Integration

4.1 Orientation

Recall the formula for change of variables in a multiple integral

$$\int_{\varphi(D)} f(y_1, \dots, y_n) dy_1 \cdots dy_n$$

$$= \int_D f(\varphi(x_1, \dots, x_n)) |J\varphi(x_1, \dots, x_n)| dx_1 \cdots dx_n \tag{4.1.1}$$

Here $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are two sets of coordinates on $\mathbf{R}^n$ related by a diffeomorphism φ from a bounded domain D in $\mathbf{R}^n$ to a bounded domain $\varphi(D)$, f is a real continuous function on D,

$$J\varphi = \det \left(\frac{\partial(y_i \circ \varphi)}{\partial x_j} \right)$$

is the Jacobian determinant of φ, and $\int$ refers to the Riemann integral. Let us interpret (4.1.1) in terms of differential forms. We have

$$d\varphi \left(\frac{\partial}{\partial x_j}\Big|_p \right) = \sum_i \frac{\partial(y_i \circ \varphi)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_j}\Big|_p$$

and

$$(d\varphi)^*(dy_i|_p) = \sum_j \frac{\partial(y_i \circ \varphi)}{\partial x_j}\Big|_p dx_j|_p,$$

so, in view of Exercise 6 in Chapter 2,

$$\varphi^*(dy_1 \wedge \cdots \wedge dy_n) = (J\varphi)\, dx_1 \wedge \cdots \wedge dx_n.$$

If we define, as we do, the left hand side of (4.1.1) as the integral of the n-form $\omega = f dy_1 \wedge \cdots \wedge dy_n$ over $\varphi(D)$, that formula says that

$$\int_{\varphi(D)} \omega = \pm \int_D \varphi^* \omega \tag{4.1.2}$$

where the sign is positive or negative according to whether the sign of the Jacobian determinant is positive or negative throughout D. In general, a diffeomorphism between open subsets of $\mathbf{R}^n$ is called *orientation-preserving* if its Jacobian determinant is everywhere positive. The above discussion shows that integration of n-forms on bounded domains is not invariant under diffeomorphisms in general, but only under those that preserve orientation. This suggests that if we want to transfer these ideas to smooth manifolds via local charts, and define integration of n-forms on there in a manner independent of local coordinates, we should try to sort out a consistent sign for the transition maps.

Let M be a smooth manifold. A smooth atlas for M is called *oriented* if all the transition maps are orientation-preserving, and M is called *orientable* if it admits an oriented atlas. If M is orientable, two oriented atlases are said to define the same orientation if their union is an oriented atlas; this defines an equivalence relation on the set of oriented atlases, and a choice of equivalence class is called an *orientation* for M.

If M is orientable, an oriented atlas for M defines an orientation on each tangent space induced from the canonical orientation of $\mathbf{R}^n$ via the local charts. For these reason, an orientation on M can also be viewed as a coherent choice of orientations on the tangent spaces to M.

4.1.3 Proposition *A smooth manifold M of dimension n is orientable if and only if it has a nowhere vanishing n-form.*

Proof. Let $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$ be the canonical n-form on $\mathbf{R}^n$. The basic fact we need is that a diffeomorphism τ of $\mathbf{R}^n$ is orientation-preserving if and only if $\tau^*\omega_0 = \omega_0$.

Assume first ω is a nowhere vanishing n-form on M. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for M. For all α, $\varphi_\alpha^*\omega_0 = f_\alpha\omega$ where f_α is a nowhere zero smooth function on U_α. Thus f_α is everywhere positive or everywhere negative on U_α; in the latter case, we replace φ_α by $\psi_\alpha = \tau \circ \varphi_\alpha$ where $\tau(x_1, \ldots, x_n) = (-x_1, \ldots, x_n)$. Since $\psi_\alpha^*\omega_0 = \varphi_\alpha^*\tau^*\omega_0 = -\varphi_\alpha^*\omega_0 = -f_\alpha\omega$, this shows that, by replacing $\mathcal{A}$ with an equivalent atlas, we may assume that $f_\alpha > 0$ for all α. Now $(\varphi_\beta\varphi_\alpha^{-1})^*\omega_0 = f_\beta f_\alpha^{-1}\omega_0$ with $f_\beta f_\alpha^{-1} > 0$ for all α, β, which proves that $\mathcal{A}$ is oriented.

Conversely, assume $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an oriented atlas for M. Define $\omega_\alpha = \varphi_\alpha^*\omega_0$. Then ω_α is a nowhere vanishing n-form on U_α, and ω_α, ω_β are positive multiples of one another on $U_\alpha \cap U_\beta$. It follows that $\omega := \sum_\alpha \rho_\alpha\omega_\alpha$ is a well defined, nowhere vanishing n-form on M, where $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. □

In view of the proof of Proposition 4.1.3, on an orientable manifold M of dimension n, there exists a bijection between equivalence classes of oriented atlases and equivalence classes of nowhere vanishing n-forms, where two nowhere vanishing n-forms on M are said to be equivalent if they differ by a positive smooth function. On a connected manifold, the sign of a nowhere zero function cannot change, so on a connected orientable manifold there are exactly two possible orientations.

4.1.4 Example Let M be the pre-image of a regular value of a smooth map $f : \mathbf{R}^{n+1} \to \mathbf{R}$. Then M is an (embedded) submanifold of $\mathbf{R}^{n+1}$ and we show in the following that M is orientable by constructing a nowhere vanishing n-form on M. Let $U_i = \{p \in M \mid \frac{\partial f}{\partial x_i}(p) \neq 0\}$ for $i = 1, \ldots, n+1$. Then $\{U_i\}$ cover M and we can take $(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$ as local coordinates on U_i. Define a nowhere vanishing n-form on U_i by

$$\omega_i = (-1)^i \left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Since f is constant on M, $\sum_k \frac{\partial f}{\partial x_k} dx_k = 0$, so we have on U_j that

$$dx_j = -\left(\frac{\partial f}{\partial x_j}\right)^{-1} \sum_{k \neq j} \frac{\partial f}{\partial x_k} dx_k.$$

Now one easily checks that

$$\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$$

and hence the ω_i can be pieced together to yield a global n-form on M.

Let M be an orientable smooth manifold and fix an orientation for M, say given by an oriented atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$. We want to define the integral of a compactly supported n-form ω on M. For that purpose, consider first the special case in which the support of ω is contained in the domain of some local chart, say, $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$. Then we set

$$\int_M \omega = \int_{U_\alpha} \omega = \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^* \omega$$

Note that choosing another local chart in $\mathcal{A}$ whose domain contains the support of ω yields the same result due to (4.1.2). In the general case, we choose a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and put

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.$$

Let us check that this definition is independent of the choices made. Namely, let $\{(V_\beta, \psi_\beta)\}$ be another oriented atlas defining the same orientation, and let $\{\lambda_\beta\}$ be a partition of unity subordinate to $\{V_\beta\}$. Note that $\rho_\alpha\lambda_\beta\omega$ has support contained in $U_\alpha \cap V_\beta$, so, by the special case,

$$\int_{U_\alpha} \rho_\alpha\lambda_\beta\omega = \int_{V_\beta} \rho_\alpha\lambda_\beta\omega.$$

It follows that

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega &= \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha\lambda_\beta\omega \\
&= \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha\lambda_\beta\omega \\
&= \sum_\beta \int_{V_\beta} \lambda_\beta\omega,
\end{aligned}$$

as we wished.

4.1.5 Exercise Let $f : M \to N$ be a diffeomorphism between connected oriented manifolds of dimension n, and let ω be a compactly supported n-form on N. Prove that

$$\int_M f^*\omega = \pm \int_N \omega$$

where the sign is "$+$" if f is orientation-preserving and "$-$" if f is orientation-reversing. (Hint: Use (4.1.2).)

4.1.6 Exercise Let M be a connected orientable manifold of dimension n and denote by $-M$ the same manifold with the opposite orientation. Show that

$$\int_{-M} \omega = -\int_M \omega$$

for every compactly supported n-form ω on M.

4.2 Stokes' theorem

Stokes' theorem for manifolds is the exact generalization of the classical theorems of Green, Gauss and Stokes of Vector Calculus. In order to proceed, we need to develop a notion of boundary.

Manifolds with boundary

In the same way as manifolds are modeled on $\mathbf{R}^n$, manifolds with boundary are modeled on the *upper half space*

$$\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \mid x_n > 0\}.$$

A *smooth manifold with boundary* of dimension n is given by a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ where φ_α maps U_α homemomorphically onto an open subset of $\mathbf{R}^n_+$ and the transition maps are diffeomorphisms between open subsets of $\mathbf{R}^n_+$. Recall a function from an arbitrary subset A of $\mathbf{R}^n$ is called *smooth* if it admits a smooth extension to an open neighborhood of A.

In particular, $\mathbf{R}^n_+$ is itself a manifold with boundary. There is a natural decomposition of $\mathbf{R}^n_+$ into the *boundary*

$$\partial\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \mid x_n = 0\}$$

and its complement, the *interior*, and both are smooth manifolds in the previous (restricted) sense, with a natural diffeomorphism $\partial\mathbf{R}^n_+ \approx \mathbf{R}^{n-1}$. For an open subset U of $\mathbf{R}^n_+$, we also put $\partial U = U \cap \partial\mathbf{R}^n_+$.

4.2.1 Lemma *Let $\tau : U \to V$ be a diffeomorphism between open subsets of $\mathbf{R}^n_+$ with everywhere positive Jacobian determinant. Then τ restricts to a diffeomorphism $\bar\tau : \partial U \to \partial V$ with everywhere positive Jacobian determinant.*

Proof. By the inverse function theorem, τ maps points in $U \setminus \partial U$ to points in $V \setminus \partial V$. Therefore $\tau(\partial U) = \partial V$. Write $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. By assumption the Jacobian matrix of $\tau = (\tau_1, \ldots, \tau_n)$ at $(x', 0) \in \partial\mathbf{R}^n_+$ has positive determinant and block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$C = \left(\frac{\partial\tau_n}{\partial x_1}(x', 0), \ldots, \frac{\partial\tau_n}{\partial x_{n-1}}(x', 0)\right) = (0, \ldots, 0)$$

since $\tau_n(x', 0) = 0$ for all x', and

$$D = \frac{\partial\tau_n}{\partial x_n}(x', 0) > 0$$

since τ maps the upper half space into itself. It follows that A, which is the Jacobian of $\bar\tau$ at $(x', 0)$, also has positive determinant, as desired. □

Let M be a smooth manifold with boundary. It follows from Lemma 4.2.1 that the *boundary* of M, namely, the subset ∂M consisting of points of M mapped to $\partial\mathbf{R}^n_+$ under coordinate charts, is well defined. Moreover, it is a smooth manifold of dimension $(n-1)$, and an oriented atlas for M induces an oriented atlas for ∂M by restricting the coordinate charts. Note also that $M \setminus \partial M$ is a smooth manifold of dimension n.

4.2.2 Examples (a) The unit ball B^n in $\mathbf{R}^n$ is a smooth manifold with boundary S^{n-1}.

(b) The Möbius band is smooth manifold with boundary a circle S^1.

In general, for an oriented smooth manifold with boundary, we will always use the so called *induced orientation* on its boundary. Namely, if in $\mathbf{R}^n_+$ we use the standard orientation given by $dx_1 \wedge \cdots \wedge dx_n$, then the induced orientation on $\partial\mathbf{R}^n_+$ is specified by $(-1)^n\, dx_1 \wedge \cdots \wedge dx_{n-1}$ (the sign is required to make the statement of Stokes' theorem right). On an oriented smooth manifold with boundary M, for any local chart (U, φ) in an oriented atlas of M, we declare the restriction of φ to $\partial U \to \partial\mathbf{R}^n_+$ to be orientation-preserving.

4.2.3 Remark A smooth manifold M in the old sense is a smooth manifold with boundary with $\partial M = \varnothing$. Indeed, we can always find an atlas for M whose local charts have images in $\mathbf{R}^n_+ \setminus \partial\mathbf{R}^n_+$.

Let M be a smooth manifold with boundary of dimension n. The tangent space to M at a point p is an n-dimensional vector space defined in the same way as in the case of a smooth manifold (even in case $p \in \partial M$). The definition of the tangent bundle also works, and TM is itself a manifold with boundary. More generally, tensor bundles and differential forms are also defined. If M is in addition oriented, the integral of compactly supported n-forms is defined similarly to above.

Statement and proof of the theorem

4.2.4 Theorem *Let ω be an $(n-1)$-form with compact support on an oriented smooth n-manifold M with boundary and give ∂M the induced orientation. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In the right hand side of Stokes' theorem, ω is viewed as $\iota^*\omega$, where $\iota : \partial M \to M$ is the inclusion, and the integral vanishes if $\partial M = \varnothing$. In the case $n = 1$, the integral on the right hand side is a finite sum and the result reduces to the Fundamental Theorem of Calculus.

Proof of Theorem 4.2.4. We first consider two special cases.

Case 1: M is an open subset U of $\mathbf{R}^n$. View ω as n-form on $\mathbf{R}^n$ which is zero on the complement of U. Write $\omega = \sum_i a_i\, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$. Then $d\omega = \sum_i (-1)^{i-1} \frac{\partial a_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$. By Fubini's theorem,

$$\begin{aligned}
\int_U d\omega &= \int_{\mathbf{R}^n} d\omega \\
&= \sum_i (-1)^{i-1} \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i}\, dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_n \\
&= 0
\end{aligned}$$

because

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i}\, dx_i \\
&= a_i(\ldots, x_{i-1}, \infty, x_{i+1}, \ldots) - a_i(\ldots, x_{i-1}, -\infty, x_{i+1}, \ldots) \\
&= 0,
\end{aligned}$$

as a_i has compact support. Since M has no boundary, this case is settled.

Case 2: M is an open subset U of $\mathbf{R}^n_+$. View ω as n-form on $\mathbf{R}^n_+$ which is zero on the complement of U. Write $\omega = \sum_i a_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ as before, but note that while the a_i are smooth on (a neighborhood) of $\mathbf{R}^n_+$, the linear forms dx_i are defined on the entire $\mathbf{R}^n$. Since a_i has compact support, $\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i}\, dx_i = 0$, so by Fubini's theorem

$$\begin{aligned}
\int_U d\omega &= \int_{\mathbf{R}^n_+} d\omega \\
&= (-1)^{n-1} \int_{\mathbf{R}^{n-1}} \left(\int_0^\infty \frac{\partial a_n}{\partial x_n}\, dx_n \right) dx_1 \cdots dx_{n-1} \\
&= (-1)^{n-1} \int_{\mathbf{R}^{n-1}} -a_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} \\
&= \int_{\partial \mathbf{R}^n_+} \omega \\
&= \int_{\partial U} \omega,
\end{aligned}$$

finishing this case.

General case: M is an arbitrary manifold with boundary of dimension n. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas for M and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\omega = \sum_\alpha \rho_\alpha \omega$ where each term has compact support. By linearity, it suffices to prove Stokes' formula for $\rho_\alpha \omega$ which has support contained in U_α. Since U_α is diffeomorhic to an open set in $\mathbf{R}^n$ or $\mathbf{R}^n_+$, cases 1 and 2 imply that the formula holds on U_α, so

$$\int_M d\rho_\alpha \omega = \int_{U_\alpha} d\rho_\alpha \omega = \int_{\partial U_\alpha} \rho_\alpha \omega = \int_{\partial M} \rho_\alpha \omega,$$

which concludes the proof of the theorem. □

4.3 De Rham Cohomology

De Rham theory, named after Georges de Rham, is a cohomology theory in the realm of smooth manifolds and "constitutes in some sense the most perfect example of a cohomology theory" (Bott and Tu). The de Rham complex of a smooth manifold is defined as a differential invariant, but turns out to be a topological invariant (although we will not prove that).

The most basic invariant of a topological space X is perhaps its number of connected components. In term of continuous functions, a component is characterized by the property that on it every locally constant continuous function is globally constant. If we define $H^0(X)$ to be the vector space of real valued locally constant continuous functions on X, then $\dim H^0(X)$ is the number of connected components of X. Of course, in case $X = M$ is a smooth manifold and we define $H^0(M)$ to be the vector space of real valued locally constant *smooth* functions on M, again $\dim H^0(X)$ is number of connected components of M.

In seeking to define $H^k(M)$ for $k > 0$, assume for simplicity M is an open subset of $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$. In this case, the locally constant smooth functions f on M are exactly those satisfying

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i = 0.$$

Therefore $H^0(M)$ appears as the space of solutions of a differential equation. In case $k > 0$, points and functions are replaced by k-dimensional submanifolds and k-forms, respectively. For instance, if $k = 1$, a 1-form $\omega = \sum_i a_i, dx_i$ defines a function on smooth maps

$$\gamma \mapsto \int_\gamma \omega$$

and we look for locally constant functions, namely, those left unchanged under a small perturbation of γ keeping the endpoints fixed. In general, if we homotope γ to a nearby curve with endpoints fixed, the difference between the line integrals is given by the integral of $d\omega$ along the spanned surface, owing to Stokes' theorem. Therefore the condition of local constancy is here $d\omega = 0$ or, equivalently, the system of partial differential equations

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0 \tag{4.3.1}$$

for all i, j. On the other hand, $\int_\gamma df = \gamma(q) - \gamma(p)$ where p, q are the endpoints of γ, so 1-forms of type df yield trivial solutions of (4.3.1). This suggest that $H^1(M)$ be defined as the vector space of locally constant line integrals modulo the trivially constant ones, and similarly for bigger k.

4.3.2 Definition Let M be a smooth manifold. A k-form ω on M is called *closed* if $d\omega = 0$, and it is called *exact* if $\omega = d\eta$ for some $(k-1)$-form η on M. These conditions define subspaces of the real vector space of k-forms on M. Since $d^2 = 0$, every exact form is closed. The *k-th de Rham cohomology space* of M is the quotient vector space

$$H^k(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\}.$$

4.3.3 Examples (a) For any smooth manifold M of dimension n, there are no exact 0-forms and all n-forms are closed. Moreover $H^0(M) = \mathbf{R}^p$ where p is the number of connected components of M, and $H^k(M) = 0$ for $k > n$ since in this case there are no nonzero k-forms.

(b) Let $\omega = f(x)dx$ be a 1-form on $\mathbf{R}$. Then $\omega = dg$ where $g(x) = \int_0^x f(t)\,dt$. Therefore every 1-form on $\mathbf{R}$ is exact and hence $H^1(\mathbf{R}) = 0$. It follows from Poincaré lemma to be proved in the next section that $H^k(\mathbf{R}^n) = 0$ for all $k > 0$.

(c) Owing to Stokes' theorem, an n-form ω on an n-dimensional oriented manifold M (without boundary) can be of the form $d\eta$ for a *compactly supported* $(n-1)$-form η only if $\int_M \omega = 0$; in particular, if M is compact, ω can be exact only if $\int_M \omega = 0$. On the other hand, if M is compact and orientable, let $(U, x_1, \ldots, x_n)$ be a positively oriented local coordinate system and let f be a non-negative smooth function with compact support contained in U. Then $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ defines an n-form on M with $\int_M \omega > 0$ and hence $H^n(M) \neq 0$. We will see later that "integration over M" defines an isomorphism $H^n(M) \cong \mathbf{R}$ for compact connected orientable M.

(d) The 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

on $M = \mathbf{R}^2 \setminus \{(0,0)\}$ is easily checked to be closed by a direct calculation. Let $\iota : S^1 \to M$ be the unit circle. If ω is exact, $\omega = df$ for some $f \in C^\infty(M)$, then $d(\iota^* f) = \iota^* df = \iota^*\omega$, and also $\iota^*\omega$ is exact, but $\int_{S^1} \iota^*\omega = 2\pi \neq 0$, so this cannot happen, owing to (c). It follows that $H^1(M) \neq 0$.

(e) Consider $M = S^1$. The polar cooordinate function θ on S^1 is defined only locally, but any two determinations of the angle differ by a constant multiple of 2π, so its differential is a well defined 1-form called the "angular form" and usually denoted by $d\theta$, although it is not globally exact (be careful!). It is easily seen that $\iota^*\omega = d\theta$, where ω is as in (d), and so $H^1(S^1) \neq 0$. We next show that $\int_{S^1} : \Omega^1(S^1) \to \mathbf{R}$ induces an isomorphism $H^1(S^1) \to \mathbf{R}$. Every 1-form is closed, so we need only to identify its kernel with the exact 1-forms. Since $d\theta$ never vanishes, any 1-form α on S^1 can be written as $\alpha = f\,d\theta$ where $f \in C^\infty(S^1)$. Now $\int_{S^1} \alpha = 0$ says that $\int_0^{2\pi} f(e^{it})\,dt = 0$, so

$$\tilde{g}(t) = \int_0^t f(e^{is})\,ds$$

is a smooth, 2π-periodic function on $\mathbf{R}$ which induces $g \in C^\infty(S^1)$ such that $g(e^{it}) = \tilde{g}(t)$ for all $t \in \mathbf{R}$. It is clear that $dg = \alpha$, completing the argument.

4.3.4 Exercise Prove that the restriction of ω from Example 4.3.3(d) to the half-plane $x > 0$ is exact.

Induced maps in cohomology

Let $f : M \to N$ be smooth. Since $d(f^*\omega) = f^*(d\omega)$ for any $\omega \in \Omega^*(N)$, $f^*\omega$ is closed if ω is closed, and it is exact if ω is exact. Thus there is an induced homomorphism

$$f^* : H^k(N) \to H^k(M)$$

for each $k \geq 0$. In addition, if $g : N \to P$ is smooth, then

$$(g \circ f)^* = f^* \circ g^*.$$

Of course, the identity map $\mathrm{id} : M \to M$ induces the identity map in cohomology. Such properties show that de Rham cohomology defines a family of contravariant functors and, in particular, a diffeomorphism $f : M \to N$ induces an isomorphism between all the corresponding cohomology spaces. Thus de Rham cohomology is a differential invariant of smooth manifolds. We will prove later that it is a homotopy invariant.

4.4 Homotopy-invariance of cohomology

Let $f, g : M \to N$ be smooth maps between smooth manifolds. A (smooth) *homotopy* between f and g is a smooth map $F : M \times [0,1] \to N$ such that

$$\begin{cases} F(p,0) &= f(p) \\ F(p,1) &= g(p) \end{cases}$$

for $p \in M$. If there exists a homotopy between f and g, we say that they f and g are *homotopic*.

4.4.1 Proposition *Let f, g be homotopic maps. Then the induced maps in de Rham cohomology*

$$f^*, g^* : H^k(N) \to H^k(M)$$

are equal.

The proof of this propositon is given below. First, we need to make some remarks. For $t \in [0,1]$, consider the inclusions i_t given by

$$i_t(p) = (p,t)$$

for $p \in M$, and consider the natural projection $\pi : M \times [0,1] \to M$ given by $\pi(p,t) = p$. Then, obviously,

$$\pi \circ i_t = \mathrm{id}_M$$

implying that

$$i_t^*\pi^* = \mathrm{id} \quad \text{in } \Omega^k(M) \text{ and } H^k(M).$$

We consider the projection $t : M \times [0,1] \to [0,1]$. Then there exists a "vertical" vector field $\frac{\partial}{\partial t}$ and a 1-form dt. Note that $\ker d\pi$ is spanned by $\frac{\partial}{\partial t}$.

4.4.2 Lemma *Let $\omega \in \Omega^k(M \times [0,1])$. Then we can write*

$$\omega = \zeta + dt \wedge \eta \tag{4.4.3}$$

where $\zeta \in \Omega^k(M \times [0,1])$ has the property that it vanishes if some of its arguments belongs to $\ker d\pi$, and $\eta \in \Omega^{k-1}(M \times [0,1])$ has the same property.

Proof. Set $\eta = i_{\frac{\partial}{\partial t}}\omega$ and $\zeta = \omega - dt \wedge \eta$. Since

$$i_{\frac{\partial}{\partial t}}\eta = i_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial t}}\omega = 0,$$

it is clear that η has the claimed property. Similarly,

$$\begin{aligned} i_{\frac{\partial}{\partial t}}\zeta &= i_{\frac{\partial}{\partial t}}\omega - i_{\frac{\partial}{\partial t}}(dt \wedge \eta) \\ &= \eta - i_{\frac{\partial}{\partial t}} dt \wedge \eta + dt \wedge i_{\frac{\partial}{\partial t}}\eta \\ &= \eta - \eta + 0 \\ &= 0, \end{aligned}$$

as desired, where we have used that interior multiplication is an anti-derivation. □

We define the homotopy operator

$$H_k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$$

by the formula

$$(H_k\omega)_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_{(p,t)}(di_t(v_1), \ldots, di_i(v_{k-1}))\, dt,$$

where ω is decomposed as in (4.4.3) and $p \in M$, $v_1, \ldots, v_{k-1} \in T_pM$. Note that H_k is "integration along the fiber of π". For simplicity, we henceforth drop the subscript and just write H for the homotopy operator.

Proof of Propostion 4.4.1. Let $\omega \in H^k(M \times [0,1])$. We first claim that

$$dH\omega + Hd\omega = i_1^*\omega - i_0^*\omega. \tag{4.4.4}$$

The proof is by direct computation: since this is a pointwise identity, we can work in a coordinate system. Let $(U, x_1, \ldots, x_n)$ be a coordinate system in M. Then $(U \times [0,1], x_1 \circ \pi, \ldots, x_n \circ \pi, t)$ is a coordinate system in $M \times [0,1]$ and we can write

$$\omega|_{U\times[0,1]} = \sum_I a_I dx_I + dt \wedge \sum_J b_J dx_J$$

where a_I, b_J are smooth functions on $U \times [0,1]$ and I, J are increasing multi-indices. In $U \times [0,1]$, we have:

$$H\omega = \sum_J \left(\int_0^1 b_J dt\right) dx_J,$$

$$dH\omega = \sum_{J,i} \left(\int_0^1 \frac{\partial b_J}{\partial x_i} dt \right) dx_i \wedge dx_J,$$

$$d\omega = \sum_{I,i} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I + \sum_I \frac{\partial a_I}{\partial t} dt \wedge dx_I - dt \wedge \sum_{J,i} \frac{\partial b_J}{\partial x_i} dx_i \wedge dx_J,$$

$$Hd\omega = \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t} dt \right) dx_I - \sum_{J,i} \left(\int_0^1 \frac{\partial b_J}{\partial x_i} dt \right) dx_i \wedge dx_J.$$

It follows that

$$\begin{aligned} dH\omega + Hd\omega|_p &= \sum_I \left(\int_0^1 \frac{\partial a_I}{\partial t}(p,t)\, dt \right) dx_I \\ &= \sum_I (a_I(p,1) - a_I(p,0)) dx_I \\ &= i_1^*\omega - i_0^*\omega|_p, \end{aligned}$$

as claimed.

Suppose now that $F : M \times [0,1] \to N$ is a homotopy between f and g. Let α be a closed k-form in N representing the cohomology class $[\alpha] \in H^k(N)$. Applying identity (4.4.4) to $\omega = F^*\alpha$ yields

$$dHF^*\alpha + HF^*d\alpha = i_1^*F^*\alpha - i_0^*F^*\alpha.$$

Since $d\alpha = 0$ and $F \circ i_0 = f$, $F \circ i_1 = g$, we get

$$d\,(HF^*\alpha) = g^*\alpha - f^*\alpha.$$

Hence $g^*\alpha$ and $f^*\alpha$ are cohomologous. □

Two smooth manifolds M and N are said to have the same *homotopy type* (in the smooth sense) and are called *homotopy equivalent* (in the smooth sense) if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ and $f \circ g$ are smoothly homotopic to the identity maps on M and N, respectively. Each of the maps f and g is then called a *homotopy equivalence*, and f and g are called *inverses up to homotopy* or *homotopy inverses*. A manifold homotopy equivalent to a point is called *contractible*.

4.4.5 Corollary *A homotopy equivalence between smooth manifolds induces an isomorphism in de Rham cohomology.*

4.4.6 Corollary (Poincaré Lemma) *The de Rham cohomology of $\mathbf{R}^n$ (or a star-shaped open subset of $\mathbf{R}^n$) is $\mathbf{R}$ in dimension zero and zero otherwise:*

$$H^k(\mathbf{R}^n) = \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

Consider an inclusion $\iota : A \to M$. A map $r : M \to A$ satisfying $r \circ \iota = \mathrm{id}_A$ is called a *retraction*. A special case of homotopy equivalence is the case in which $\iota \circ r : M \to M$ is homotopic to id_M; if that happens, r is called a *deformation retract* of M onto A.

4.4.7 Exercise Check that $r : \mathbf{R}^2 \setminus \{0\} \to S^1$ given by $r(x) = \frac{x}{||x||}$ is a deformation retract. Compare with Examples 4.3.3(d) and (e).

4.4.8 Lemma *There exists no smooth retraction $r : B^n \to \partial B^n$ from the closed ball onto its boundary.*

Proof. The case $n = 1$ is easy as the closed interval B^1 is connected and its boundary is disconnected. Assume $n \geq 2$ and suppose, to the contrary, that such a retraction r exists. Recall that $\partial B^n = S^{n-1}$ and there exists an $(n-1)$-form ω on ∂B^n such that $\int_{\partial B^n} \omega \neq 0$ (Example 4.1.4 and 4.3.3(c)). Since r is the identity along ∂B^n,

$$\int_{\partial B^n} r^*\omega = \int_{\partial B^n} \omega \neq 0.$$

On the other hand, by Stokes' theorem,

$$\int_{\partial B^n} r^*\omega = \int_{B^n} dr^*\omega = \int_{B^n} r^*d\omega = 0,$$

as $d\omega = 0$, which is a contradiction. □

4.4.9 Theorem (Smooth Brouwer's fixed point theorem) *Let $f : B^n \to B^n$ be a smooth map. Then there exists $p \in B^n$ such that $f(p) = p$. In other words, every smooth self-map of the closed n-ball admits a fixed point.*

Proof. Suppose, on the contrary, that $f(x) \neq x$ for all $x \in B^n$. The half-line originating at x and going through $f(x)$ meets ∂B^n at a unique point; call it $r(x)$. It is easy to see that this defines a smooth retraction $r : B^n \to \partial B^n$ which is prohibited by Lemma 4.4.8. □

4.4.10 Remark The theorem is not true in the case of the open n-ball, as is easily seen.

For the next result, consider the unit sphere $\iota : S^n \to \mathbf{R}^{n+1}$. It is useful to have an explicit expression for a non-zero element in $H^n(S^n)$ (Example 4.1.4):

$$\omega = (-1)^i \frac{1}{x_i} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots dx_{n+1} \tag{4.4.11}$$

on $x_i \neq 0$ for $i = 1, \ldots, n+1$.

4.4.12 Theorem (Hairy ball theorem) *Let X be a smooth vector field on S^{2m}. Then there exists $p \in S^{2m}$ such that $X_p = 0$. In other words, every smooth vector field on an even-dimensional sphere has a zero.*

Proof. Suppose, on the contrary, that X never vanishes. By rescaling, we may assume that X is a unit vector field with respect to the metric induced from Euclidean space. Set

$$F_t : S^{2m} \to S^{2m}, \quad F_t(p) = \cos t\, x + \sin t\, X(p).$$

It is clear that F_t defines a homotopy between the identity map and the antipodal map of S^{2m}:

$$F_0 = \mathrm{id}_{S^{2m}} \quad \text{and} \quad F_\pi = -\mathrm{id}_{S^{2m}}.$$

Note that

$$F_\pi^*(x_i \circ \iota) = -x_i \circ \iota.$$

It follows that

$$F_\pi^* \omega = (-1)^{2m+1} \omega = -\omega,$$

where ω is as in (4.4.11). On the other hand,

$$F_0^* \omega = \omega,$$

and by Proposition 4.4.1, $F_0^*\omega$ and $F_\pi^*\omega$ are cohomologous, which contradicts the fact that ω is not cohomologous to zero. □

4.4.13 Remark Theorems 4.4.9 and 4.4.12 can be extended to the continuous category by using appropriate approximation results.

We close this section computing the de Rham cohomology of the n-sphere. The argument is a nice presentation of the "Mayer-Vietoris principle" in a very special case.

4.4.14 Proposition *The de Rham cohomology of S^n vanishes except in dimensions 0 and n.*

Proof. We may assume $n > 1$. We prove first that $H^1(S^n) = 0$. Let ω be a closed 1-form on S^n. We must show that ω is exact. Decompose S^n into the union of two open sets U and V, where U in a neighborhood of the northern hemisphere diffeomorphic to an open n-ball, V is a neighborhood of the southern hemisphere diffeomorphic to an open n-ball, and $U \cap V$ is a neighborhood of the equator which is diffeomorphic to $S^{n-1} \times (-1, 1)$. Since U and V are contractible, $\omega|_U = df$ for a smooth function f on U and $\omega|_V = dg$ for a smooth function g on V. In general on $U \cap V$, f and g do not agree, but the difference $h := f|_{U\cap V} - g|_{U\cap V}$ has $dh = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$. Since $n > 1$, S^{n-1} is connected and thus h is a constant. Setting

$$k := \begin{cases} f & \text{on } U, \\ g + h & \text{on } V, \end{cases}$$

defines a smooth function on S^n such that $dk = \omega$, as we wished.

We proceed by induction. Let ω be a closed k-form on S^n for $1 < k < n$. We shall prove that ω is exact using the same decomposition $S^n = U \cap V$ as above and the induction hypothesis. As above, $\omega|_U = d\alpha$ for a $(k-1)$-form α on U an $\omega|_V = d\beta$ for a $(k-1)$-form β on V. Let $\gamma = \alpha|_{U\cap V} - \beta|_{U\cap V}$. Then $d\gamma = 0$. Since γ is a closed $(k-1)$-form on $U \cap V$ and $U \cap V$ is homotopy equivalent to S^{n-1}, by the induction hypothesis, $\gamma = d\xi$ for a $(k-2)$-form on $U \cap V$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. Setting

$$\eta := \begin{cases} \alpha - d(\rho_V \xi) & \text{on } U, \\ \beta + d(\rho_U \xi) & \text{on } V, \end{cases}$$

defines a $(k-1)$-form on S^n such that $d\eta = \omega$. This completes the induction step and the proof of the theorem. □

4.5 Degree theory

Our first aim is to prove that the top dimensional de Rham cohomology of a compact connected orientable smooth manifold is one-dimensional. We start with a lemma in Calculus.

4.5.1 Lemma *Let f be a smooth function on $\mathbf{R}^n$ with support in the open cube $C = (-1, 1)^n$ and*

$$\int_{\mathbf{R}^n} f\, dx_1 \cdots dx_n = 0.$$

Then there exist smooth functions $f_1, \ldots, f_n$ on $\mathbf{R}^N$ with support in C such that

$$f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

4.5.2 Lemma *Let ω be an n-form on $\mathbf{R}^n$ with support contained in the open cube C such that $\int_{\mathbf{R}^n} \omega = 0$. Then there exists an $(n-1)$-form η on $\mathbf{R}^n$ with support contained in C such that $d\eta = \omega$.*

Proof. The Poincaré lemma yields η with $d\eta = \omega$ but does not give information about the support of η. Instead, write $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ for $f \in C^\infty(\mathbf{R}^n)$. Then $\mathrm{supp} f \subset C$ and $\int_{\mathbf{R}^n} f\, dx_1 \cdots dx_n = 0$, so $f = \sum_i \frac{\partial f_i}{\partial x_i}$ as in Lemma 4.5.1, and thus $\omega = d\eta$ where $\eta = \sum_i (-1)^{i-1} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. □

4.5.3 Proposition *If M is a compact connected orientable smooth manifold of dimension n, then $H^n(M) = \mathbf{R}$.*

Proof. By compactness, there is a finite cover $\{U_1, \ldots, U_m\}$ by coordinate neighborhoods diffeomorphic to the open cube C. Let ω_0 be a bump n-form as in Example 4.3.3(c) with support contained in U_1 and total integral 1. Then ω_0 defines non-zero cohomology class in $H^n(M) = 0$. We shall prove that any n-form ω on M is cohomologous to a multiple of ω_0, namely, $\omega = c\,\omega_0 + d\eta$ for some $c \in \mathbf{R}$ and some $(n-1)$-form η. Using a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, we can write $\omega = \sum_{i=1}^m \rho_i\omega$ where $\rho_i\omega$ is an n-form with support in U_i. By linearity, it suffices to prove the result for $\rho_i\omega$, so we may assume from the outset that the support of ω is contained in U_k, for some $k = 1, \ldots, m$.

Owing to the connectedness of M, we can find a sequence $U_{i_1}, \ldots, U_{i_r}$ such that $U_{i_1} = U_1$, $U_{i_r} = U_k$ and $U_{i_j} \cap U_{i_{j+1}} \neq \varnothing$ for all $j = 1, \ldots, r-1$. For all $j = 1, \ldots, r-1$, choose an n-form ω_j with support in $U_{i_j} \cap U_{i_{j+1}}$ and total integral 1. Now $\omega_0 - \omega_1$ has support in $U_{i_1} = U_1$ and total integral zero, so by Lemma 4.5.2, there exists η_1 with support in U_1 such that

$$\omega_0 - \omega_1 = d\eta_1.$$

Next, $\omega_1 - \omega_2$ has support in U_{i_2} and total integral zero, so the lemma yields η_2 with support in U_{i_2} such that

$$\omega_1 - \omega_2 = d\eta_2.$$

Continuing, we find η_j with support in U_{i_j} such that

$$\omega_{j-1} - \omega_j = d\eta_j$$

for all $j = 1, \ldots, r-1$. Adding up, we get

$$\omega_0 - \omega_{r-1} = d\eta$$

where $\eta = \sum_{j=1}^{r-1} \eta_j$. Now $U_{i_r} = U_k$ contains the support of ω and ω_{r-1}, and $\omega - c\omega_{r-1}$ has total integral zero, where $c = \int_M \omega$. By applying the lemma again,

$$\omega - c\omega_{r-1} = d\zeta$$

and hence

$$\omega = c\omega_0 + d(\zeta - c\eta)$$

as required. $\square$

4.5.4 Corollary *Let M be a compact connected oriented smooth manifold of dimension n. Then "integration over M"*

$$\int_M : H^n(M) \to \mathbf{R}$$

is a well defined linear isomorphism which is positive precisely on the cohomology classes defined by nowhere vanishing n-forms belonging to the orientation of M.

Proof. By Stokes formula, the integral of an exact form is zero, so the integral of an n-forms depends only on its cohomology class and thus the map is well defined. By the theorem, $H^n(M)$ is one dimensional and there exist bump n-forms with non-zero integral, so the map is an isomorphism.

Let ω be a nowhere vanishing n-form belonging to the orientation of M, choose an oriented atlas $\{(U_\alpha, \varphi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha))\}$ and a partition of unity $\{\rho_\alpha\}$ subrodinate to $\{U_\alpha\}$. Then $\omega = \sum_\alpha \rho_\alpha \omega$, where $\rho_\alpha\omega$ has support in U_α and on which its local representation is of the form $f_\alpha\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$ for a non-negative smooth function f_α on U_α. It follows that

$$\int_M \omega = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (f_\alpha \circ \varphi_\alpha^{-1})\, dx_1 \cdots dx_n > 0$$

since $f_\alpha \geq 0$ and it is positive somewhere. Conversely, if ω' is an n-form with $c = \int_M \omega' > 0$, then ω' is cohomologous to $c\omega$, and $c\omega$ and ω are nowhere vanishing n-forms defining the same orientation on M. □

Let $f : M \to N$ be a smooth map between compact connected oriented manifolds of the same dimension. Let ω_M, ω_N be n-forms on M, N, respectively, with total integral one. Then $f^* : H^n(N) \to H^*(M)$ carries ω_N to a multiple of ω_M; this number is called the *degree* of f, denoted $\deg f$. It follows from Proposition 4.4.1 that homotopic maps have the same degree.

4.5.5 Proposition *Let $f : M \to N$ be a smooth.*

a. The degree of f is an integer.

b. For all $\omega \in \Omega^n(N)$,

$$\int_M f^*\omega = (\deg f) \int_N \omega$$

c. If $q \in N$ is a regular value of f, then

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sgn}(\det df_p) \qquad \textit{(finite sum)}$$

Proof. (b) follows from the commutativity of the diagram

$$\begin{tikzcd} H^n(N) \arrow[r, "f^*"] \arrow[d, "\int_N"'] & H^n(M) \arrow[d, "\int_M"] \\ \mathbf{R} \arrow[r, "\deg f"'] & \mathbf{R} \end{tikzcd}$$

and (a) follows from (c). Let us prove (c).

Since q is a regular value and $\dim M = \dim N$, f is a local diffeomorphism at each $p \in f^{-1}(q)$. In particular, $f^{-1}(q)$ is discrete and thus finite, due to the compactness of M. Write $f^{-1}(q) = \{p_1, \ldots, p_m\}$ and choose open neighborhoods $\tilde{U}_i$ of p_i and V_i of q such that $f : \tilde{U}_i \to V_i$ is a diffeomorphism for all $i = 1, \ldots, m$. Setting $V = \cap_{i=1}^m V_i$ and $U_i = \tilde{U}_i \cap f^{-1}(V)$, now $f : U_i \to V$ is a diffeomorphism for all i. Moreover, $f(M \setminus \cup_{i=1}^m U_i)$ is a compact subset of N disjoint from q, so by further shrinking V we can ensure that $f^{-1}(V) = \cup_{i=1}^m U_i$.

Let α be an n-form on N with total integral one and support contained in V. Then $f^*\alpha$ is an n-form on M with support in $\cup_{i=1}^m U_i$. In view of Exercise 4.1.5

$$\int_{U_i} f^*\alpha = \mathrm{sgn}(\det df_{p_i}) \int_V \alpha = \mathrm{sgn}(\det df_{p_i})$$

where we consider the determinant of the Jacobian matrix of f at p_i relative to orientation-preserving local charts around p_i and q, so it sign is $+1$ if $df_{p_i} : T_{p_i}M \to T_qN$ preserves orientation and -1 if it reverses orientation. It follows that

$$\deg f = \int_M f^*\alpha = \sum_{i=1}^p \int_{U_i} f^*\alpha = \sum_{i=1}^p \mathrm{sgn}(\det df_{p_i}),$$

as desired. □

4.5.6 Corollary *The degree of a non-surjective map is zero.*

4.5.7 Remark There always exist a regular value of f by Sard's theorem.

4.5.8 Example Consider S^1 as the set of unit complex numbers. Then $f : S^1 \to S^1$ given by $f(z) = z^n$ is smooth and maps the angular form $d\theta$ to $n\,d\theta$, hence it has degree n.

4.5.9 Example Let $f : S^1 \to \mathbf{R}^2$ be a smooth map. Its image is a circle in the plane. Fix a point q not in this circle. The *winding number* $W(f,q)$ of f with respect to q is the degree of the map $F : S^1 \to S^1$ given by

$$F(x) = \frac{x-q}{||x-q||}.$$

Note that $W(f,q_1) = W(f,q_2)$ if q_1 and q_2 lis in the same connected component of the complement of the image of f.

4.5.10 Example Let f, $g : S^1 \to \mathbf{R}^3$ be two smooth maps. Their images yield two circles in $\mathbf{R}^3$ which we suppose to be disjoint. The *linking number* $\mathrm{Lk}(f,g)$ is the degree of the map $F : S^1 \times S^1 \to S^2$ given by

$$F(x,y) = \frac{f(x)-g(y)}{||f(x)-g(y)||}.$$

If f_t, $g_t : S^1 \to \mathbf{R}^3$ are homotopies of f, g such that f_t and g_t have disjoint images for all t, then $\mathrm{Lk}(f_t, g_t)$ is independent of t.

In case f, $g : S^1 \to S^3$, one chooses $q \in S^3$ not in the image of those maps and performs stereographic projection $S^3 \setminus \{q\} \to \mathbf{R}^3$ to define their linking number. Moving q continuously yield homotopies of f, g, so by connectedness of S^3, this definition does not depend on the choice of q.

4.6 Maxwell's equations

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and mathematician James Clerk Maxwell who published an early form of those equations between 1861 and 1862.

The *electric field*

$$\vec{E}(t) = (E_1, E_2, E_3)$$

and the *magnetic field*

$$\vec{B}(t) = (B_1, B_2, B_3)$$

are vector fields on $\mathbf{R}^3$. *Maxwell's equations* are

$$\begin{aligned} \operatorname{div} \vec{E} &= 4\pi\rho \\ \operatorname{div} \vec{B} &= 0 \\ \operatorname{curl} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \operatorname{curl} \vec{E} &= \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{J} \end{aligned}$$

where ρ is the *electric charge density* and $\vec{J}$ is the *electric current*.

Spacetime is $\mathbf{R}^4$ with coordinates (x_1, x_2, x_3, t) and an inner product of signature $(+++-)$. The *electromagnetic field* is $F \in \Omega^2(\mathbf{R}^4)$ given by

$$\begin{aligned} F = &(E_1 dx_1 + E_2 dx_2 + E_3 dx_3) \wedge dt \\ &+ B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2 \end{aligned}$$

We use the Hodge star (Exercise 10 in Chapter 2) to write

$$\begin{aligned} *F = &-(B_1 dx_1 + B_2 dx_2 + B_3 dx_3) \wedge dt \\ &+ E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2 \end{aligned}$$

The *source* is $J \in \Omega^3(\mathbf{R}^4)$ given by

$$\mathcal{J} = \rho dx_1 \wedge dx_2 \wedge dx_3 - dt \wedge (J_1 dx_2 \wedge dx_3 + J_2 dx_3 \wedge dx_1 + J_3 dx_1 \wedge dx_2) \tag{4.6.1}$$

Now Maxwell's equations are equivalent to

$$\begin{aligned} dF &= 0 \\ d * F &= 4\pi\mathcal{J} \end{aligned}$$

The second equation says in particular that $\mathcal{J}$ is exact, thus $d\mathcal{J} = 0$. Computing $d\mathcal{J}$ from (4.6.1) we get the *law of conservation of charge*

$$\frac{d\rho}{dt} + \operatorname{div} \vec{J} = 0.$$

4.7 Problems

§ 4.1

1 Let M be a smooth manifold of dimension n and let $f : M \to \mathbf{R}^{n+1}$ be an immersion. Prove that M is orientable if and only if there exists a nowhere vanishing smooth vector field X along f such that $X_{f(p)}$ is normal to $df_p(T_pM)$ in $\mathbf{R}^{n+1}$.

2 Prove that $\mathbf{R}P^n$ is orientable if and only if n is odd.

3 Show that the global n-form constructed in Example 4.1.4 for the case S^n can be given as the restriction of

$$\alpha = \sum_{i=1}(-1)^{i-1}x_i \, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

to S^n.

4 (Integration on a Riemannian manifold) Let (M, g) be a Riemannian manifold of dimension n.

a. On any coordinate neighborhood U, construct a *local orthonormal frame* $E_1, \ldots, E_n$, that is, a set of n smooth vector fields on U which is orthonormal at every point of U. (Hint: Apply the Gram-Schmidt process to the coordinate vector fields.)

b. Let $\omega_1, \ldots, \omega_n$ be the 1-forms dual to an orthonormal frame on U. This is called a *local orthonormal coframe* on U. Suppose now $\omega_1', \ldots, \omega_n'$ is a local orthonormal coframe on U'. Prove that

$$\omega_1 \wedge \cdots \wedge \omega_n = \pm\omega_1' \wedge \cdots \wedge \omega_n'$$

at each point of $U \cap U'$.

c. Deduce that in case M is orientable, the locally defined n-forms $\omega_1 \wedge \cdots \wedge \omega_n$ can be pieced together to yield a globally defined nowhere vanishing n-form vol_M on M satisfying

$$\mathrm{vol}_M(E_1, \ldots, E_n) = 1$$

for every local orthonormal frame $E_1, \ldots, E_n$. This form is called the *volume form* of the oriented Riemannian manifold M and its integral is called the *volume* of M.

d. Show that for a positively oriented basis $v_1, \ldots, v_n$ of T_pM, we have

$$(\mathrm{vol}_M)_p(v_1, \ldots, v_n) = \sqrt{\det\left(g_p(v_i, v_j)\right)}.$$

Deduce that, in local coordinates $(U, \varphi = (x^1, \ldots, x^n))$,

$$\mathrm{vol}_M = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

5 Consider the unit sphere S^n in $\mathbf{R}^{n+1}$ as a Riemannian manifold where, for each $p \in S^n$, the inner product on the tangent space T_pS^n is obtained by restriction of the standard scalar product in $\mathbf{R}^{n+1}$. Recall the n-form α on S^n given in Exercise 3. Let X be the outward unit normal vector field along S^n.

a. Show that

$$\alpha = \iota_X(dx_1 \wedge \cdots \wedge dx_{n+1}).$$

b. Deduce from (a) that α is the volume form of S^n with respect to some orientation.

c. In case $n = 2$, compute the volume of S^2.

§ 4.2

6 Let $f : M \to N$ be a smooth map between compact connected orientable manifolds of the same dimension. Assume M is the boundary ∂P of a smooth manifold P and f extends to a smooth map $F : P \to N$. Prove that $\deg f = 0$.

§ 4.3

7 Let α and β be closed differential forms. Show that $\alpha \wedge \beta$ is closed. In addition, if β is eact, show that $\alpha \wedge \beta$ is exact.

8 Let $\alpha = (2x + y \cos xy)\, dx + (x \cos xy)\, dy$ be a 1-form on $\mathbf{R}^2$. Show that α is exact by finding a smooth function f on $\mathbf{R}^2$ such that $df = \alpha$.

9 Prove that T^2 and S^2 are not diffeomorphic by using de Rham cohomology.

10 *a.* Prove that every 1-form on the open subset A in $\mathbf{R}^3$ given by

$$1 < \left(\sum_{i=1}^{3} x_i^2\right)^{1/2} < 2$$

is closed.

b. Give an example of a 2-form on A which is closed but not exact.

c. Prove that A is not diffeomorphic to the open ball in $\mathbf{R}^3$.

§ 4.4

11 Prove that if a 1-form ω on a smooth manifold M has $\int_\gamma \omega = 0$ for every smooth curve $\gamma : (a, b) \to M$, then ω is exact. (Hint: Show that $f(p) = \int_{p_0}^{p} \omega$ is well defined and satisfies $df = \omega$.)

12 Prove that $H^1(M) = 0$ for a simply-connected smooth manifold M. (Hint: By approximation results, a smooth manifold is simply-connected if and only if every smooth closed curve is smoothly homotopic to a point.)

§ 4.5

13 (Fundamental theorem of algebra) Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$ be a complex polynomial.

a. Consider the extended complex plane $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and show that $z : \bar{\mathbf{C}} \backslash \{\infty\} \to \mathbf{C} \cong \mathbf{R}^2$, $\frac{1}{z} : \bar{\mathbf{C}} \backslash \{0\} \to \mathbf{C} \cong \mathbf{R}^2$ define a smooth atlas on $\bar{\mathbf{C}}$. (Hint: Use Proposition 1.2.8.) Use stereographic projection from the north and south poles to construct a diffeomorphism $S^2 \cong \bar{\mathbf{C}}$.

b. Extend f to a map $\tilde{f} : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ by putting $\tilde{f}(\infty) = \infty$. Check that $\tilde{f}$ is smooth using the atlas constructed in (a).

c. Show that $\tilde{f}$ is smoothly homotopic to $g : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ where $g(z) = z^k$. What is the degree of g?

d. Deduce from (c) that f is surjective. In particular, there exists $z_0 \in \mathbf{C}$ such that $f(z_0) = 0$.

14 Define the *Hopf map* $\pi : S^3 \to S^2$ by $\pi(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$, where we view $S^3 \subset \mathbf{C}^2$ and $S^2 \subset \mathbf{C} \times \mathbf{R}$.

a. Show that the level sets of π are circles of the form $\{e^{it} \cdot p \mid t \in \mathbf{R}\}$ for some $p \in S^3$.

b. Compute the linking number of $\pi^{-1}(0, 1)$ and $\pi^{-1}(0, -1)$.

15 Let M be a compact connected orientable surface (2-dimensional manifold) in $\mathbf{R}^3$. Consider the Riemannian metric obtained by restriction of the scalar product of R^3 to the tangent spaces of M.

a. According to Exercise 1, there exists a smooth normal unit vector field along M in $\mathbf{R}^3$. Use the canonical parallelism in $\mathbf{R}^3$ to view this vector field as a smooth map $g : M \to S^2$; this map is called the *Gaussian map* of M; check that it is uniquely defined, up to sign.

b. For $p \in M$, the differential $dg_p : T_pM \to T_{g(p)}S^2$ where T_pM and $T_{g(p)}S^2$ can again be identified under the canonical parallelism in R^3. The *Gaussian curvature* $\kappa(p)$ of M at p is the determinant $\det(dg_p)$, and does not depend on the choice of sign in (a). Prove that

$$\kappa \operatorname{vol}_M = g^* \operatorname{vol}_{S^2}.$$

c. Use (b) and the Gauss-Bonnet theorem to conclude that the degree of the Gauss map is half the Euler characteristic of M:

$$\deg g = \frac{1}{2} \chi(M).$$