CHAPTER 2

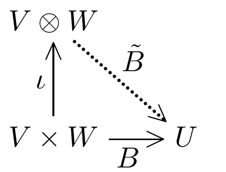
Tensor fields and differential forms

2.1 Multilinear algebra

Let V be a real vector space. In this section, we construct the tensor algebra $T(V)$ and the exterior algebra $\Lambda(V)$ over V. Elements of $T(V)$ are called tensors on V. Later we will apply this constructions to the tangent space T_pM of a manifold M and let p vary in M, similarly to the definition of the tangent bundle.

Tensor algebra

All vector spaces are real. Let V and W be vector spaces. It is less important what the tensor product of V and W *is* than what it *does*. Namely, a tensor product of V and W is a vector space $V \otimes W$ together with a bilinear map $\iota : V \times W \to V \otimes W$ such that the following *universal property* holds: for every vector space U and every bilinear map $B : V \times W \to U$, there exists a unique linear map $\tilde{B} : V \otimes W \to U$ such that $\tilde{B} \circ \iota = B$.



There are different ways to construct $V \otimes W$. It does not actually matter which one we choose, in view of the following exercise.

2.1.1 Exercise Prove that the tensor product of V and W is uniquely defined by the universal property. In other words, if $(V \otimes_1 W, \iota_1)$, $(V \otimes_2 W, \iota_2)$ are two tensor products, then there exists an isomorphism $\ell : V \otimes_1 W \to V \otimes_2 W$ such that $\ell \circ \iota_1 = \iota_2$.

We proceed as follows. Start with the canonical isomorphism $V^{**} \cong V$ between V and its bidual. It says that we can view an element v in V as the linear map on V^* given by $f \mapsto f(v)$. Well, we can extend this idea and consider the space $\mathrm{Bil}(V,W)$ of bilinear forms on $V \times W$. Then there is a natural map $\iota : V \times W \to \mathrm{Bil}(V,W)^*$ given by $\iota(v,w)(b) = b(v,w)$ for $b \in \mathrm{Bil}(V,W)$. We claim that $(\mathrm{Bil}(V,W)^*, \iota)$ satisfies the universal property: given a bilinear map $B : V \times W \to U$, there is an associated map $U^* \to \mathrm{Bil}(V,W)$, $f \mapsto f \circ B$; let $\tilde{B} : \mathrm{Bil}(V,W)^* \to U^{**} = U$ be its transpose.

2.1.2 Exercise Check that $\tilde{B} \circ \iota = B$.

Now that $V \otimes W$ is constructed, we can forget about its definition and keep in mind its properties only (in the same way as when we work with real numbers and we do not need to know that they are equivalence classes of Cauchy sequences), namely, the universal property and those listed in the sequel. Henceforth, we write $v \otimes w = \iota(v,w)$ for $v \in V$ and $w \in W$.

2.1.3 Proposition *Let V and W be vector spaces. Then:*

a. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$;

b. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$;

c. $av \otimes w = v \otimes aw = a(v \otimes w)$;

for all v, v_1, $v_2 \in V$; w, w_1, $w_2 \in W$; $a \in \mathbf{R}$.

2.1.4 Proposition *Let U, V and W be vector spaces. Then there are canonical isomorphisms:*

a. $V \otimes W \cong W \otimes V$;

b. $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$;

c. $V^* \otimes W \cong \mathrm{Hom}(V,W)$; *in particular,* $\dim V \otimes W = (\dim V)(\dim W)$.

2.1.5 Exercise Prove Propositions 2.1.3 and 2.1.4.

2.1.6 Exercise Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be bases for V and W, respectively. Prove that $\{e_i \otimes f_j \ : \ i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$ is a basis for $V \otimes W$.

2.1.7 Exercise Let $A = (a_{ij})$ be a real $m \times n$ matrix, viewed as an element of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$. Use the canonical inner product in $\mathbf{R}^n$ to identify $(\mathbf{R}^n)^* \cong \mathbf{R}^n$. What is the element of $\mathbf{R}^n \otimes \mathbf{R}^m$ that corresponds to A?

Taking $V = W$ and using Proposition 2.1.4(b), we can now inductively form the tensor nth power $\otimes^n V = \otimes^{n-1} V \otimes V$ for $n \geq 1$, where we adopt the convention that $\otimes^0 V = \mathbf{R}$. The *tensor algebra* $T(V)$ over V is the direct sum

$$T(V) = \bigoplus_{r,s \geq 0} V^{r,s}$$

where

$$V^{r,s} = (\otimes^r V) \otimes (\otimes^s V^*)$$

is called the *tensor space of type* (r, s). The elements of $T(V)$ are called *tensors*, and those of $V^{r,s}$ are called *homogeneous of type* (r, s). The multiplication $\otimes$, read "tensor", is the $\mathbf{R}$-linear extension of

$$\begin{aligned}&(u_1 \otimes \cdots \otimes u_{r_1} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^*) \otimes (v_1 \otimes \cdots \otimes v_{r_2} \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*) \\ &\quad = \quad u_1 \otimes \cdots \otimes u_{r_1} \otimes v_1 \otimes \cdots \otimes v_{r_2} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^* \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*.\end{aligned}$$

$T(V)$ is a non-commutative, associative *graded* algebra, in the sense that tensor multiplication is compatible with the natural grading:

$$V^{r_1,s_1} \otimes V^{r_2,s_2} \subset V^{r_1+r_2,s_1+s_2}.$$

Note that $V^{0,0} = \mathbf{R}$, $V^{1,0} = V$, $V^{0,1} = V^*$, so real numbers, vectors and linear forms are examples of tensors.

Exterior algebra

Even more important to us will be a certain quotient of the subalgebra $T^+(V) = \bigoplus_{k\geq 0} V^{k,0}$ of $T(V)$. Let $\mathfrak{I}$ be the two-sided ideal of $T^+(V)$ generated by the set of elements of the form

$$v \otimes v \tag{2.1.8}$$

for $v \in V$. The *exterior algebra* over V is the quotient

$$\Lambda(V) = T^+(V)/\mathfrak{I}.$$

The induced multiplication is denoted by $\wedge$, and read "wedge" or "exterior product". In particular, the class of $v_1 \otimes \cdots \otimes v_k$ modulo $\mathfrak{I}$ is denoted $v_1 \wedge \cdots \wedge v_k$. This is also a graded algebra, where the space of elements of degree k is

$$\Lambda^k(V) = V^{k,0}/\mathfrak{I} \cap V^{k,0}.$$

Since $\mathfrak{I}$ is generated by elements of degree 2, we immediately get

$$\Lambda^0(V) = \mathbf{R} \quad \text{and} \quad \Lambda^1(V) = V.$$

$\Lambda(V)$ is not commutative, but we have:

2.1.9 Proposition *$\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$ for $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^\ell(V)$.*

Proof. Since $v \otimes v \in \mathfrak{I}$ for all $v \in V$, we have $v \wedge v = 0$. Since $\mathbf{R}$ is not a field of characteristic two, this relation is equivalent to $v_1 \wedge v_2 = -v_2 \wedge v_1$ for all $v_1, v_2 \in V$.

By linearity, we may assume that $\alpha = u_1 \wedge \cdots \wedge u_k$, $\beta = v_1 \wedge \cdots \wedge v_\ell$. Now

$$\begin{aligned}
\alpha \wedge \beta &= u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_\ell \\
&= -u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge u_k \wedge v_2 \cdots \wedge v_\ell \\
&= u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge v_2 \wedge u_k \wedge v_3 \cdots \wedge v_\ell \\
&= \cdots \\
&= (-1)^\ell u_1 \wedge \cdots \wedge u_{k-1} \wedge v_1 \wedge \cdots \wedge v_\ell \wedge u_k \\
&= (-1)^{2\ell} u_1 \wedge \cdots \wedge u_{k-2} \wedge v_1 \wedge \cdots \wedge v_\ell \wedge u_{k-1} \wedge u_k \\
&= \cdots \\
&= (-1)^{k\ell} \beta \wedge \alpha,
\end{aligned}$$

as we wished. ☐

2.1.10 Lemma *If* $\dim V = n$, *then* $\dim \Lambda^n(V) = 1$ *and* $\Lambda^k(V) = 0$ *for* $k > n$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Since

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k} \; : \; i_1, \ldots, i_k \in \{1, \ldots, n\}\} \tag{2.1.11}$$

is a basis of $V^{k,0}$ (see Exercise 2.1.6), the image of this set under the projection $V^{k,0} \to \Lambda^k(V)$ is a set of generators of $\Lambda^n(V)$. Taking into account Proposition 2.1.9 yields $\Lambda^k(V) = 0$ for $k > n$ and that $\Lambda^n(V)$ is generated by $e_1 \wedge \cdots \wedge e_n$, so we need only show that this element in not zero.

Suppose, on the contrary, that $e_1 \otimes \cdots \otimes e_n \in \mathfrak{I}$. Then $e_1 \otimes \cdots \otimes e_n$ is a linear combination of elements of the form $\alpha \otimes v \otimes v \otimes \beta$ where $v \in V$, $\alpha \in V^{k,0}$, $\beta \in V^{\ell,0}$ and $k + \ell + 2 = n$. Writing α (resp. β) in terms of the basis (2.1.11), we may assume that the only appearing base elements are of the form $e_1 \otimes \cdots \otimes e_k$ (resp. $e_{n-\ell+1} \otimes \cdots \otimes e_n$). It follows that we can write

$$e_1 \otimes \cdots \otimes e_n = \sum_{k=0}^{n-2} c_k \, e_1 \otimes \cdots \otimes e_k \otimes v_k \otimes v_k \otimes e_{k+3} \otimes \cdots \otimes e_n \tag{2.1.12}$$

where $c_k \in \mathbf{R}$ and $v_k \in V$ for all k. Finally, write $v_k = \sum_{i=1}^n a_{ik} e_i$ for $a_k \in \mathbf{R}$. The coefficient of

$$e_1 \otimes \cdots \otimes e_k \otimes e_{k+2} \otimes e_{k+1} \otimes e_{k+3} \otimes \cdots \otimes e_n$$

on the right hand side of (2.1.12) is

$$\sum_{k=0}^{n-2} c_k \, a_{k+2,k} a_{k+1,k},$$

thus zero. However, the coefficient of $e_1 \otimes \cdots \otimes e_n$ on the right hand side is

$$\sum_{k=0}^{n-2} c_k\, a_{k+1,k} a_{k+2,k},$$

hence also zero, a contradiction. □

2.1.13 Proposition *If $\{e_1, \ldots, e_n\}$ be a basis of V, then*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \;:\; i_1 < \cdots < i_k\}$$

is a basis of $\Lambda^k(V)$ for all $0 \le k \le n$; in particular, $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Fix $k \in \{0, \ldots, k\}$. The above set is clearly a set of generators of $\Lambda^k(V)$ and we need only show linear independence. Suppose

$$\sum a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} = 0,$$

which we write as

$$\sum a_I e_I = 0$$

where the I denote increasing k-multi-indices, and $e_\varnothing = 1$. Multiply through this equation by e_J, where J is an increasing $n-k$-multi-index, and note that $e_I \wedge e_J = 0$ unless I is the multi-index J^c complementary to J, in which case $e_{J^c} \wedge e_J = \pm e_1 \wedge \cdots \wedge e_n$. Since $e_1 \wedge \cdots \wedge e_n \neq 0$ by Lemma 2.1.10, this shows that $a_I = 0$ for all I. □

2.2 Tensor bundles

Cotangent bundle

In the same way as the fibers of the tangent bundle of M are the tangent spaces T_pM for $p \in M$, the fibers of the cotangent bundle of M will be the dual spaces T_pM^*. Indeed, form the disjoint union

$$T^*M = \dot{\bigcup}_{p \in M} T_pM^*.$$

There is a natural projection $\pi^* : T^*M \to M$ given by $\pi(\tau) = p$ if $\tau \in T_pM^*$. Recall that every local chart (U, φ) of M induces a local chart $\tilde{\varphi} : \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^{2n}$ of TM, where $\tilde{\varphi}(v) = (\varphi(\pi(v)), d\varphi(v))$, and thus a map $\tilde{\varphi}^* : (\pi^*)^{-1}(U) \to \mathbf{R}^n \times (\mathbf{R}^n)^* = \mathbf{R}^{2n}$, $\tilde{\varphi}^*(\tau) = (\varphi(\pi^*(\tau)), ((d\varphi)^*)^{-1}(\tau))$, where $(d\varphi)^*$ denotes the transpose map of $d\varphi$ and we have identified $\mathbf{R}^n = \mathbf{R}^{n*}$ using the canonical Euclidean inner product. The collection

$$\{((\pi^*)^{-1}(U), \tilde{\varphi}^*) \mid (U, \varphi) \in \mathcal{A}\}, \tag{2.2.1}$$

for an atlas $\mathcal{A}$ of M, satisfies the conditions of Proposition 1.2.8 and defines a Hausdorff, second-countable topology and a smooth structure on T^*M such that $\pi^* : TM \to M$ is smooth.

A section of T^*M is a map $\omega : M \to T^*M$ such that $\pi^* \circ \omega = \mathrm{id}_M$. A smooth section of T^*M is also called a *differential form of degree* 1 or *differential* 1*-form*. For instance, if $f : M \to \mathbf{R}$ is a smooth function then $df_p : T_pM \to \mathbf{R}$ is an element of T_pM^* for all $p \in M$ and hence defines a differential 1-form df on M.

2.2.2 Exercise Prove that the differential of a smooth function on M indeed gives a a smooth section of T^*M by using the atlas (2.2.1).

If $(U, x_1, \ldots, x_n)$ is a system of local coordinates on M, the differentials $dx_1, \ldots, dx_n$ yield local smooth sections of T^*M that form the dual basis to $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ at each point (recall (1.3.8)). Therefore any section ω of T^*M can be locally written as $\omega|_U = \sum_{i=1}^n a_i dx_i$, and one proves similarly to Proposition 1.6.4 that ω is smooth if and only if the a_i are smooth functions on U, for every coordinate system $(U, x_1, \ldots, x_n)$.

Tensor bundles

We now generalize the construction of the tangent and cotangent bundles using the notion of tensor algebra. Let M be a smooth manifold. Set:

$$\begin{array}{lcll} T^{r,s}(M) & = & \bigcup_{p\in M} (T_pM)^{r,s} & \textit{tensor bundle of type } (r,s) \textit{ over } M; \\ \Lambda^k(M) & = & \bigcup_{p\in M} \Lambda^k(T_pM^*) & \textit{exterior } k\textit{-bundle over } M; \\ \Lambda(M) & = & \bigcup_{p\in M} \Lambda(T_pM^*) & \textit{exterior algebra bundle over } M. \end{array}$$

Then $T^{r,s}(M)$, $\Lambda^k(M)$ and $\Lambda(M)$ admit natural structures of smooth manifolds such that the projections onto M are smooth. If $(U, x_1, \ldots, x_n)$ is a coordinate system on M, then the bases $\{\frac{\partial}{\partial x_i}|_p\}_{i=1}^n$ of T_pM and $\{dx_i|_p\}_{i=1}^n$ of T_pM^*, for $p \in U$, define bases of $(T_pM)^{r,s}$, $\Lambda^k(T_pM^*)$ and $\Lambda(T_pM)$. For instance, a section ω of $\Lambda^k(M)$ can be locally written as

$$\omega|_U = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{2.2.3}$$

where the $a_{i_1,\ldots,i_k}$ are functions on U.

2.2.4 Exercise Check that $T^{1,0}(M) = TM$, $T^{0,1}(M) = T^*M = \Lambda^1(M)$ and $\Lambda^0(M) = M \times \mathbf{R}$.

The smooth sections of $T^{r,s}(M)$, $\Lambda^k(M)$, $\Lambda^*(M)$ are respectively called *tensor fields of type* (r,s), *differential* k*-forms*, *differential forms* on M. For instance, a section ω of $\Lambda^k(M)$ is a differential k-form if and only if the functions a_i in all its local representations (2.2.3) are smooth.

We will denote the space of differential k-forms on M by $\Omega^k(M)$ and the space of all differential forms on M by $\Omega^*(M)$. Note that $\Omega^*(M)$ is a graded algebra over $\mathbf{R}$ with wedge multiplication and a module over the ring $C^\infty(M)$.

A differential k-form ω on M has been defined to be an object that, at each point $p \in M$, yields a map ω_p that can be evaluated on k tangent vectors $v_1, \ldots, v_k$ at p to yield a real number, with some smoothness assumption. The meaning of the next proposition is that we can *equivalently* think of ω as being an object that, evaluated at k vector fields $X_1, \ldots, X_k$ yields the smooth function $\omega(X_1, \ldots, X_k) : p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$. We first prove a lemma.

Hereafter, it shall be convenient to denote the $C^\infty(M)$-module of smooth vector fields on M by $\mathfrak{X}(M)$.

2.2.5 Lemma *Let*

$$\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k\ factors} \to C^\infty(M)$$

be a $C^\infty(M)$-multilinear map. Then the value of $\omega(X_1, \ldots, X_k)$ at any given point p depends only on the values of $X_1, \ldots, X_k$ at p.

Proof. For simplicity of notation, let us do the proof for $k = 1$; the case $k > 1$ is similar. We first show that if $X|_U = X'|_U$ for some open subset U of M, then $\omega(X)|_U = \omega(X')|_U$. Indeed let $p \in U$ be arbitrary, take an open neighborhood V of p such that $\bar{V} \subset U$ and a smooth function $\lambda \in C^\infty(M)$ with $\lambda|_{\bar{V}} = 1$ and $\operatorname{supp} \lambda \subset U$ (Exercise 1.5.1). Then

$$\begin{aligned} \omega(X)(p) &= \omega_p(X_p) \\ &= \omega_p(\lambda(p)X_p) \\ &= \omega(\lambda X)(p) \\ &= \omega(\lambda X')(p) \\ &= \omega_p(\lambda(p)X'_p) \\ &= \omega_p(X'_p) \\ &= \omega(X')(p), \end{aligned}$$

where in the fourth equality we have used that $\lambda X = \lambda X'$ as vector fields on M.

Finally, we prove that $\omega(X)(p)$ depends only on $X(p)$. By linearity, it suffices to prove that $X(p) = 0$ implies $\omega(X)(p)$. Let $(W, x_1, \ldots, x_n)$ be a coordinate system around p and write $X|_W = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for $a_i \in C^\infty(W)$. By assumption, $a_i(p) = 0$ for all i. Let λ be a smooth function on M with support contained in W and such that it is equal to 1 on an open neighborhood

U of p with $\bar{U} \subset W$. Define also

$$\tilde{X}_i = \begin{cases} \lambda \frac{\partial}{\partial x_i} & \text{on } W \\ 0 & \text{on } M \setminus \bar{U} \end{cases} \quad \text{and} \quad \tilde{a}_i = \begin{cases} \lambda a_i & \text{on } W \\ 0 & \text{on } M \setminus \bar{U}. \end{cases}$$

Then $\tilde{X} := \sum_{i=1}^n \tilde{a}_i \tilde{X}_i$ is a globally defined smooth vector field on M such that $\tilde{X}|_U = X|_U$ and we can apply the result in the previous paragraph to write

$$\begin{aligned} \omega(X)(p) &= \omega(\tilde{X})(p) \\ &= \left(\sum_{i=1}^n \tilde{a}_i \omega(\tilde{X}_i) \right)(p) \\ &= \sum_{i=1}^n \tilde{a}_i(p) \omega(\tilde{X}_i)(p) \\ &= 0 \end{aligned}$$

because $\tilde{a}_i(p) = a_i(p) = 0$ for all i. □

2.2.6 Proposition *$\Omega^*(M)$ is canonically isomorphic as a $C^\infty(M)$-module to the $C^\infty(M)$-module of the $C^\infty(M)$-multilinear maps*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ factors}} \to C^\infty(M) \tag{2.2.7}$$

Proof. Let $\omega \in \Omega^k(M)$. Then $\omega_p \in \Lambda^k(T_pM^*) \cong \Lambda^k(T_pM)^* \cong A_k(T_pM)$ for every $p \in M$, owing to Exercises 7 and 4, namely, ω_p can be considered to be an alternating k-multilinear form on T_pM. Therefore, for vector fields $X_1, \ldots, X_k$ on M,

$$\tilde{\omega}(X_1, \ldots, X_k)(p) := \omega_p(X_1(p), \ldots, X_k(p))$$

defines a smooth function on M, $\tilde{\omega}(X_1, \ldots, X_k)$ is $C^\infty(M)$-linear in each argument X_i, thus $\tilde{\omega}$ is a $C^\infty(M)$-multilinear map as in (2.2.7).

Conversely, let $\tilde{\omega}$ be a $C^\infty(M)$-multilinear map as in (2.2.7). Due to Lemma 2.2.5, we have $\tilde{\omega}_p \in A_k(T_pM) \cong \Lambda^k(T_pM^*)$, namely, $\tilde{\omega}$ defines a section ω of $\Lambda^k(M)$: given $v_1, \ldots, v_k \in T_pM$, choose $X_1, \ldots, X_k \in \mathfrak{X}(M)$ such that $X_i(p) = v_i$ for all i and put

$$\omega_p(v_1, \ldots, v_k) := \tilde{\omega}(X_1, \ldots, X_k)(p).$$

The smoothness of the section ω follows from the fact that, in a coordinate system $(U, x_1, \ldots, x_n)$, we can write $\omega|_U = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $a_{i_1 \cdots i_k}(q) = \omega_q(\frac{\partial}{\partial x_{i_1}}\big|_q, \ldots, \frac{\partial}{\partial x_{i_k}}\big|_q) = \tilde{\omega}(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}})(q)$ for all $q \in U$, and thus $a_{i_1 \cdots i_k} \in C^\infty(U)$. It follows that ω is a differential k-form on M. □

Henceforth we will not distinguish between differential k-forms and alternating multilinear maps (2.2.7). Similarly to Proposition 2.2.6:

2.2.8 Proposition *The $C^\infty(M)$-module of tensor fields of type (r,s) on M is canonically isomorphic to the $C^\infty(M)$-module of $C^\infty(M)$-multilinear maps*

$$\underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{r\ factors}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{s\ factors}\to C^\infty(M).$$

2.3 The exterior derivative

Recall that $\Lambda^0(M)=M\times\mathbf{R}$, so a smooth section of this bundle is a map $M\to M\times\mathbf{R}$ of the form $p\mapsto(p,f(p))$ where $f\in C^\infty(M)$. This shows that $\Omega^0(M)\cong C^\infty(M)$. Furthermore, we have seen that the differential of $f\in C^\infty(M)$ can be viewed as a differential 1-form $df\in\Omega^1(M)$, so we have an operator $C^\infty(M)\to\Omega^1(M)$, $f\mapsto df$. In this section, we extend this operator to an operator $d:\Omega^*(M)\to\Omega^*(M)$, called *exterior differentiation*, mapping $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all $k\geq 0$. It so happens that d plays an *extremely* important rôle in the theory of smooth manifolds.

2.3.1 Theorem *There exists a unique $\mathbf{R}$-linear operator $d:\Omega^*(M)\to\Omega^*(M)$ with the following properties:*

a. $d\left(\Omega^k(M)\right)\subset\Omega^{k+1}(M)$ *for all $k\geq 0$ (d has* degree $+1$*);*

b. $d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^{\deg\omega}\,\omega\wedge d\eta$ *for every ω, $\eta\in\Omega^*(M)$ (d is an* anti-derivation*);*

c. $d^2=0$*;*

d. df *is the differential of f for every $f\in C^\infty(M)\cong\Omega^0(M)$.*

Proof. We start with uniqueness, so let d be as in the statement. The first case is when M is a coordinate neighborhood $(U,x_1,\ldots,x_n)$. Then any $\omega\in\Omega^k(U)$ can be written as $\omega=\sum_I a_I dx_I$, where I runs over increasing multi-indices $(i_1,\ldots,i_k)$ and $a_I\in C^\infty(U)$, and we get

$$\begin{aligned} d\omega &= \sum_I d(a_I\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}) \quad \text{(by } \mathbf{R}\text{-linearity)}\\ &= \sum_I d(a_I)\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ &\qquad +\sum_{r=1}^k a_I\,dx_{i_1}\wedge\cdots\wedge d(dx_{i_r})\wedge\cdots\wedge dx_{i_k} \quad \text{(by (b))} \qquad (2.3.2)\\ &= \sum_I\sum_{r=1}^n\frac{\partial a_I}{\partial x_r}dx_r\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} \quad \text{(by (c) and (d).)} \end{aligned}$$

Next we go to the case of a general manifold M and show that d is a *local* operator, in the sense that $(d\omega)|_U=0$ whenever $\omega|_U=0$ and U is an open subset of M. So assume $\omega|_U=0$, take an arbitrary point $p\in U$, and choose $\lambda\in C^\infty(M)$ such that $0\leq\lambda\leq 1$, λ is flat equal to 1 on $M\setminus U$ and has support disjoint from $\bar{V}$, where V is a neighborhood of p with $\bar{V} \subset U$. Then $\omega = \lambda\omega$ on the entire M so that, using (b) we get

$$(d\omega)_p = d(\lambda\omega)_p = d\lambda_p \wedge \underbrace{\omega_p}_{=0} + \underbrace{\lambda(p)}_{=0} d\omega_p = 0,$$

as wished.

To continue, we verify that d induces an operator d_U on $\Omega^*(U)$ satisfying (a)-(d) for every open subset U of M. So given $\omega \in \Omega^k(U)$ and $p \in U$, construct $\tilde{\omega} \in \Omega^k(M)$ which coincides with ω on a neighborhood V of p with $\bar{V} \subset U$, as usual by means of a bump function, and define $(d_U\omega)_p := (d\tilde{\omega})_p$. The definition is independent of the chosen extension, as d is a local operator. It is easy to check that d_U indeed satisfies (a)-(d); for instance, for (b), note that $\tilde{\omega} \wedge \tilde{\eta}$ is an extension of $\omega \wedge \eta$ and hence $d_U(\omega \wedge \eta)_p = (d(\tilde{\omega} \wedge \tilde{\eta}))_p = (d\tilde{\omega})_p \wedge \tilde{\eta}_p + (-1)^{\deg\tilde{\omega}}\tilde{\omega}_p \wedge (d\tilde{\eta})_p = (d_U\omega)_p \wedge \eta_p + (-1)^{\deg\omega}\omega_p \wedge (d_U\eta)_p$. Note also that the collection $\{d_U\}$ is *natural with respect to restrictions*, in the sense that if $U \subset V$ are open subsets of M then $d_V|_U = d_U$.

Finally, for $\omega \in \Omega^*(M)$ and a coordinate neighborhood $(U, x_1, \ldots, x_n)$, on one hand $d_U(\omega|_U)$ is uniquely defined by formula (2.3.2). On the other hand, ω itself is an extension of $\omega|_U$, and hence $(d\omega)_p = (d_U(\omega|_U))_p$ for every $p \in U$. This proves that $d\omega$ is uniquely defined.

To prove existence, we first use formula (2.3.2) to define an $\mathbf{R}$-linear operator d_U on $\Omega^k(U)$ for every coordinate neighborhood U of M. It is clear that d_U satisfies (a) and (d); let us prove that it also satisfies (b) and (c). So let $\omega = \sum_I a_I dx_I \in \Omega^k(U)$. Then $d_U\omega = \sum_I da_I \wedge dx_I$ and

$$\begin{aligned} d_U^2\omega &= \sum_{I,r} d_U\left(\frac{\partial a_I}{\partial x_r} dx_r \wedge dx_I\right) \\ &= \sum_{I,r,s} \frac{\partial^2 a_I}{\partial x_s \partial x_r} dx_s \wedge dx_r \wedge dx_I \\ &= 0, \end{aligned}$$

since $\frac{\partial^2 a_I}{\partial x_s \partial x_r}$ is symmetric and $dx_s \wedge dx_r$ is skew-symmetric in r, s. Let also $\eta = \sum_J b_J dx_J$. Then $\omega \wedge \eta = \sum_{I,J} a_I b_J dx_I \wedge dx_J$ and

$$\begin{aligned} d_U(\omega \wedge \eta) &= \sum_{I,J} d_U(a_I b_J dx_I \wedge dx_J) \\ &= \sum_{I,J,r} \frac{\partial a_I}{\partial x_r} b_J dx_r \wedge dx_I \wedge dx_J + \sum_{I,J,s} a_I \frac{\partial b_J}{\partial x_s} dx_s \wedge dx_I \wedge dx_J \\ &= \left(\sum_{I,r} \frac{\partial a_I}{\partial x_r} dx_r \wedge dx_I \right) \wedge \left(\sum_J b_J dx_J \right) \\ &\qquad + (-1)^{|I|} \left(\sum_I a_I dx_I \right) \wedge \left(\sum_{J,s} \frac{\partial b_J}{\partial x_s} dx_s \wedge dx_J \right) \\ &= d_U\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d_U \eta, \end{aligned}$$

where we have used Proposition 2.1.9 in the third equality to write $dx_s \wedge dx_I = (-1)^{|I|} dx_I \wedge dx_s$.

We finish by noting that the operators d_U for each coordinate system U of M can be pieced together to define a global operator d. Indeed for two coordinate systems U and V, the operators d_U and d_V induce two operators on $\Omega^*(U \cap V)$ satisfying (a)-(d) by the remarks above which must coincide by the uniqueness part. Note also that the resulting d satisfies (a)-(d) since it locally coincides with some d_U. □

2.3.3 Remark We have constructed the exterior derivative d as an operator between sections of vector bundles which, locally, is such that the local coordinates of $d\omega$ are linear combinations of partial derivatives of the local coordinates of ω (cf. 2.3.2). For this reason, d is called a *differential operator*.

Pull-back

A nice feature of differential forms is that they can always be pulled-back under a smooth map. In contrast, the push-forward of a vector field under a smooth map need not exist if the map is not a diffeomorphism.

Let $f : M \to N$ be a smooth map. The differential $df_p : T_pM \to T_{f(p)}$ at a point p in M has a transpose map $(df_p)^* : T_{f(p)}N^* \to T_pM^*$ and there is an induced algebra homomorphism $\delta f_p := \Lambda((df_p)^*) : \Lambda(T_{f(p)}N^*) \to \Lambda(T_pM^*)$ (cf. Problem 6). For varying $p \in M$, this yields a smooth map $\delta f : \Lambda^*(N) \to \Lambda^*(M)$. Recall that a differential form ω on N is a section of $\Lambda^*(N)$. The *pull-back of ω under f* is the section of $\Lambda^*(M)$ given by $f^*\omega =$

$\delta f \circ \omega \circ f$, so that the following diagram is commutative:

$$\begin{array}{ccc} \Lambda^*(M) & \xleftarrow{\delta f} & \Lambda^*(N) \\ {\scriptstyle f^*\omega}\uparrow & & \vdots\uparrow{\scriptstyle \omega} \\ M & \xrightarrow[f]{} & N \end{array}$$

(We prove below that $f^*\omega$ is smooth, so that it is in fact a differential form on M.) In more detail, we have

$$(f^*\omega)_p = \delta f(\omega_{f(p)})$$

for all $p \in M$. In particular, if ω is a k-form, then $(f^*\omega)_p \in \Lambda^k(T_pM^*) = \Lambda^k(T_pM)^* = A_k(T_pM)$ and

$$(f^*\omega)_p(v_1, \dots, v_k) = \omega_{f(p)}(df_p(v_1), \dots, df_p(v_k)) \tag{2.3.4}$$

for all $v_1, \dots, v_k \in T_pM$.

2.3.5 Exercise Let $f : M \to N$ be a smooth map.

a. In the case of 0-forms, that is smooth functions, check that $f^*(g) = g \circ f$ for all $g \in \Omega^0(N) = C^\infty(N)$.

b. In the case $\omega = dg \in \Omega^1(N)$ for some $g \in C^\infty(N)$, check that $f^*(dg) = d(g \circ f)$.

2.3.6 Proposition *Let $f : M \to N$ be a smooth map. Then:*

a. $f^* : \Omega^*(N) \to \Omega^*(M)$ *is a homomorphism of algebras;*

b. $d \circ f^* = f^* \circ d$;

c. $(f^*\omega)(X_1, \dots, X_k)(p) = \omega_{df(p)}(df(X_1(p)), \dots, df(X_k(p)))$ *for all* $\omega \in \Omega^*(N)$ *and all* $X_1, \dots, X_k \in \mathfrak{X}(M)$.

Proof. Result (c) follows from (2.3.4). For (a), it only remains to prove that $f^*\omega$ is actually a *smooth* section of $\Lambda^*(M)$ for a differential form $\omega \in \Omega^*(M)$. So let $p \in M$, choose a coordinate system $(V, y_1, \dots, y_n)$ of N around $f(p)$ and a neighborhood U of p in M with $f(U) \subset V$. Since f^* is linear, we may assume that ω is a k-form. As ω is smooth, we can write

$$\omega|_V = \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

It follows from Exercise 2.3.5 that

$$f^*\omega|_U = \sum_I (a_I \circ f)\, d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_k} \circ f), \tag{2.3.7}$$

which indeed is a smooth form on V. Finally, (b) is proved using (2.3.7):

$$\begin{aligned} d(f^*\omega)_p &= d\left(\sum_I (a_I\circ f)\, d(x_{i_1}\circ f)\wedge\cdots\wedge d(x_{i_k}\circ f)\right)\Big|_p \\ &= \sum_I \left(d(a_I\circ f)\wedge d(x_{i_1}\circ f)\wedge\cdots\wedge d(x_{i_k}\circ f)\right)|_p \\ &= f^*\left(\sum_I da_I\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)\Big|_p \\ &= f^*(d\omega)_p, \end{aligned}$$

as desired. □

2.4 The Lie derivative of tensors

In section 1.6, we defined the Lie derivative of a smooth vector field Y on M with respect to another smooth vector field X by using the flow $\{\varphi_t\}$ of X to identify different tangent spaces of M along an integral curve of X. The same idea can be used to define the Lie derivative of a differential form ω or tensor field S with respect to X. The main point is to understand the action of $\{\varphi_t\}$ on the space of differential forms or tensor fields.

So let $\{\varphi_t\}$ denote the flow of a vector field X on M, and let ω be a differential form on M. Then the pull-back $\varphi_t^*\omega$ is a differential form and $t\mapsto(\varphi_t^*\omega)_p$ is a smooth curve in T_pM^*, for all $p\in M$. The *Lie derivative* of ω with respect to X is the section $L_X\omega$ of $\Lambda(M)$ given by

(2.4.1)
$$(L_X\omega)_p = \frac{d}{dt}\Big|_{t=0}(\varphi_t^*\omega)_p.$$

We prove below that $L_X\omega$ is smooth, so it indeed yields a differential form on M. In view of (2.3.4), it is clear that the Lie derivative preserves the degree of a differential form.

We extend the definition of Lie derivative to an arbitrary tensor field S of type (r,s) as follows. Suppose

$$S_{\varphi_t(p)} = v_1\otimes\cdots\otimes v_r\otimes v_1^*\otimes\cdots\otimes v_s^*.$$

Then we define $(\varphi_t^*S)_p\in(T_pM)^{r,s}$ to be

$$d\varphi_{-t}(v_1)\otimes\cdots\otimes d\varphi_{-t}(v_r)\otimes\delta\varphi_t(v_1^*)\otimes\cdots\otimes\delta\varphi_t(v_s^*)$$

and put

(2.4.2)
$$(L_XS)_p = \frac{d}{dt}\Big|_{t=0}(\varphi_t^*S)_p.$$

Before stating properties of the Lie derivative, it is convenient to introduce two more operators. For $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$ with $k \geq 1$, the *interior multiplication* $\iota_X\omega \in \Omega^{k-1}(M)$ is the $(k-1)$-differential form given by

$$\iota_X\omega(X_1,\ldots,X_k) = \omega(X, X_1,\ldots,X_k)$$

for $X_1,\ldots,X_k \in \mathfrak{X}(M)$, and ι_X is zero on 0-forms.

2.4.3 Exercise Prove that $\iota_X\omega$ is indeed a *smooth* section of $\Lambda^{k-1}(M)$ for $\omega \in \Omega^k(M)$. Prove also that ι_X is an *anti-derivation* in the sense that

$$\iota_X(\omega \wedge \eta) = \iota_X\omega \wedge \eta + (-1)^k\omega \wedge \iota_X\eta$$

for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. (Hint: For the last assertion, it suffices to check the identity at one point.)

Let V be a vector space. The *contraction* $c_{i,j} : V^{r,s} \to V^{r-1,s-1}$ is the linear map that operates on basis vectors as

$$v_1 \otimes \cdots \otimes v_r \otimes v_1^* \otimes \cdots \otimes v_s^*$$
$$\mapsto v_j^*(v_i)\, v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_r \otimes v_1^* \otimes \cdots \hat{v_j^*} \otimes \cdots \otimes v_s^*.$$

It is easy to see that $c_{i,j}$ extends to a map $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$.

2.4.4 Exercise Let V be a vector space. Recall the canonical isomorphism $V^{1,1} \cong \mathrm{Hom}(V,V) = \mathrm{End}(V)$ (Proposition 2.1.4). Check that $c_{1,1} : V^{1,1} \to V^{0,0}$ is trace map $\mathrm{tr} : \mathrm{End}(V) \to \mathbf{R}$.

2.4.5 Proposition *Let X be a smooth vector field on M. Then:*

a. $L_X f = X(f)$ *for all* $f \in C^\infty(M)$.

b. $L_X Y = [X,Y]$ *for all* $X \in \mathfrak{X}(M)$.

c. L_X *is a type-preserving* $\mathbf{R}$*-linear operator on the space* $\mathcal{T}(M)$ *of tensor fields on* M.

d. $L_X : \mathcal{T}(M) \to \mathcal{T}(M)$ *is a derivation, in the sense that*

$$L_X(S \otimes S') = (L_X S) \otimes S' + S \otimes (L_X S')$$

e. $L_X : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r,s}(M)$ *commutes with contractions:*

$$L_X(c(S)) = c(L_x S)$$

for any contraction $c : \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$.

f. L_X *is a degree-preserving* $\mathbf{R}$*-linear operator on the space of differential forms* $\Omega(M)$ *which is a derivation and commutes with exterior differentiation.*

g. $L_X = \iota_X \circ d + d \circ \iota_X$ *on* $\Omega(M)$ (Cartan's magical formula) *h. For $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$, we have:*

$$L_{X_0}\omega(X_1, \ldots, X_k) = X_0(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, [X_0, X_i], X_{i+1}, \ldots, X_k).$$

i. Same assumption as in (h), we have:

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} X_i\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

Proof. (a) follows from differentiation of $(\varphi_t^* f)_p = f(\varphi_t(p))$ at $t = 0$. (b) was proved in section 1.6. (c) is clear from the definition. For (d), note that φ_t^* defines an automorphism of the tensor algebra $T(T_pM)$, namely, $\varphi_t^*(S \otimes S')|_p = (\varphi_t^* S)_p \otimes (\varphi_t^* S')_p$. The result follows by differentiation of this identity and the using the fact that multiplication is $\mathbf{R}$-bilinear.

(e) follows from the fact that φ_t^* commutes with contractions. As a consequence, which we will use below, if $\omega \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$ then $\omega(Y) = c(Y \otimes \omega)$ so

$$\begin{aligned} X(\omega(Y)) &= L_X(c(Y \otimes \omega)) \qquad \text{(using (a))} \\ &= c(L_X(Y \otimes \omega)) \\ &= c(L_X Y \otimes \omega + Y \otimes L_X\omega) \qquad \text{(using (d))} \\ &= \omega([X, Y]) + L_X\omega(Y) \qquad \text{(using (b));} \end{aligned}$$

in other words,

$$L_X\omega(Y) = X(\omega(Y)) - \omega([X, Y]). \tag{2.4.6}$$

For (f), we first remark that L_X is a derivation as a map from $\Omega(M)$ to non-necessarily smooth sections of $\Lambda(M)$: this is a pointwise check, and follows from (d), viewing the exterior algebra $\Lambda(T_pM)$ as a quotient of tensor algebra $T^+(T_pM)$. Next, check that L_X commutes with d on functions using (2.4.6):

$$\begin{aligned} L_X(df)(Y) &= X(df(Y)) - df([X, Y]) \\ &= X(Y(f)) - [X, Y](f) \\ &= Y(X(f)) \\ &= d(X(f))(Y) \\ &= d(L_X f)(Y) \end{aligned}$$

for all $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$. To continue, note that L_X is a local operator: formula (2.4.1) shows that $L_X\omega|_U$ depends only on $\omega|_U$, for any open subset U of M, and the same applies for (2.4.1). Finally, to see that $L_X\omega$ is smooth for any $\omega \in \Omega(M)$, we may assume that ω has degree k and work in a coordinate system $(U, x_1, \ldots, x_n)$, where ω has a local representation as in (2.2.3). Using the above collected facts:

$$L_X\omega|_U = \sum_{i_1<\cdots<i_k} X(a_{i_1\cdots i_k})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ \sum_{j=1}^{k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge d(X(x_{i_j})) \wedge \cdots \wedge dx_{i_k}$$

as wished.

To prove (g), let $P_X = d\circ\iota_X + \iota_X\circ d$. Then P_X and L_X are local operators, derivations of $\Omega(M)$, that coincide on functions and commmute with d. Since any differential form is locally a sum of wedge products of functions and differentials of functions, it follows that $L_X = P_X$.

The case $k = 1$ in (h) is formula (2.4.6). The proof for $k > 1$ is completely analogous.

Finally, (i) is proved by induction on k. The initial case $k = 0$ is immediate. Assuming (i) holds for $k - 1$, one proves it for k by starting with (h) and using (g) and the induction hypothesis. □

2.4.7 Exercise Carry out the calculations to prove (h) and (i) in Proposition 2.4.5.

2.5 Vector bundles

The tangent, cotangent and and all tensor bundles we have constructed so far are smooth manifolds of a special kind in that they have a fibered structure over another manifold. For instance, TM fibers over M so that the fiber over any point p in M is the tangent space T_pM. Moreover, there is some control on how the fibers vary with the point. In case of TM, this is reflected on the way a chart $(\pi^{-1}(U), \tilde\varphi)$ is constructed from a given chart (U, φ) of M. Recall that $\tilde\varphi : \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n$ where $\tilde\varphi(v) = (\varphi(\pi(v)), d\varphi(v))$. So $\tilde\varphi$ induces a diffeomorphism $\cup_{p\in U} T_pM \to \varphi(U) \times \mathbf{R}^n$ so that each fiber T_pM is mapped linearly and isomorphically onto $\{\varphi(p)\} \times \mathbf{R}^n$. We could also compose this map with $\varphi^{-1} \times \mathrm{id}$ to get a diffeomorphism

$$TM|_U := \cup_{p\in U} T_pM \to \varphi(U) \times \mathbf{R}^n \to U \times \mathbf{R}^n.$$

Of course each T_pM is abstractly isomorphic to $\mathbf{R}^n$, where $n = \dim M$, but here we are saying that the part of TM consisting of fibers lying over points in U is diffeomorphic to a product $U \times \mathbf{R}^n$ in such a way that T_pM corresponds to $\{p\} \times \mathbf{R}^n$. This is the idea of a vector bundle.

2.5.1 Definition A (smooth) *vector bundle* of rank k over a smooth manifold M is a smooth manifold E, called the *total space*, together with a smooth projection $\pi : E \to M$ such that:

a. $E_p := \pi^{-1}(p)$ is a vector space of dimension k for all $p \in M$;

b. M can be covered by open sets U such that there exists a diffeomorphism $E|_U = \pi^{-1}(U) \to U \times \mathbf{R}^k$ mapping E_p linearly and isomorphically onto $\{p\} \times \mathbf{R}^k$ for all $p \in U$.

The *trivial vector bundle* of rank k over M is the direct product $M \times \mathbf{R}^k$ with the projection onto the first factor. A vector bundle of rank $k = 1$ is also called a *line bundle*.

An equivalent definition of vector bundle, more similar in spirit to the definition of smooth manifold, is as follows.

2.5.2 Definition A (smooth) *vector bundle* of rank k over a smooth manifold M is a set E, called the *total space*, together with a projection $\pi : E \to M$ with the following properties:

a. M admits a covering by open sets U such that there exists a bijection $\varphi_U : E|_U = \pi^{-1}(U) \to U \times \mathbf{R}^k$ satisfying $\pi = \pi_1 \circ \varphi_U$, where $\pi_1 : U \times \mathbf{R}^k \to U$ is the projection onto the first factor. Such a φ_U is called a *local trivialization*.

b. Given local trivializations φ_U, φ_V with $U \cap V \neq \varnothing$, the change of local trivialization or transition function

$$\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbf{R}^k \to (U \cap V) \times \mathbf{R}^k$$

has the form

$$(x, a) \mapsto (x, g_{UV}(x)a)$$

where

$$g_{UV} : U \cap V \to \mathbf{GL}(k, \mathbf{R})$$

is smooth.

2.5.3 Exercise Prove that the family of transition functions $\{g_{UV}\}$ in Definition 2.5.2 satisfies the *cocycle conditions*:

$$\begin{aligned} g_{UU}(x) &= \mathrm{id} \qquad (x \in U) \\ g_{UV}(x)g_{VW}(x)g_{WU}(x) &= \mathrm{id} \qquad (x \in U \cap V \cap W) \end{aligned}$$

2.5.4 Exercise Let M be a smooth manifold.

a. Prove that for a vector bundle $\pi : E \to M$ as in Definition 2.5.2, the total space E has a natural structure of smooth manfifold such that π is smooth and the local trivializations are diffeomorphisms.

b. Prove that Definitions 2.5.1 and 2.5.2 are equivalent.

2.5.5 Example In this example, we construct a very important example of vector bundle which is not a tensor bundle, called the *tautological (line) bundle* over $\mathbf{R}P^n$. Recall that a point p in real projective space $M = \mathbf{R}P^n$ is a 1-dimensional subspace of $\mathbf{R}^{n+1}$ (Example 1.2.7). Set $E = \dot\cup_{p\in M} E_p$ where E_p is the subspace of $\mathbf{R}^{n+1}$ corresponding to p, namely, E_p consists of vectors $v \in \mathbf{R}^{n+1}$ such that $v \in p$. Let $\pi : E \to M$ map E_p to p. We will prove that this is a smooth vector bundle by constructing local trivializations and using Definition 2.5.2. Recall the atlas $\{\varphi_i\}_{i=1}^{n+1}$ of Example 1.2.7. Set

$$\tilde\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbf{R} \qquad v \mapsto (\pi(v), x_i(v)).$$

This is a bijection and the cocycle

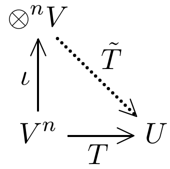
$$g_{ij}(x_1, \ldots, x_{n+1}) = x_i/x_j \in \mathbf{GL}(1, \mathbf{R}) = \mathbf{R} \setminus \{0\}$$

is smooth on $U_i \cap U_j$, as wished.

2.6 Problems

§ 2.1

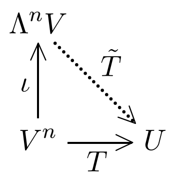
1 Let V be a vector space and let $\iota : V^n \to \otimes^n V$ be defined as $\iota(v_1, \ldots, v_n) = v_1 \otimes \cdots \otimes v_n$, where $V^n = V \times \cdots \times V$ (n factors on the right hand side). Prove that $\otimes^n V$ satisfies the following universal property: for every vector space U and every n-multilinear map $T : V^n \to U$, there exists a unique linear map $\tilde T : \otimes^n V \to U$ such that $\tilde T \circ \iota = T$.



2 Prove that $\otimes^n V$ is canonically isomorphic to the dual space of the space n-multilinear forms on V^n. (Hint: Use Problem 1.)

3 Let V be a vector space. An n-multilinear map $T : V^n \to U$ is called *alternating* if $T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (\operatorname{sgn}\sigma) T(v_1, \ldots, v_n)$ for every $v_1, \ldots, v_n \in V$ and every permutation σ of $\{1, \ldots, n\}$, where sgn denotes the sign ± 1 of the permutation.

Let $\iota : V^n \to \Lambda^n(V)$ be defined as $\iota(v_1, \ldots, v_n) = v_1 \wedge \cdots \wedge v_n$. Note that ι is alternating. Prove that $\otimes^n V$ satisfies the following universal property: for every vector space U and every alternating n-multilinear map $T : V^n \to U$, there exists a unique linear map $\tilde T : \Lambda^n(V) \to U$ such that $\tilde T \circ \iota = T$.



4 Denote the vector space of all alternating multilinear forms $V^n \to \mathbf{R}$ by $A_n(V)$. Prove that $\Lambda^n(V)$ is canonically isomorphic to $A_n(V)^*$.

5 Prove that $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$.

6 Let V and W be vector spaces and let $T : V \to W$ be a linear map.

$a.$ Show that T naturally induces a linear map $\Lambda^k(T) : \Lambda^k(V) \to \Lambda^k(W)$. (Hint: Use Problem 3.)

$b.$ Show that the maps $\Lambda^k(V)$ for $0 \leq k \leq n$ induce an algebra homomorphism $\Lambda(T) : \Lambda(V) \to \Lambda(W)$.

$c.$ Let now $V = W$ and $n = \dim V$. The operator $\Lambda^n(T)$ is multiplication by a scalar, as $\dim \Lambda^n(V) = 1$; define the *determinant* of T to be this scalar. Any $n \times n$ matrix $A = (a_{ij})$ can be viewed as a linear operator on $\mathbf{R}^n$. Prove that

$$\det A = \sum_\sigma (\operatorname{sgn} \sigma)\, a_{i,\sigma(i)} \cdots a_{n,\sigma(n)},$$

where $\operatorname{sgn} \sigma$ is the sign of the permutation σ and σ runs over the set of all permutations of the set $\{1, \ldots, n\}$. Prove also that the determinant of the product of two matrices is the product of their determinants.

$d.$ Using Problem 7(a) below, prove that the transpose map $\Lambda^k(T)^* = \Lambda^k(T^*)$.

7 Let V be vector space.

$a.$ Prove that there is a canonical isomorphism

$$\Lambda^k(V^*) \cong \Lambda^k(V)^*$$

given by

$$v_1^* \wedge \cdots \wedge v_k^* \mapsto \left(\, u_1 \wedge \cdots \wedge u_k \mapsto \det(v_i^*(u_j)) \,\right).$$

$b.$ Let $\alpha, \beta \in V^* \cong \Lambda^1(V^*) \cong A_1(V)$. Show that $\alpha \wedge \beta \in \Lambda^2(V^*)$, viewed as an element of $\Lambda^2(V)^* \cong A_2(V)$ is given by

$$\alpha \wedge \beta\, (u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

for all $u, v \in V$.

8 Let V be a vector space.

$a.$ In analogy with the exterior algebra, construct the *symmetric algebra* $Sym(V)$, a commutative graded algebra, as a quotient of $T(V)$.

$b.$ Determine a basis of the homogeneous subspace $Sym^n(V)$.

$c.$ State and prove that $Sym(V)$ satisfies a certain universal property.

d. Show that the $Sym^n(V)$ is canonically isomorphic to the dual of the space $S_n(V)$ of symmetric n-multilinear forms $V^n \to \mathbf{R}$.

In view of (d), $Sym(V^*)$ is usually defined to be the space $\mathcal{P}(V)$ of *polynomials* on V.

9 An element of $\Lambda^n(V)$ is called *decomposable* if it lies in the subspace $\Lambda^1(V) \wedge \cdots \wedge \Lambda^1(V)$ (n factors).

a. Show that in general not every element of $\Lambda^n(V)$ is decomposable.

b. Show that, for $\dim V \leq 3$, every homogeneous element in $\Lambda(V)$ is decomposable.

c. Let ω be a differential form. Is $\omega \wedge \omega = 0$?

§ 2.2

10 Let M be a smooth manifold. A *Riemannian metric* g on M is an assignment of positive definite inner product g_p on each tangent space T_pM which is smooth in the sense that $g(X,Y)(p) = g_p(X(p), Y(p))$ defines a smooth function for every X, $Y \in \mathfrak{X}(M)$. A *Riemannian manifold* is a smooth manifold equipped with a Riemannian metric.

a. Show that a Riemannian metric g on M is the same as a tensor field of $\tilde{g}$ type $(0,2)$ which is *symmetric* in the sense that $\tilde{g}(Y,X) = \tilde{g}(X,Y)$ for every $X, Y \in \mathfrak{X}(M)$.

b. Fix a local coordinate system $(U, x_1, \ldots, x_n)$ on M.

(i) Let g be a Riemannian metric on M. Show that $g|_U = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \in C^\infty(U)$ e $g_{ij} = g_{ji}$.

(ii) Conversely, given functions $g_{ij} = g_{ji} \in C^\infty(U)$ show how to define a Riemannian metric on U.

c. Use part (b)(ii) and a partition of unity to prove that every smooth manifold can be equipped with a Riemannian metric.

d. On a Riemannian manifold M there exists a natural diffeomorphism $TM \approx T^*M$ taking fibers to fibers. (Hint: There exist linear isomorphisms $v \in T_pM \mapsto g_p(v, \cdot) \in T_pM^*$).

§ 2.3

11 Consider $\mathbf{R}^3$ wit coordinates (x, y, z). In each case, decide whether $d\omega = 0$ or there exists η such that $d\eta = \omega$.

a. $\omega = yzdx + xzdy + xydz$.

b. $\omega = xdx + x^2y^2dy + yzdz$.

c. $\omega = 2xy^2 dx \wedge dy + zdy \wedge dz$.

§ 2.4

12 Let M and N be smooth manifolds where M is connected, and consider the projection $\pi : M \times N \to N$ onto the second factor. Prove that a k-form ω on $M \times N$ is of the form $\pi^*\eta$ for some k-form η on N if and only if $\iota_X\omega = L_X\omega = 0$ for every $X \in \mathfrak{X}(M \times N)$ satisfying $d\pi \circ X = 0$.

13 Let M be a smooth manifold.

$a.$ Prove that $\iota_X\iota_X = 0$ for every $X \in \mathfrak{X}(M)$.

$b.$ Prove that $\iota_{[X,Y]}\omega = L_X\iota_Y\omega - L_Y\iota_X\omega$ for every $X, Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$.

§ 2.5

14 The *Whitney sum* $E_1 \oplus E_2$ of two vector bundles $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ is a vector bundle $\pi : E = E_1 \oplus E_2 \to M$ where $E_p = (E_1)_p \oplus (E_2)_p$ for all $p \in M$.

$a.$ Show that $E_1 \oplus E_2$ is indeed a vector bundle by expressing its local trivializations in terms of those of E_1 and E_2 and checking the conditions of Definition 2.5.2.

$b.$ Similarly, construct the tensor product bundle $E_1 \otimes E_2$ and the dual bundle E^*.