(U_0 \times V_0)$, and $\tau : \mathfrak{g} \to \mathfrak{m}$ denotes the projection along $\mathfrak{h}$. Hence the change of charts $\psi^{g'} \circ (\psi^g)^{-1}$ is smooth.

The local representation of π is τ in the above charts is τ, namely, there is a commutative diagram

$$\begin{tikzcd} g\exp V_0 \exp U_0 \arrow[r, "\pi"] \arrow[d, "f^{-1}\circ L_{g^{-1}}"'] & V^g \\ U_0\times V_0 \arrow[r, "\tau"'] & V_0 \arrow[u, "L_g\circ\pi\circ\exp"'] \end{tikzcd}$$

which shows that π is a submersion. Similarly, the commutative diagram

$$\begin{tikzcd} G\times g\exp V_0 \arrow[r] \arrow[d, "\mathrm{id}\times\pi"'] & G \arrow[d, "\pi"] \\ G\times V^g \arrow[r, "\lambda|_{G\times V^g}"'] & G/H \end{tikzcd}$$

proves that λ is smooth. The uniqueness of the smooth structure follows from Proposition 3.6.4 below. $\square$

Let M be a smooth manifold and let G be a Lie group. An *action* of G on M is a smooth map $\mu : G\times M \to M$ such that $\mu(1,p) = p$ and $\mu(g,\lambda(g',p)) = \mu(gg',p)$ for all $p \in M$ and g, $g' \in G$. For brevity of notation, in case μ is fixed and clear from the context, we will simply write $\mu(g,p) = gp$.

An action of G is M is called *transitive* if for every p, $q \in M$ there exists $g \in G$ such that $gp = q$. In this case, M is called *homogeneous under G*, *G-homogeneous*, or simply a *homogeneous manifold*. Examples of homogeneous manifolds are given by the quotients G/H, where H is closed Lie subgroup of G, according to Theorem 3.6.3. Conversely, the next proposition that every homogeneous manifold is of this form. For an action of G on M and $p \in M$, the *isotropy group* at p is the subgroup G_p of G consisting of elements that fix p, namely, $G_p = \{\, g \in G \mid gp = p\,\}$. Plainly, G_p is a closed subgroup of G, and so a Lie subgroup of G, owing to Theorem 3.7.1 below.

3.6.4 Proposition *Let $\mu : G\times M \to M$ be a transitive action of a Lie group G on a smooth manifold M. Fix $p_0 \in M$ and let $H = G_{p_0}$ be the isotropy group at p_0. Define a map*

$$f : G/H \to M, \qquad f(gH) = \mu(g,p_0).$$

Then f is well-defined and a diffeomorphism.

Proof. It is easy to see that f is well-defined, bijective and smooth. We can write $f\circ\pi = \omega$, where $\omega : G \to M$ is the "orbit map" $\omega(g) = gp_0$. For $X \in \mathfrak{g}$, we have

$$d\omega_1(X) = \frac{d}{ds}\Big|_{s=0}(\exp sX)p_0 = d(\exp(-sX))\frac{d}{dt}\Big|_{t=s}(\exp tX)p_0,$$

so $X \in \ker d\omega_1$ if and only if $\exp tX \in H$ for all $t \in \mathbf{R}$ if and only if X belongs to the Lie algebra $\mathfrak{h}$ of H, due to (3.3.10). Since $df_{1H} \circ d\pi_1 = d\omega_1$ and $\ker d\pi_1 = \mathfrak{h}$, this implies that f is an immersion at $1H$, and thus an immersion everywhere by the equivariance property $f \circ \lambda_g = \mu_g \circ f$ for all $g \in G$.

This already implies that $\dim G/H \leq \dim M$ and that $(G/H, f)$ is a submanifold of M, but the strict inequality cannot hold as f is bijective and the image of a smooth map from a smooth manifold into a strictly higher dimensional smooth manifold has null measure (this result follows from the statement that the image of a smooth map $\mathbf{R}^n \to \mathbf{R}^{n+k}$ with $k > 0$ has null measure, and the second countability of smooth manifolds). It follows that f is a local diffeomorphism and hence a diffeomorphism. □

3.6.5 Examples (a) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$ and view elements of $\mathbf{R}^n$ as column-vectors ($n \times 1$ matrices). Then $GL(n, \mathbf{R})$ acts on $\mathbf{R}^n$ by left-multiplication:

$$GL(n, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n \tag{3.6.6}$$

The basis $\{e_i\}$ is orthonormal with respect to the standard scalar product in $\mathbf{R}^n$. The orthogonal group $O(n)$ precisely consists of those elements of $GL(n, \mathbf{R})$ whose action on $\mathbf{R}^n$ preserves the lengths of vectors. In particular, the action (3.6.6) restricts to an action

$$O(n) \times S^{n-1} \to S^{n-1} \tag{3.6.7}$$

which is smooth, since S^{n-1} is an embedded submanifold of $\mathbf{R}^n$. The action (3.6.7) is transitive due to the facts that any unit vector can be completed to an orthonormal basis of $\mathbf{R}^n$, and any two orthonormal bases of $\mathbf{R}^n$ differ by an orthogonal transformation. The isotropy group of (3.6.7) at e_1 consists of transformations that leave the orthogonal complement $e_1^\perp$ invariant, and indeed any orthogonal transformation of $e_1^\perp \cong \mathbf{R}^{n-1}$ can occur. It follows that the isotropy group is isomorphic to $O(n-1)$ and hence

$$S^{n-1} = O(n)/O(n-1)$$

presents the unit sphere as a homogeneous space, where a the diffeomorphism is given by $gO(n-1) \mapsto g(e_1)$. If we use only orientation-preserving transformations on $\mathbf{R}^n$, also the elements of the isotropy group will act by orientation-preserving transformations and hence

$$S^{n-1} = SO(n)/SO(n-1).$$

(b) The group $SO(n)$ also acts transtively on the set of lines through the origin in $\mathbf{R}^n$. Besides the orthogonal transformations of $e_1^\perp$, the isotropy group at the line $\mathbf{R}e_1$ now also contains transformations that map e_1 to $-e_1$. It follows that

$$\mathbf{R}P^n = SO(n)/O(n-1)$$

where $O(n-1)$ is identified with the subgroup of $SO(n)$ consisting of matrices of the form

$$\begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}$$

where $A \in O(n-1)$.

(c) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{C}^n$. It is a unitary basis with respect to the standard Hermitian inner product in $\mathbf{C}^n$. Similarly to (a), one shows that $U(n)$ and $SU(n)$ act transitively on the set of unit vectors of $\mathbf{C}^n$, namely, the sphere S^{2n-1}. More interesting is to consider the set $\mathbf{C}P^{n-1}$ of one-dimensional complex subspaces of C^n. This set is homogeneous under $SU(n)$ and the isotorpy group at the line $\mathbf{C}e_1$ consists of matrices of the form

$$\begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix}$$

where $A \in U(n-1)$. It follows from Theorem 3.6.3 that $\mathbf{C}P^{n-1}$ is a smooth manifold and

$$\mathbf{C}P^{n-1} = SU(n)/U(n-1)$$

as a homogeneous manifold.

(d) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$, and let $V_k(\mathbf{R}^n)$ be the set of orthonormal k-frames in $\mathbf{R}^n$, that is, ordered k-tuples of orthonormal vectors in $\mathbf{R}^n$. There is an action

$$O(n) \times V_k(\mathbf{R}^n) \to V_k(\mathbf{R}^n), \quad g \cdot (v_1, \ldots, v_k) = (gv_1, \ldots, gv_k)$$

which is clearly transitive. The isotropy group at $(e_1, \ldots, e_k)$ is the subgroup of $O(n)$ consisting of matrices of the form

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \tag{3.6.8}$$

where $A \in O(n-k)$. The resulting homogeneous space

$$V_k(\mathbf{R}^n) = O(n)/O(n-k)$$

is called the *Stiefel manifold* of k-frames in $\mathbf{R}^n$. Note that the restricted action of $SO(n)$ on $V_k(\mathbf{R}^n)$ is also transitive and

$$V_k(\mathbf{R}^n) = SO(n)/SO(n-k).$$

3.7 Additional results

In this section, we state without proofs some important, additional results about basic Lie theory, and add some remarks.

Closed subgroups

3.7.1 Theorem *Let G be a Lie group, and let A be a closed (abstract) subgroup of G. Then A admits a unique manifold structure whch makes it into a Lie group; moreover, the topology in this manifold structure must be the relative topology.*

3.7.2 Corollary *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then $A = \ker\varphi$ is a closed Lie subgroup of G with Lie algebra $\mathfrak{a} = \ker d\varphi$.*

Proof. A is a closed subgroup and hence a Lie subgroup of G by Theorem 3.7.1. The rest follows from (3.3.9) and (3.3.10). □

Continuous homomorphisms

3.7.3 Theorem *Let $\varphi : G \to H$ be a* continuous *homomorphism between Lie groups. Then φ is smooth.*

3.7.4 Definition A *topological group* is an abstract group equipped with a topology such that the group operations are continuous maps.

3.7.5 Corollary *A Hausdorff second countable locally Euclidean group G can have at most one smooth structure making it into a Lie group.*

Proof. Let $[\mathcal{A}]$ and $[\mathcal{B}]$ two such smooth structures on G. The identity map $(G,[\mathcal{A}]) \to (G,[\mathcal{B}])$ is a homomorphism and a homeomorphism, and hence a diffeomorphism by Theorem 3.7.3. □

Hilbert's fifth problem is the fifth mathematical problem posed by David Hilbert in his famous address to the International Congress of Mathematicians in 1900. One (restricted) interpretation of the problem in modern language asks whether a connected (Hausdorff second countable) locally Euclidan group admits a smooth structure which makes it into a Lie group. In 1952, A. Gleason proved that a locally compact group satisfying the "no-small subgroups" (NSS) condition (compare Problem 11) is a Lie group, and then immediately afterwards Montgomery and Zippin used Gleason's result to prove inductively that locally Euclidean groups of any dimension satisfy NSS. The two papers appeared together in the same issue of the Annals of Mathematics. Here one says that a topological group satisfies *NSS* if there exists a neighborhood of the identity which contains no subgroups other than the trivial group. (Actually, the above is not quite the full story; Gleason assumed a weak form of finite dimensionality in his original argument that NSS implies Lie, but shortly thereafter Yamabe showed that finite dimensionality was not needed in the proof.)

Theorem of Ado

A (real) *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbf{R})$; if, in addition, φ is injective, it is called a *faithful* representation.

A faithful representation of a Lie algebra $\mathfrak{g}$ can be thought of a "linear picture" of $\mathfrak{g}$ and allows one to view $\mathfrak{g}$ as a Lie algebra of matrices.

3.7.6 Theorem (Ado) *Every Lie algebra admits a faithful representation.*

3.7.7 Theorem *There is a bijective correspondence between isomorphism classes of Lie algebras and isomorphism classes of simply-connected Lie groups.*

Proof. If $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbf{R})$ by Theorem 3.7.6. Owing to Theorem 3.3.5, there is a connected Lie subgroup of $\mathbf{GL}(n, \mathbf{R})$ with Lie algebra $\mathfrak{g}$. Due to Theorem 3.4.1 and Lemma 3.4.4(c), there is also a simply-connected Lie group with Lie algebra $\mathfrak{g}$. Two simply-connected Lie groups with isomorphic Lie algebras are isomorphic in view of Theorem 3.4.5. □

Theorem of Yamabe

3.7.8 Theorem (Yamabe) *An arcwise connected subgroup of a Lie group is a Lie subgroup.*

3.7.9 Corollary *Let G be a connected Lie group and let A and B be connected Lie subgroups. Then the subgroup (A, B) generated by the commutators*

$$S = \{aba^{-1}b^{-1} \ : \ a \in A, \ b \in B\}$$

is a Lie subgroup of G. In particular, the commutator of G, (G, G), is a Lie subgroup of G.

Proof. As a continuous image of $A \times B$, S is arcwise connected, and so is $T = S \cup S^{-1}$, since $S \cap S^{-1} \ni 1$. As a continuous image of $T \times \cdots \times T$ (n factors) also T^n is arcwise connected and hence so is $(A, B) = \cup_{n \geq 1} T^n$, since it is an increasing union of arcwise connected subsets. The result follows from Yamabe's theorem 3.7.8. □

3.7.10 Example In general, the subgroup (A, B) does not have to be closed for closed connected subgroups A and B of G, even if G is simply-connected. Indeed, take G to be the simply-connected covering of $\mathbf{SL}(4, \mathbf{R})$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be one-dimensional and respectively spanned by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Then A and B are closed one-dimensional subgroups isomorphic to $\mathbf{R}$ but their commutator is a dense line in a torus.

3.8 Problems

§ 3.1

1 Let $\alpha,\ \beta : (-\epsilon, \epsilon) \to G$ be smooth curves in a Lie group G such that $\alpha(0) = \beta(0) = 1$, and consider $\gamma(t) = \alpha(t)\beta(t)$. Prove that $\dot\gamma(0) = \dot\alpha(0) + \dot\beta(0)$. (Hint: consider the multiplication map $\mu : G \times G \to G$ and show that $d\mu(v, w) = d\mu((v, 0) + (0, w)) = v + w$ for $v, w \in T_1G$.)

2 *a.* Show that

$$\mathbf{SO}(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ : \ a,\ b \in \mathbf{R},\ a^2 + b^2 = 1 \right\}.$$

Deduce that $\mathbf{SO}(2)$ is diffeomorphic to S^1.

b. Show that

$$\mathbf{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \ : \ \alpha,\ \beta \in \mathbf{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Deduce that $\mathbf{SU}(2)$ is diffeomorphic to S^3.

3 Let

$$H^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \quad x, y, z \in \mathbf{R} \right\}.$$

a. Prove that H^3 is closed under matrix multiplication and it has the structure of a Lie group (the so called *Heisenberg group*).

b. Show that $A = \frac{\partial}{\partial x}$, $B = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $C = \frac{\partial}{\partial z}$ are left-invariant vector fields. Compute their Lie brackets.

c. Describe the Lie algebra of H^3.

4 Classify all real Lie algebras of dimension two and three.

5 Let $G = \mathbf{O}(n)$.

a. Show that $G^\circ \subset \mathbf{SO}(n)$.

b. Prove that any element in $\mathbf{SO}(n)$ is conjugate in G to a matrix of the form

$$\begin{pmatrix} R_{t_1} & & & & & \\ & \ddots & & & & \\ & & R_{t_p} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where R_t is the 2×2 block

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and $t_1, \ldots, t_p \in \mathbf{R}$.

c. Deduce from the above that $\mathbf{SO}(n)$ is connected. Conclude that $\mathbf{O}(n)$ has two connected components and $\mathbf{SO}(n)$ is the identity component.

§ 3.2

6 Show that

$$\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in \mathbf{R}$.

7 Give examples of matrices A, $B \in \mathfrak{gl}(2, \mathbf{R})$ such that $e^{A+B} \neq e^A e^B$.

8 In this problem, we show that the exponential map in a Lie group does not have to be surjective.

a. Show that every element g in the image of $\exp : \mathfrak{g} \to G$ has a square root, namely, there is $h \in G$ such that $h^2 = g$.

b. Prove that $\operatorname{trace} A^2 \geq -2$ for any $A \in \mathbf{SL}(2, \mathbf{R})$ (Hint: A satisfies its characteristic polynomial $X^2 - 2(\operatorname{trace} X)X + (\det X)I = 0$.)

c. Deduce from the above that $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ does not lie in the image of $\exp : \mathfrak{sl}(2, \mathbf{R}) \to \mathbf{SL}(2, \mathbf{R})$.

9 Let $X \in \mathfrak{sl}(2, \mathbf{R})$. Show that

$$e^X = \begin{cases} \cosh(-\det X)^{1/2} I + \frac{\sinh(-\det X)^{1/2}}{(-\det X)^{1/2}} X & \text{if } \det X < 0, \\ \cos(\det X)^{1/2} I + \frac{\sin(\det X)^{1/2}}{(\det X)^{1/2}} X & \text{if } \det X > 0, \\ I + X & \text{if } \det X = 0. \end{cases}$$

10 (*Polar decomposition of matrices*)

a. Prove that any $g \in \mathbf{GL}(n, \mathbf{R})$ can be written as $g = hk$ where $h \in \mathbf{O}(n)$ and k is a positive-definite symmetric matrix.

b. Prove that the exponential map defines a bijection between the space of real symmetric matrices and the set of real positive-definite symmetric matrices. (Hint: Prove it first for diagonal matrices.)

c. Deduce from the above that $\mathbf{GL}(n, \mathbf{R})$ is diffeomorphic to $\mathbf{O}(n) \times \mathbf{R}^{\frac{n(n+1)}{2}}$.

11 Let G be a Lie group. Prove that it does not have small subgroups; i.e., prove the existence of an open neighborhood of 1 such that $\{1\}$ is the only subgroup of G entirely contained in U.

12 For a connected Lie group, prove that the second-countability of its topology is a consequence of the other conditions in the definition of a Lie group. (Hint: Use Proposition 3.2.5).

§ 3.3

13 Check that

$$A + iB \in \mathbf{GL}(n, \mathbf{C}) \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbf{GL}(2n, \mathbf{R})$$

defines an injective homomorphism φ of $\mathbf{GL}(n, \mathbf{C})$ onto a closed subgroup of $\mathbf{GL}(2n, \mathbf{R})$. Check also that φ restricts to an injective homomorphism of $\mathbf{U}(n)$ onto a closed subgroup of $\mathbf{SO}(2n)$.

14 Prove that a discrete normal subgroup of a connected Lie group is central.

15 Determine the center of $\mathbf{SU}(n)$.

16 Construct a diffeomorphism between $\mathbf{U}(n)$ and $S^1 \times \mathbf{SU}(n)$. Is it an isomorphism of Lie groups?

§ 3.4

17 Consider $G = \mathbf{SU}(2)$ and its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$.

a. Check that

$$\mathfrak{g} = \left\{ \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} : x,\ y,\ z \in \mathbf{R} \right\}.$$

b. Identify $\mathfrak{g}$ with $\mathbf{R}^3$ and check that $\det : \mathfrak{g} \to \mathbf{R}$ corresponds to the usual quadratic form on $\mathbf{R}^3$. Check also that Ad_g preserves this quadratic form for all $g \in G$.

c. Deduce form the above that there is a smooth homomorphism $\mathbf{SU}(2) \to \mathbf{SO}(3)$ which is the simply-connected covering of $\mathbf{SO}(3)$.

§ 3.5

18 Prove that the kernel of the adjoint representation of a connected Lie group coincides with its center.

19 Let A be a connected subgroup of a connected Lie group G. Prove that A is a normal subgroup of G if and only if the Lie algebra $\mathfrak{a}$ of A is an ideal of the Lie algebra $\mathfrak{g}$ of G.

§ 3.6

20 a. Let $\mathrm{Gr}_k(\mathbf{R}^n)$ be the set of k-dimensional subspaces of $\mathbf{R}^n$. Prove that

$$\mathrm{Gr}_k(\mathbf{R}^n) = SO(n)/S(O(k) \times O(n-k)).$$

This is called the *Grassmann manifold* of k-planes in $\mathbf{R}^n$.

b. Consider now the set $\mathrm{Gr}_k^+(\mathbf{R}^n)$ of *oriented* k-dimensional subspaces of $\mathbf{R}^n$, and prove that

$$\mathrm{Gr}_k(\mathbf{R}^n) = SO(n)/SO(k) \times SO(n-k).$$

This is called the Grassmann manifold of oriented k-planes in $\mathbf{R}^n$.

c. Define the Grassmann manifold $\mathrm{Gr}_k(\mathbf{C}^n)$ of k-planes in $\mathbf{C}^n$ and prove that

$$\begin{aligned}\mathrm{Gr}_k(\mathbf{C}^n) &= U(n)/\left[U(k) \times U(n-k)\right] \\ &= SU(n)/S(U(k) \times SU(n-k)).\end{aligned}$$