CHAPTER A

Covering manifolds

In this appendix, we summarize some properties of covering spaces in the context of smooth manifolds.

A.1 Topological coverings

Recall that a (topological) *covering* of a space X is another space $\tilde{X}$ with a continuous map $\pi : \tilde{X} \to X$ such that X is a union of evenly covered open set, where a connected open subset U of X is called *evenly covered* if

$$\pi^{-1}U = \cup_{i \in I} \tilde{U}_i \qquad \text{(A.1.1)}$$

is a disjoint union of open sets $\tilde{U}_i$ of $\tilde{X}$, each of which is mapped homeomorphically onto U under π. In particular, the fibers of π are discrete subsets of $\tilde{X}$. It also follows from the definition that $\tilde{X}$ has the Hausdorff property if X does. Further it is usual, as we shall do, to require that X and $\tilde{X}$ be connected, and then the index set I can be taken the same for all evenly covered open sets.

A.1.2 Examples *(a)* $\pi : \mathbf{R} \to S^1$, $\pi(t) = e^{it}$ *is a covering.*

(b) $\pi : S^1 \to S^1$, $\pi(z) = z^n$ *is a covering for any nonzero integer* n.

(c) $\pi : (0, 3\pi) \to S^1$, $\pi(t) = e^{it}$ *is a local homemeomorphism which is not a covering, since* $1 \in S^1$ *does not admit evenly covered neighborhoods.*

A.2 Fundamental groups

Covering spaces are closely tied with fundamental groups. The *fundamental group* $\pi_1(X, x_0)$ of a topological space X with basepoint x_0 is defined as follows. As a set, it consists of the homotopy classes of continuous loops based at x_0. The concatenation of such loops is compatible with the equivalence relation given by homotopy, so it induces a group operation on $\pi_1(X, x_0)$ making it into a group. If X is arcwise connected, the isomorphism class of the fundamental group is independent of the choice of basepoint (indeed for x_0, $x_1 \in X$ and c a continuous path from x_0 to x_1, conjugation by c^{-1} induces an isomorphism from $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$) and thus is sometimes denoted by $\pi_1(X)$. Finally, a continuous map $f : X \to Y$ between topological spaces with $f(x_0) = y_0$ induces a homomorphism $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ so that the assignment $(X, x_0) \to \pi_1(X, x_0)$ is functorial. Of course the fundamental group is trivial if and only if the space is simply-connected.

Being locally Euclidean, a smooth manifold is locally arcwise connected and locally simply-connected. A connected space X with such local connectivity properties admits a simply-connected covering space, which is unique up to isomorphism; an isomorphism between coverings $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $\pi_2 \circ f = \pi_1$. More generally, there exists a bijective correspondance between isomorphism classes of coverings $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$ given by $(\tilde{X}, \tilde{x}_0) \mapsto \pi_\#(\pi_1(\tilde{X}, \tilde{x}_0))$; moreover, a change of basepoint in $\tilde{X}$ corresponds to passing to a conjugate subgroup $\pi_1(X, x_0)$.

A.3 Smooth coverings

Suppose $\pi : \tilde{M} \to M$ is a covering where M is a smooth manifold. Then there is a natural structure of smooth manifold on $\tilde{M}$ such that the projection π is smooth. In fact, for every chart (U, π) of M where U is evenly covered as in (A.1.1), take a chart $(\tilde{U}_i, \varphi \circ \pi|_{\tilde{U}_i})$ for $\tilde{M}$. This gives an atlas of $\tilde{M}$, which is smooth because for another chart (V, ψ) of M, V evenly covered by $\cup_{i \in I} \tilde{V}_i$ and $\tilde{U}_i \cap \tilde{V}_j \neq \varnothing$ for some $i, j \in I$, we have that the transition map

$$(\psi \circ \pi|_{\tilde{V}_j})(\varphi \circ \pi|_{\tilde{U}_i})^{-1} = \psi \circ \varphi^{-1}$$

is smooth. We already know that $\tilde{M}$ is a Hausdorff space. It is possible to choose a countable basis of connected open sets for M which are evenly covered. The connected components of the preimages under π of the elements of this basis form a basis of connected open sets for $\tilde{M}$, which is countable as long as the index set I is countable, but this follows from the countability of the fundamental group $\pi_1(M)$[1]. Now, around any point in $\tilde{M}$, π admits a local representation as the identity, so it is a local diffeomorphism. Note that we have indeed proved more: M can be covered by evenly covered neighborhoods U such that the restriction of π to a connected component of $\pi^{-1}U$ is a diffeomorphism onto U. This is the definition of a *smooth covering*. Note that a topologic covering whose covering map is smooth need not be a smooth covering (e.g. $\pi : \mathbf{R} \to \mathbf{R}$, $\pi(x) = x^3$).

[1] Ref?

Next, we can formulate basic results in covering theory for a smooth covering $\pi : \tilde{M} \to M$ of a smooth manifold M. Fix basepoints $\tilde{p} \in \tilde{M}$, $p \in M$ such that $\pi(\tilde{p}) = p$. We say that a map $f : N \to M$ admits a lifting if there exists a map $\tilde{f} : N \to \tilde{M}$ such that $\pi \circ \tilde{f} = f$.

A.3.1 Theorem (Lifting criterion) *Let $q \in f^{-1}(p)$. A smooth map $f : N \to M$ admits a smooth lifting $\tilde{f} : N \to \tilde{M}$ with $\tilde{f}(q) = \tilde{p}$ if and only if $f_{\#}(\pi_1(N, q)) \subset \pi_{\#}(\pi_1(\tilde{M}, \tilde{p}))$. In that case, if N is connected, the lifting is unique.*

Taking $f : N \to M$ to be the universal covering of M in Theorem A.3.1 shows that the universal covering of M covers any other covering of M and hence justifies its name.

A.4 Deck transformations

For a topological covering $\pi : \tilde{X} \to X$, a *deck transformation* or *covering transformation* is an isomorphism $\tilde{X} \to \tilde{X}$, namely, a homeomorphism $f : \tilde{X} \to \tilde{X}$ such that $\pi \circ f = \pi$. The deck transformations form a group under composition. It follows from uniqueness of liftings that a deck transformation is uniquely determined by its action on one point. In particular, the only deck transformation admitting fixed points is the identity. Since a smooth covering map $\pi : \tilde{M} \to M$ is a local diffeomorphism, in this case the equation $\pi \circ f = \pi$ implies that deck transformations are diffeomorphisms of $\tilde{M}$.

An action of a (discrete) group on a topological space (resp. smooth manifold) is a homomorphism from the group to the group of homeomorphisms (resp. diffeomorphisms) of the space (resp. manifold). For a smooth manifold M, we now recall the canonical action of $\pi_1(M, p)$ on its universal covering $\tilde{M}$ by deck transformations. First we remark that by the lifting criterion, given $q \in M$ and $\tilde{q}_1$, $\tilde{q}_2 \in \pi^{-1}(q)$, there is a unique deck transformation mapping $\tilde{q}_1$ to $\tilde{q}_2$. Now let γ be a continuous loop in M based at p representing an element $[\gamma] \in \pi_1(M, p)$. By the remark, it suffices to describe the action of $[\gamma]$ on a point $\tilde{p} \in \pi^{-1}(p)$, which goes as follows: lift γ uniquely to a path $\tilde{\gamma}$ starting at $\tilde{p}$; then $[\gamma] \cdot \tilde{p}$ is by definition the endpoint of $\tilde{\gamma}$, which sits in the fiber $\pi^{-1}(p)$. The definiton independs of the choice made, namely, if we change γ to a homotopic curve, we get the same result. This follows from Theorem A.3.1 applied to the homotopy, as it is defined on a square and a square is simply-connected. Since $\pi : \tilde{M} \to M$ is the universal covering, every deck transformation is obtained in this way from an element of $\pi_1(M, p)$.

As action of a (discrete) group Γ on a topological space X is called *free* if no nontrivial element of Γ has fixed points, and it is called *proper* if any two points x, $y \in X$ admit open neighborhoods $U \ni x$, $V \ni y$ such that $\{ \gamma \in \Gamma \mid \gamma U \cap V \neq \varnothing \}$ is finite. The action of $\pi_1(M, p)$ on the universal covering $\tilde{M}$ by deck transformations has both properties. In fact, we have already remarked it is free. To check properness, let $\tilde{p}$, $\tilde{q} \in \tilde{M}$. If these points lie in the same orbit of $\pi_1(M, p)$ or, equivalently, the same fiber of π, the required neighborhoods are the connected components of $\pi^{-1}(U)$ containing $\tilde{p}$ and $\tilde{q}$, resp., where U is an evenly covered neighborhood of $\pi(\tilde{p}) = \pi(\tilde{q})$. On the other hand, if $\pi(\tilde{p}) =: p \neq q := \pi(\tilde{p})$, we use the Hausdorff property of M to find disjoint evenly covered neighborhoods $U \ni p$, $V \ni q$ and then it is clear that the connected component of $\pi^{-1}(U)$ containing $\tilde{p}$ and the connected component of $\pi^{-1}(V)$ containing $\tilde{q}$ do the job.

Conversely, we have:

A.4.1 Theorem *If the group Γ acts freely and properly on a smooth manifold $\tilde{M}$, then the quotient space $M = \Gamma\backslash\tilde{M}$ endowed with the quotient topology admits a unique structure of smooth manifold such that the projection $\pi : \tilde{M} \to M$ is a smooth covering.*

Proof. The action of Γ on $\tilde{M}$ determines a partition into equivalence classes or *orbits*, namely $\tilde{p} \sim \tilde{q}$ if and only if $\tilde{q} = \gamma\tilde{p}$ for some $\gamma \in \Gamma$. The orbit through $\tilde{p}$ is denoted $\Gamma(\tilde{p})$. The quotient space $\Gamma\backslash\tilde{M}$ is also called *orbit space*.

The quotient topology is defined by the condition that $U \subset M$ is open if and only if $\pi^{-1}(U)$ is open in $\tilde{M}$. In particular, for an open set $\tilde{U} \subset \tilde{M}$ we have $\pi^{-1}(\pi(\tilde{U})) = \cup_{\gamma\in\Gamma}\gamma(\tilde{U})$, a union of open sets, showing that $\pi(\tilde{U})$ is open and proving that π is an open map. In particular, π maps a countable basis of open sets in $\tilde{M}$ to a countable basis of open sets in M.

The covering property follows from the fact that Γ is proper. In fact, let $\tilde{p} \in \tilde{M}$. From the definition of proeprness, we can choose a neighborhood $\tilde{U} \ni \tilde{p}$ such that $\{ \gamma \in \Gamma \mid \gamma\tilde{U} \cap \tilde{U} \neq \varnothing \}$ is finite. Using the Hausdorff property of $\tilde{M}$ and the freeness of Γ, we can shrink $\tilde{U}$ so that this set becomes empty. Now the map π identifies all disjoint homeomorphic open sets γU for $\gamma \in \Gamma$ to a single open set $\pi(U)$ in M, which is then evenly covered.

The Hausdorff property of M also follows from properness of Γ. Indeed, let p, $q \in M$, $p \neq q$. Choose $\tilde{p} \in \pi^{-1}(p)$, $\tilde{q} \in \pi^{-1}(q)$ and neighborhoods $\tilde{U} \ni \tilde{p}$, $\tilde{V} \ni \tilde{q}$ such that $\{ \gamma \in \Gamma \mid \gamma\tilde{U} \cap \tilde{V} \neq \varnothing \}$ is finite. Note that $\tilde{q} \notin \Gamma(\tilde{p})$, so by the Hausdorff property for $\tilde{M}$, we can shrink $\tilde{U}$ so that this set becomes empty. Since π is open, $U := \pi(\tilde{U})$ and $V := \pi(\tilde{V})$ are now disjoint neighborhoods of p and q, respectively.

Finally, we construct a smooth atlas for M. Let $p \in M$ and choose an evenly covered neighborhood $U \ni p$. Write $\pi^{-1}U = \cup_{i\in I}\tilde{U}_i$ as in (A.1.1). By shrinking U we can ensure that $\tilde{U}_i$ is the domain of a local chart $(\tilde{U}_i, \tilde{\varphi}_i)$ of $\tilde{M}$. Now $\varphi_i := \tilde{\varphi}_i \circ (\pi|_{\tilde{U}_i})^{-1} : U \to \mathbf{R}^n$ defines a homeomorphism onto the open set $\tilde{\varphi}_i(\tilde{U}_i)$ and thus a local chart (U, φ_i) of M. The domains of such charts cover M and it remains only to check that the transition maps are smooth. So let V be another evenly covered neighborhood of p with $\pi^{-1}V = \cup_{j\in I}\tilde V_j$ and associated local chart $\psi_j := \tilde\psi_j \circ (\pi|_{\tilde V_j})^{-1} : U \to \mathbf{R}^n$ where $(\tilde V_j, \tilde\psi_j)$ is a local chart of $\tilde M$. Then

$$\psi_j \circ \varphi_i^{-1} = \tilde\psi_j \circ (\pi|_{\tilde V_j})^{-1} \circ \pi \circ \tilde\varphi_i^{-1} \tag{A.4.2}$$

However, $(\pi|_{\tilde V_j})^{-1} \circ \pi$ is realized by a unique element $\gamma \in \Gamma$ in a neighborhood of $\tilde p_i = \pi|_{\tilde U_i}^{-1}(p)$. Since Γ acts by diffeomorphisms, this shows that the transtion map (A.4.2) is smooth and finishes the proof. $\square$