APPENDIX A

Smooth manifolds

A.1 Introduction

The theory of smooth manifolds can be thought of a natural and very useful extension of the differential calculus on $\mathbf{R}^n$ in that its main theorems, which in our opinion are well represented by the inverse (or implicit) function theorem and the existence and uniqueness result for ordinary differential equations, admit generalizations. At the same time, from a differential-geometric viewpoint, manifolds are natural generalizations of the surfaces in three-dimensional Euclidean space.

We will review some basic notions from the theory of smooth manifolds and their smooth mappings.

A.2 Basic notions

A *topological manifold of dimension n* is a Hausdorff, second-countable topological space M which is locally modeled on the Euclidean space $\mathbf{R}^n$. The latter condition refers to the fact that M can be covered by a family of open sets $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that each U_α is homeomorphic to an open subset of $\mathbf{R}^n$ via a map $\varphi_\alpha : U_\alpha \to \mathbf{R}^n$. The pairs $(U_\alpha, \varphi_\alpha)$ are then called *local charts* or *coordinate systems*, and the family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is called a *topological atlas* for M. A *smooth atlas* for M (smooth in this book always means C^∞) is a topological atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ which also satisfies the following compatibility condition: the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth for every α, $\beta \in \mathcal{A}$ (of course, this condition is void if α, β are such that $U_\alpha \cap U_\beta = \varnothing$). A *smooth structure* on M is a smooth atlas which is maximal in the sense that one cannot enlarge it by adjoining new local charts of M while having it satisfy the above compatibility condition. (This maximality condition is really a technical one, and one easily sees that every smooth atlas can be enlarged to a unique maximal one.) Formally speaking, a smooth manifold consists of the topological space together with the smooth structure. The basic idea behind these definitions is that one can carry notions and results of differential calculus in $\mathbf{R}^n$ to smooth manifolds by using the local charts, and the compatibility condition between the local charts ensures that what we get for the manifolds is well defined.

A.2.1 Remark Sometimes it happens that we start with a set M with no prescribed topology and attempt to introduce a topology and smooth structure at the same time. This can be done by constructing a family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $\mathcal{A}$ is a countable index set, each $\varphi : U_\alpha \to \mathbf{R}^n$ is a bijection onto an open subset of $\mathbf{R}^n$ and the maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are homeomorphisms and smooth maps between open sets of $\mathbf{R}^n$ for all α, $\beta \in \mathcal{A}$ and for some fixed n. It is easy to see then the collection

$$\{\, \varphi_\alpha^{-1}(W) \mid W \text{ open in } \mathbf{R}^n,\ \alpha \in \mathcal{A} \,\}$$

forms a basis for a second-countable, locally Euclidean topology on M. Note that this topology needs not to be automatically Hausdorff, so one has to check that in each particular case and then the family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is a smooth atlas for M. Further, instead of considering maximal smooth atlases, one can equivalently define a smooth structure on M to be an equivalence class of smooth atlases, where one defines two smooth atlases to be equivalent if their union is a smooth atlas.

A.2.2 Examples (a) The Euclidean space $\mathbf{R}^n$ itself is a smooth manifold. One simply uses the identity map of $\mathbf{R}^n$ as a coordinate system. Similarly, any n-dimensional real vector space V can be made into a smooth manifold of dimension n simply by using a global coordinate system on V given by a basis of the dual space V^*.

(b) The complex n-space $\mathbf{C}^n$ is a real $2n$-dimensional vector space, so it has a structure of smooth manifold of dimension $2n$.

(c) The *real projective space* $\mathbf{R}P^n$ is the set of all lines through the origin in $\mathbf{R}^{n+1}$. Since every point of $\mathbf{R}^{n+1} \setminus \{0\}$ lies in a unique such line, these lines can obviously be seen as defining equivalence classes in $\mathbf{R}^{n+1} \setminus \{0\}$, so $\mathbf{R}P^n$ is a quotient topological space of $\mathbf{R}^{n+1} \setminus \{0\}$. One sees that the projection $\pi : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R}P^n$ is an open map, so it takes a countable basis of $\mathbf{R}^{n+1}\setminus\{0\}$ to a countable basis of $\mathbf{R}P^n$. It is also easy to see that this topology is Hausdorff. Moreover, $\mathbf{R}P^n$ is compact, since it is the image of the unit sphere of $\mathbf{R}^{n+1}$ under π (every line through the origin contains a point in the unit sphere). We next construct a atlas of $\mathbf{R}P^n$ by defining local charts whose domains cover it all. Suppose $(x_0, \ldots, x_n)$ are the standard coordinates in $\mathbf{R}^{n+1}$. Note that the set of all $\ell \in \mathbf{R}P^n$ that are not parallel to the cordinate hyperplane $x_i = 0$ form an open subset U_i of $\mathbf{R}P^n$. A line cannot be parallel to all coordinate hyperplanes, so $\bigcup_{i=0}^n U_i = \mathbf{R}P^n$. Now every line $\ell \in U_i$ must meet the hyperplane $x_i = 1$ at a unique point; define $\varphi_i : U_i \to \mathbf{R}^n$ by setting $\varphi_i(\ell)$ to be this point and identifying the hyperplane $x_i = 1$ with $\mathbf{R}^n$. Of course, the expression of φ_i is

$$\varphi_i(\ell) = \Big(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\Big),$$

where $(x_0, x_1, \ldots, x_n)$ are the coordinates of any point $p \in \ell$. This shows that $\varphi_i \circ \pi$ is continuous, so φ_i is continous by the definition of quotient topology. Clearly, φ_i is injective, and one sees that its inverse is also continous. Finally, we check the compatibility between the local charts, namely, that $\varphi_j \circ \varphi_i^{-1}$ is smooth for all i, j. For simplicity of notation, we assume that $i = 0$ and $j = 1$. We have that

$$\varphi_1 \circ \varphi_0^{-1}(x_1, \ldots, x_n) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right),$$

so it is smooth since $x_1 \neq 0$ on $\varphi_0(U_0 \cap U_1)$.

(d) The *complex projective space* $\mathbf{C}P^n$ is the set of all complex lines through the origin in $\mathbf{C}^{n+1}$. One puts s structure of smooth manifold on it in a similar way as it is done for $\mathbf{R}P^n$. Now the local charts map into $\mathbf{C}^n$, so the dimension of $\mathbf{C}P^n$ is $2n$.

(e) If M and N are smooth manifolds, one defines a smooth structure on the product topological space $M \times N$ as follows. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is an atlas of M and $\{(V_\beta, \varphi_\beta)\}_{\beta\in\mathcal{B}}$ is an atlas of N. Then the family $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \varphi_\beta)\}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$, where

$$\varphi_\alpha \times \varphi_\beta(p, q) = (\varphi_\alpha(p), \varphi_\beta(q)),$$

defines an atlas of $M \times N$. It follows that $\dim M \times N = \dim M + \dim N$.

(f) The *general linear group* $\mathbf{GL}(n, \mathbf{R})$ is the set of all $n \times n$ non-singular real matrices. Since the set of $n \times n$ real matrices can be identified with a $\mathbf{R}^{n^2}$ and as such the determinant becomes a continuous function, $\mathbf{GL}(n, \mathbf{R})$ can be viewed as the open subset of $\mathbf{R}^{n^2}$ where the determinant does not vanish and hence acquires the structure of a smooth manifold of dimension n^2.

(g) Similarly as above, the *complex general linear group* $\mathbf{GL}(n, \mathbf{C})$, which is the set of all $n \times n$ non-singular complex matrices, can be viewed as an open subset of $\mathbf{C}^{2n^2}$ and hence admits the structure of a smooth manifold of dimension $2n^2$. ⋆

Before giving more examples of smooth manifolds, we introduce a new definition. Let N be a smooth manifold of dimension $n + k$. A subset M of N is called an *embedded submanifold* of N of dimension n if M has the topology induced from N and, for every $p \in M$, there exists a local chart (U, φ) of N such that $\varphi(U \cap M) = \varphi(U) \cap \mathbf{R}^n$, where we view $\mathbf{R}^n$ as a subspace of $\mathbf{R}^{n+k}$ in the standard way. We say that (U, φ) is a local chart of M *adapted* to N. Note that an embedded submanifold M of N is a smooth manifold in its own right in that an atlas of M is furnished by the restrictions of the local charts of N to M. Namely, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is an atlas of N, then $\{(U_\alpha \cap M, \varphi_\alpha|_{U_\alpha \cap M})\}_{\alpha \in \mathcal{A}}$ becomes an atlas of M. Note that the compatibility condition for the local charts of M is automatic.

A.2.3 Examples (a) Suppose M is a smooth manifold and $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas of M. Any open subset U of M carries a natural structure of embedded submanifold of M of the same dimension as M simply by restricting the local charts of M to U, namely, one takes the atlas $\{(U_\alpha \cap U, \varphi_\alpha|_{U_\alpha \cap U})\}$.

(b) Let $f : U \to \mathbf{R}^m$ be a smooth mapping, where U is an open subset of $\mathbf{R}^n$. Then the graph of f is a smooth submanifold M of $\mathbf{R}^{n+m}$ of dimension n. In fact, an adapted local chart is given by $\varphi : U \times \mathbf{R}^m \to U \times \mathbf{R}^m$, $\varphi(p, q) = (p, q - f(p))$, where $p \in \mathbf{R}^m$ and $q \in \mathbf{R}^n$. More generally, if a subset M of $\mathbf{R}^{m+n}$ can be covered by open sets each of which is the graph of a smooth mapping from an open subset of $\mathbf{R}^n$ into $\mathbf{R}^m$, then M is an embedded submanifold of $\mathbf{R}^{n+m}$.

(c) The *n-sphere*

$$S^n = \{ (x_1, \dots, x_{n+1}) \mid x_1^2 + \cdots x_{n+1}^2 = 1 \}$$

is an n-dimensional embedded submanifold of $\mathbf{R}^{n+1}$ since each open hemisphere given by an equation of type $x_i > 0$ or $x_i < 0$ is the graph of a smooth mapping $\mathbf{R}^n \to \mathbf{R}$.

(d) The product of n-copies of the circle S^1 is a n-dimensional manifold called the *n-torus* and it is denoted by T^n. ⋆

A smooth mapping between two smooth manifolds is defined to be a continuous mapping whose local representations with respect to charts on both manifolds is smooth. Namely, let M and N be two smooth manifolds and let $\Omega \subset M$ be open. A continuous map $f : \Omega \to N$ is called *smooth* if and only if

$$\psi \circ f \circ \varphi^{-1} : \varphi(\Omega \cap U) \to \psi(V)$$

is smooth as a map between open sets of Euclidean spaces, for every local charts (U, φ) of M and (V, ψ) of N. Clearly, the composition of two smooth maps is again smooth. Also, a map $f : M \to N$ is smooth if and only if M can be covered by open sets such that the restriction of f to each of which is smooth. We denote the space of smooth functions from M to N by $C^\infty(M, N)$. If $N = \mathbf{R}$, we also write $C^\infty(M, \mathbf{R}) = C^\infty(M)$.

A smooth map $f : M \to N$ between smooth manifolds is called a *diffeomorphism* if it is invertible and the inverse $f^{-1} : N \to M$ is also smooth. Also, $f : M \to N$ is called a *local diffeomorphism* if every $p \in M$ admits an open neighborhood U such that $f(U)$ is open and f defines a diffeomorphism from U onto $f(U)$.

A.2.4 Example Let $\mathbf{R}$ denote as usual the real line with it standard topology and standard smooth structure. We denote by M the manifold whose underlying topological space is $\mathbf{R}$ and whose smooth structure is defined as follows. Let $f(x) = \sqrt[3]{x}$. Then f defines a homeomorphism of $\mathbf{R}$, so we can use it to define a global chart, and we set the smooth sructure of M to be given by the maximal atlas containing this chart. Note that $\mathbf{R}$ and M are different as smooth manifolds, since f viewed as a function $\mathbf{R} \to \mathbf{R}$ is not smooth, but f viewed as a function $M \to \mathbf{R}$ is smooth (since the composition $f \circ f^{-1} : \mathbf{R} \to \mathbf{R}$ is smooth). On the other hand, M is diffeomorphic to $\mathbf{R}$. Indeed, $f : M \to \mathbf{R}$ is a diffeomorphism, since it is smooth, bijective, and its inverse $f^{-1} : \mathbf{R} \to M$ which is given by $f^{-1}(x) = x^3$ is also smooth (since $f \circ f^{-1} : \mathbf{R} \to \mathbf{R}$ is smooth). ⋆

A.3 The tangent space

We next set the task of defining the tangent space to a smooth manifold at a given point. Recall that for a surface S in $\mathbf{R}^3$, the tangent space T_pS is defined to be the subspace of $\mathbf{R}^3$ consisting of all the tangent vectors to the smooth curves in S through p. Here a curve in S is called smooth if it is smooth viewed as a curve in $\mathbf{R}^3$, and its tangent vector at a point is obtained by differentiating the curve as such. In the case of a general smooth manifold M, in the absence of a circumventing ambient space, we construct the tangent space T_pM using the only thing at our disposal, namely, the local charts. The idea is to think that T_pM is the abstract vector space whose elements are represented by vectors of $\mathbf{R}^n$ with respect to a given local chart around p, and using a different local chart gives another representation, so we need to identify all those representations via local charts by using an equivalence relation.

Let M be a smooth manifold of dimension n, and let $p \in M$. Suppose that $\mathcal{F}$ is the maximal atlas defining the smooth structure of M. The *tangent space* of M at p is the set T_pM of all pairs (a, φ) — where $a \in \mathbf{R}^n$ and $(U, \varphi) \in \mathcal{F}$ is a local chart around p — quotiented by the equivalence relation

$$(a, \varphi) \sim (b, \psi) \quad \text{if and only if} \quad d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

The fact that this is indeed an equivalence relation follows from the chain rule in $\mathbf{R}^n$. Denote the equivalence class of (a, φ) be $[a, \varphi]$. Each such equivalence class is called a *tangent vector* at p. Note that for a fixed local chart (U, φ) around p, the map

$$a \in \mathbf{R}^n \mapsto [a, \varphi] \in T_pM \tag{A.3.1}$$

is a bijection. It follows from the linearity of $d(\psi \circ \varphi^{-1})_{\varphi(p)}(a)$ that the equivalence relation $\sim$ is compatible with the vector space structure of $\mathbf{R}^n$ in the sense that if $(a, \varphi) \sim (b, \psi)$, $(a', \varphi) \sim (b', \psi)$ and $\lambda \in \mathbf{R}$, then $(\lambda a + a', \varphi) \sim (\lambda b + b', \psi)$. The bottom line is that we can use the bijection (A.3.1) to define a structure of a vector space on T_pM by declaring it to be an isomorphism. The preceding remark implies that this structure does not depend on the choice of local chart around p. Note that $\dim T_pM = \dim M$.

Let $(U, \varphi = (x_1, \ldots, x_n))$ be a local chart of M, and denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$. The *coordinate vectors* at p are defined to be

$$\left.\frac{\partial}{\partial x_i}\right|_p = [e_i, \varphi].$$

Note that

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \tag{A.3.2}$$

is a basis of T_pM.

In the case of $\mathbf{R}^n$, for each $p \in \mathbf{R}^n$ there is a canonical isomorphism $\mathbf{R}^n \to T_p\mathbf{R}^n$ given by

$$a \mapsto [a, \mathrm{id}], \tag{A.3.3}$$

where id is the identity map of $\mathbf{R}^n$. Usually we will make this identification without further comment. If we write $\mathrm{id} = (r_1, \ldots, r_n)$ as we will henceforth do, then this means that $\frac{\partial}{\partial r_i}\big|_p = e_i$.

In particular, $T_p\mathbf{R}^n$ and $T_q\mathbf{R}^n$ are canonically isomorphic for every p, $q \in \mathbf{R}^n$. In the case of a general smooth manifold M, obviously there are no such canonical isomorphisms.

Tangent vectors as directional derivatives

Let M be a smooth manifold, and fix a point $p \in M$. For each tangent vector $v \in T_pM$ of the form $v = [a, \varphi]$, where $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M, and for each $f \in C^\infty(U)$, we define the *directional derivative of f in the direction of v* to be the real number

$$\begin{aligned} v(f) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= d(f \circ \varphi^{-1})(a). \end{aligned}$$

It is a simple consequence of the chain rule that this definition does not depend on the choice of representative of v.

In the case of $\mathbf{R}^n$, $\frac{\partial}{\partial r_i}\big|_p f$ is simply the partial derivative in the direction e_i, the ith vector in the canonical basis of $\mathbf{R}^n$. In general, if $\varphi = (x_1, \ldots, x_n)$, then $x_i \circ \varphi^{-1} = r_i$, so

$$v(x_i) = d(r_i)_{\varphi(p)}(a) = a_i,$$

where $a = \sum_{i=1}^n a_i e_i$. Since $v = [a, \varphi] = \sum_{i=1}^n a_i [e_i, \varphi]$, it follows that

$$v = \sum_{i=1}^n v(x_i) \frac{\partial}{\partial x_i}\Big|_p. \tag{A.3.4}$$

If v is a coordinate vector $\frac{\partial}{\partial x_i}$ and $f \in C^\infty(U)$, we also write

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial f}{\partial x_i}\Big|_p.$$

As a particular case of (A.3.4), take now v to be a coordinate vector of another local chart $(V, \psi = (y_1, \ldots, y_n))$ around p. Then

$$\frac{\partial}{\partial y_j}\Big|_p = \sum_{i=1}^n \frac{\partial x_i}{\partial y_j}\Big|_p \frac{\partial}{\partial x_i}\Big|_p.$$

Note that the preceding formula shows that even if $x_1 = y_1$ we do not need to have $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}$.

The differential

Let $f : M \to N$ be a smooth map between smooth manifolds. Fix a point $p \in M$, and local charts (U, φ) of M around p, and (V, ψ) of N around $q = f(p)$. The *differential of f at p* is the linear map

$$df_p : T_pM \to T_qN$$

given by

$$[a,\varphi] \mapsto [d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}(a),\psi].$$

It is easy to check that this definition does not depend on the choices of local charts. Using the identification (A.3.3) , one checks that $d\varphi_p : T_pM \to \mathbf{R}^n$ and $d\psi_q : T_pM \to \mathbf{R}^n$ are linear isomorphisms and

$$df_p = (d\psi_q)^{-1}\circ d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}\circ d\varphi_p.$$

It is also a simple exercise to prove the following important proposition.

A.3.5 Proposition (Chain rule) *Let M, N, P be smooth manifolds. If $f\in C^\infty(M,N)$ and $g\in C^\infty(N,P)$, then $g\circ f\in C^\infty(M,P)$ and*

$$d(g\circ f)_p = dg_{f(p)}\circ df_p$$

for $p\in M$.

If $f\in C^\infty(M,N)$, $g\in C^\infty(N)$ and $v\in T_pM$, then it is a simple matter of unravelling the definitions to check that

$$df_p(v)(g) = v(g\circ f).$$

Now (A.3.4) together with this equation gives that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right) &= \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right)(y_i)\frac{\partial}{\partial y_i}\Big|_p \\ &= \sum_{i=1}^n \frac{\partial(y_i\circ f)}{\partial x_j}\Big|_p\frac{\partial}{\partial y_i}\Big|_p. \end{aligned}$$

The matrix

$$\left(\frac{\partial(y_i\circ f)}{\partial x_j}\Big|_p\right)$$

is called the *Jacobian matrix of f at p* relative to the given coordinate systems. Observe that the chain rule (Proposition A.3.5) is equivalent to saying that the Jacobian matrix of $g\circ f$ at a point is the product of the Jacobian matrices of g and f at the appropriate points.

Consider now the case in which $N=\mathbf{R}$ and $f\in C^\infty(M)$. Then $df_p : T_pM\to T_{f(p)}\mathbf{R}$, and upon the identification between $T_{f(p)}\mathbf{R}$ and $\mathbf{R}$, we easily see that $df_p(v)=v(f)$. Applying this to $f=x_i$, where $(U,\varphi=(x_1,\ldots,x_n))$ is a local chart around p, and using again (A.3.4) shows that

$$\{dx_1|_p,\ldots,dx_n|_p\}$$

is the basis of T_pM^* dual of the basis (A.3.2), and hence

$$df_p = \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_i}\Big|_p\right)dx_i|_p = \sum_{i=1}^n\frac{\partial f}{\partial x_i}\,dx_i|_p.$$

Finally, we discuss smooth curves on M. A *smooth curve* in M is simply a smooth map $\gamma:(a,b)\to M$ where (a,b) is an interval of $\mathbf{R}$. One can also consider smooth curves γ in M defined on a closed interval $[a,b]$. This simply means that γ admits a smooth extension to an open interval $(a-\epsilon,b+\epsilon)$ for some $\epsilon>0$.

If $\gamma : (a,b) \to M$ is a smooth curve, the *tangent vector* to γ at $t \in (a,b)$ is

$$\dot\gamma(t) = d\gamma_t\left(\left.\frac{\partial}{\partial r}\right|_t\right) \in T_{\gamma(t)}M,$$

where r is the canonical coordinate of $\mathbf{R}$. Note that an arbitrary vector $v \in T_pM$ can be considered to be the tangent vector at 0 to the curve $\gamma(t) = \varphi^{-1}(t, 0, \ldots, 0)$, where (U, φ) is a local chart around p with $\varphi(p) = 0$ and $d\varphi_p(v) = \frac{\partial}{\partial r_1}|_0$.

In the case in which $M = \mathbf{R}^n$, upon identifying $T_{\gamma(t)}\mathbf{R}^n$ and $\mathbf{R}^n$, it is easily seen that

$$\dot\gamma(t) = \lim_{h\to 0}\frac{\gamma(t+h)-\gamma(t)}{h}.$$

The tangent bundle

For a smooth manifold M, there is a canonical way of assembling together all of its tangent spaces at its various points. The resulting object turns out to admit a natural structure of smooth manifold and even the structure of a vector bundle which we will discuss later in **??**.

Let M be a smooth manifold and consider the disjoint union

$$TM = \dot{\bigcup_{p\in M}} T_pM.$$

We can view the elements of TM as equivalence classes of triples (p, a, φ), where $p \in M$, $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M such that $p \in U$, and

$$(p, a, \varphi) \sim (q, b, \psi) \quad \text{if and only if } p = q \text{ and } d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

There is a natural projection $\pi : TM \to M$ given by $\pi[p, a, \varphi] = p$, and then $\pi^{-1}(p) = T_pM$. Next, we use Remark A.2.1 to show that TM inherits from M a structure of smooth manifold of dimension $2\dim M$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a smooth atlas for M. For each $\alpha \in \mathcal{A}$, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a diffeomorphism and, for each $p \in U_\alpha$, $d(\varphi_\alpha)_p : T_pU_\alpha = T_pM \to \mathbf{R}^n$ is the isomorphism mapping $[p, a, \varphi]$ to a. Set

$$\tilde\varphi_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbf{R}^n, \qquad [p, a, \varphi] \to (\varphi_\alpha(p), a).$$

Then $\tilde\varphi_\alpha$ is a bijection and $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbf{R}^{2n}$. Moreover, the maps

$$\tilde\varphi_\beta \circ \tilde\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

are given by

$$(x, a) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(x)\,,\, d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)).$$

Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth diffeomorphism, we have that $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x$ is a linear isomorphism and $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)$ is also smooth on x. It follows that $\{(\pi^{-1}(U_\alpha), \tilde\varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ defines a topology and a smooth atlas for M so that it becomes a smooth manifold of dimension $2n$.

If $f \in C^\infty(M, N)$, then we define the *differential of* f to be the map

$$df : TM \to TN$$

that restricts to $df_p : T_pM \to T_{f(p)}N$ for each $p \in M$. Using the above atlases for TM and TN, we immediately see that $df \in C^\infty(TM, TN)$.

The inverse function theorem

We have now come to state the version for smooth manifolds of the first theorem mentioned in the introduction.

A.3.6 Theorem (Inverse function theorem) *Let $f : M \to N$ be a smooth function between two smooth manifolds M, N, and let $p \in M$ and $q = f(p)$. If $df_p : T_pM \to T_qN$ is an isomorphism, then there exists an open neighborhood W of p such that $f(W)$ is an open neighborhood of q and f restricts to a diffeomorphism from W onto $f(W)$.*

Proof. The proof is really a transposition of the inverse function theorem for smooth mappings between Euclidean spaces to manifolds using local charts. Note that M and N have the same dimension, say, n. Take local charts (U, φ) of M around p and (V, ψ) of N around q such that $f(U) \subset V$. Set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. By the inverse function theorem for smooth mappings of $\mathbf{R}^n$, there exists an open subset $\tilde{W} \subset \varphi(U)$ with $\varphi(p) \in \tilde{W}$ such that $\alpha(\tilde{W})$ is an open neighborhood of $\psi(q)$ and α restricts to a diffeomorphism from $\tilde{W}$ onto $\alpha(\tilde{W})$. It follows that $f = \psi^{-1} \circ \alpha \circ \varphi$ is a diffeomorphism from the open neighborhood $W = \varphi^{-1}(\tilde{W})$ of p onto the open neighborhood $\psi^{-1}(\alpha(\tilde{W}))$ of q. □

A.3.7 Corollary *Let $f : M \to N$ be a smooth function between two smooth manifolds M, N, and let $p \in M$ and $q = f(p)$. Then f is a local diffeomorphism at p if and only if $df_p : T_pM \to T_qN$ is an isomorphism.*

Proof. Half of the statement is just a rephrasing of the theorem. The other half is the easy part, and follows from the chain rule. □

If M is a smooth manifold of dimension n with smooth structure $\mathcal{F}$, then a map $\tau : W \to \mathbf{R}^n$, where W is an open subset of M, is a diffeomorphism onto its image if and only if $(W, \tau) \in \mathcal{F}$ by the maximality of the smooth atlas $\mathcal{F}$. It follows from this remark and the inverse function theorem that if $f : M \to N$ is a local diffeomorpshim at $p \in M$, then there exist local charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ of f is the identity.

A.4 Immersions and submanifolds

The concept of embedded submanifold that was introduced in section A.2 is too strong for some purposes. There are other, weaker notions of submanifolds one of which we discuss now. We first give the following definition. A smooth map $f : M \to N$ between smooth manifolds is called an *immersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is an injective map, and f is called simply an *immersion* if it is an immersion at every point of its domain.

Let M and N be smooth manifolds such that M is a subset of N. We say that M is an *immersed submanifold of* N or simply a *submanifold of* N if the inclusion map of M into N is an immersion. Note that embedded submanifolds are automatically immersed submanifolds, but the main point behind this definition is that the topology of M can be finer than the induced topology from N. Note also that it immediately follows from this definition that if P is a smooth manifold and $f : P \to N$ is an injective immersion, then the image $f(P)$ is a submanifold of N.

A.4.1 Example Take the 2-torus $T^2 = S^1 \times S^1$ viewed as a submanifold of $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$ and consider the map

$$f : \mathbf{R} \to T^2, \qquad f(t) = (\cos at, \sin at, \cos bt, \sin bt),$$

where a, b are non-zero real numbers. Since $f'(t)$ never vanishes, this map is an immersion and its image a submanifold of T^2. We claim that if b/a is an irrational number, then $M = f(\mathbf{R})$ is *not* an embedded submanifold of T^2. In fact, the assumption on b/a implies that M is a dense subset of T^2, but an embedded submanifold of some other manifold is always locally closed. ⋆

A.4.2 Theorem (Local form of an immersion) *Let M and N be smooth manifolds of dimensions n and $n+k$, respectively, and suppose that $f : M \to N$ is an immersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, such that $f(U) \subset V$, and set$\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective, so, up to rearranging indices, we can assume that $d(\pi_1 \circ \alpha)_{\varphi(p)} = \pi_1 \circ d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\pi_1 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$ is the projection onto the first factor. By the inverse function theorem, by shrinking U, we can assume that $\pi_1 \circ \alpha$ is a diffeomorphism from $U_0 = \varphi(U)$ onto its image V_0; let $\beta : V_0 \to U_0$ be its smooth inverse. Now we can describe $\alpha(U_0)$ as being the graph of the smooth map $\gamma = \pi_2 \circ \alpha \circ \beta : V_0 \subset \mathbf{R}^n \to \mathbf{R}^k$, where $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ is the projection onto the second factor. By Example A.2.3, $\alpha(U_0)$ is an embedded submanifold of $\mathbf{R}^{n+k}$ and the map $\tau : V_0 \times \mathbf{R}^k \to V_0 \times \mathbf{R}^k$ given by $\tau(x, y) = (x, y - \gamma(x))$ is a diffeomorphism such that $\tau(\alpha(U_0)) = V_0 \times \{0\}$. Finally, we put $\tilde{\varphi} = \pi_1 \circ \alpha \circ \varphi$ and $\tilde{\psi} = \tau \circ \psi$. Then $(U, \tilde{\varphi})$ and $(V, \tilde{\psi})$ are local charts, and for $x \in \tilde{\varphi}(U) = V_0$ we have that

$$\tilde{\psi} \circ f \circ \tilde{\varphi}(x) = \tau \circ \psi \circ f \circ \varphi^{-1} \circ \beta(x) = \tau \circ \alpha \circ \beta(x) = (x, 0).$$

□

A.4.3 Scholium *If $f : M \to N$ is an immersion at $p \in M$, then there exists an open neighborhood U of p in M such that $f|_U$ is injective and $f(U)$ is an embedded submanifold of N.*

Proof. The local injectivity of f at p is an immediate consequence of the fact that some local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$, hence, injective. Moreover, in the proof of the theorem, we have seen that $\alpha(U_0)$ is an embedded submanifold of $\mathbf{R}^{n+k}$. Since $\psi(f(U)) = \alpha(U_0)$ and ψ is a diffeomorphism, it follows that $f(U)$ is an embedded submanifold of N. □

The preceding result is particularly useful in geometry when dealing with local properties of an isometric immersion.

A smooth map $f : M \to N$ between manifolds is called an *embedding* if it is an injective immersion which is also a homeomorphism into $f(M)$ with the relative topology.

A.4.4 Scholium *If $f : M \to N$ is an embedding, then the image $f(M)$ is an embedded submanifold of N.*

Proof. In the proof of the theorem, we have seen that $\tilde{\psi}(f(U)) = V_0 \times \{0\}$. Since f is an open map into $f(M)$ with the relative topology, we can find an open subset W of N contained in V such that $W \cap f(M) = f(U)$. The result follows. □

Recall that a continuous map between locally compact, Hausdorff topological spaces is called *proper* if the inverse image of a compact subset of the counter-domain is a compact subset of the domain. It is known that proper maps are closed. Also, it is clear that if the domain is compact, then every continuous map is automatically proper. An embedded submanifold M of a smooth manifold N is called *properly embedded* if the inclusion map is proper. Now the following proposition is a simple remark.

A.4.5 Proposition *If $f : M \to N$ is an injective immersion which is also a proper map, then the image $f(M)$ is a properly embedded submanifold of N.*

If $f : M \to N$ is a smooth map between manifolds whose image lies in a submanifold P of N and P does not carry the relative topology, it may happen f viewed as a map into P is discontinuous.

A.4.6 Theorem *Suppose that $f : M \to N$ is smooth and P is an immersed submanifold of N such that $f(M) \subset P$. Consider the induced map $f_0 : M \to P$ that satisfies $i \circ f_0 = f$, where $i : P \to N$ is the inclusion.*

a. If P is an embedded submanifold of N, then f_0 is continuous.

b. If f_0 is continuous, then it is smooth.

Proof. (a) If $V \subset P$ is open, then $V = W \cap P$ for some open subset $W \subset N$. By continuity of f, we have that $f_0^{-1}(V) = f^{-1}(W)$ is open in M, hence also f_0 is continuous.

(b) Let $p \in M$ and $q = f(p) \in P$. Take a local chart $\psi : V \to \mathbf{R}^n$ of N around q. By the local form of an immersion, there exists a projection from $\mathbf{R}^n$ onto a subspace obtained by setting certain coordinates equal to 0 such that $\tau = \pi \circ \psi \circ i$ is a local chart of P defined on a neighborhood U of q. Note that $f_0^{-1}(U)$ is a neighborhood of p in M. Now

$$\tau \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ i \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ f|_{f_0^{-1}(U)},$$

and the latter is smooth. □

A submanifold P of N with the property that given any smooth map $f : M \to N$ with image lying in P, the induced map into P is also smooth will be called a *quasi-embedded submanifold*.

A.5 Submersions and inverse images

Submanifolds can also be defined by equations. In order to explain this point, we introduce the following definition. A smooth map $f : M \to N$ between manifolds is called a *submersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is a surjective map, and f is called simply a *submersion* if it is a submersion at every point of its domain.

A.5.1 Theorem (Local form of a submersion) *Let M an N be smooth manifolds of dimensions $n+k$ and n, respectively, and suppose that $f : M \to N$ is a submersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard projection of $\mathbf{R}^{n+k}$ onto $\mathbf{R}^n$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^{n+k} \to \mathbf{R}^n$ is surjective, so, up to rearranging indices, we can assume that $d(\alpha \circ \iota_1)_{\varphi(p)} = d\alpha_{\varphi(p)} \circ \iota_1 : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\iota_1 : \mathbf{R}^n \to \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k$ is the standard inclusion. Define $\tilde{\alpha} : \varphi(U) \subset \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k$ by $\tilde{\alpha}(x, y) = (\alpha(x, y), y)$. Since $d\alpha_{\varphi(p)} \circ \iota_1$ is an isomorphism, it is clear that $d\tilde{\alpha}_{\varphi(p)} : \mathbf{R}^n \oplus \mathbf{R}^k \to \mathbf{R}^n \oplus \mathbf{R}^k$ is an isomorphism. By the inverse function theorem, there exists an open neighborhood U_0 of $\varphi(p)$ contained in $\varphi(U)$ such that $\tilde{\alpha}$ is a diffeomorphism from U_0 onto its image V_0; let $\tilde{\beta} : V_0 \to U_0$ be its smooth inverse. We put $\tilde{\varphi} = \tilde{\alpha} \circ \varphi$. Then $(\varphi^{-1}(U_0), \tilde{\varphi})$ is a local chart of M around p and

$$\psi \circ f \circ \tilde{\varphi}^{-1}(x, y) = \psi \circ f \circ \varphi^{-1} \circ \tilde{\beta}(x, y) = \alpha \circ \tilde{\beta}(x, y) = x.$$

□

A.5.2 Corollary *Let $f : M \to N$ be a smooth map, and let $q \in N$ be such that $f^{-1}(q) \neq \varnothing$. If f is a submersion at all points of $P = f^{-1}(q)$, then P admits the structure of an embedded submanifold of M of dimension* $\dim M - \dim N$.

Proof. It is enough to construct local charts of M that are adapted to P and whose domains cover P. So suppose $\dim M = n + k$, $\dim N = n$, let $p \in P$ and consider local charts (U, φ) and (V, ψ) as in Theorem A.5.1 such that $p \in U$ and $q \in V$. We can assume that $\psi(q) = 0$. Now it is obvious that $\varphi(U \cap P) = \varphi(U) \cap \mathbf{R}^n$, so φ is an adapted chart around p. $\square$

A.5.3 Examples (a) Let A be a non-degenerate real symmetric matrix of order $n + 1$ and define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(p) = \langle Ap, p\rangle$ where $\langle , \rangle$ is the standard Euclidean inner product. Then $df_p : \mathbf{R}^{n+1} \to \mathbf{R}$ is given by $df_p(v) = 2\langle Ap, v\rangle$, so it is surjective if $p \neq 0$. It follows that f is a submersion on $\mathbf{R}^{n+1} \setminus \{0\}$ and $f^{-1}(r)$ for $r \in \mathbf{R}$ is an embedded submanifold of $\mathbf{R}^{n+1}$ of dimension n if it is nonempty. In particular, by taking A to be the identity matrix we get a manifold structure for S^n which coincides with the one previously constructed.

(b) Denote by V the vector space of real symmetric matrices of order n, and define $f : GL(n, \mathbf{R}) \to V$ by $f(A) = AA^t$. We first claim that f is a submersion at the identity matrix I. One easily computes that

$$df_I(B) = \lim_{h \to 0} \frac{f(I + hB) - f(I)}{h} = B + B^t,$$

where $B \in T_I GL(n, \mathbf{R}) = M(n, \mathbf{R})$. Now, given $C \in V$, df_I maps $\frac{1}{2}C$ to C, so this checks the claim. We next check that f is a submersion at any $D \in f^{-1}(I)$. Note that $DD^t = I$ implies that $f(AD) = f(A)$. This means that $f = f \circ R_D$, where $R_D : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ is the map that multiplies on the right by D. We have that R_D is a diffeomorphism of $GL(n, \mathbf{R})$ whose inverse is plainly given by $R_{D^{-1}}$. Therefore $d(R_D)_I$ is an isomorphism, so the chain rule $df_I = df_D \circ d(R_D)_I$ yields that df_D is surjective, as desired. Now $f^{-1}(I) = \{ A \in GL(n, \mathbf{R}) \mid AA^t = I \}$ is an embedded submanifold of $GL(n, \mathbf{R})$ of dimension

$$\dim GL(n, \mathbf{R}) - \dim V = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Note that $f^{-1}(I)$ is a group with respect to the multiplication of matrices; it is called the *orthogonal group* of order n and is usually denoted by $O(n)$. ⋆

A.6 Partitions of unity

In general, a locally compact, Hausdorff, second-countable topological space is paracompact (every open covering of the space admits an open locally finite refinement) and σ-compact (it is a countable union of compact subsets). The σ-compactness immediately implies that every open covering of the space admits a countable open refinement. Paracompactness can used to prove that the existence of smooth partitions of unity on smooth manifolds, an extremely useful tool in the theory. Partitions of unity are used to piece together locally defined smooth objects on the manifold to construct a global one, and conversely to represent global objects by locally finite sums of locally defined ones. Recall that a *partition of unity* subordinate to an open covering $\{U_i\}_{i \in I}$ of a smooth manifold M is a collection $\{\lambda_i\}_{i \in I}$ of nonnegative smooth functions on M such that the family of supports $\{\operatorname{supp} \lambda_i\}$ is locally finite (this means that every point of M admits an open neighborhood intersecting only finitely many members of the family), $\operatorname{supp} \lambda_i \subset U_i$ for every $i \in I$, and $\sum_{i \in I} \lambda_i = 1$.

The starting point of the construction of smooth partitions of unity is the remark that the function

$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is smooth everywhere. Therefore the function

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

is smooth, flat and equal to 0 on $(-\infty, 0]$, and flat and equal to 1 on $[1, +\infty)$. Finally,

$$h(t) = g(t+2)g(2-t)$$

is smooth, flat and equal to 1 on $[-1, 1]$ and its support lies in $(-2, 2)$. We refer to [War83, Theorem 1.11] for the proof of the existence of smooth partitions of unity subordinate to an arbitrary open covering of a smooth manifold. In the following, we will do some applications.

A.6.1 Examples (a) Suppose $\{U_i\}_{i\in I}$ is an open covering of M and for each $i \in I$ we are given $f_i \in C^\infty(U_i)$. Take a partition of unity $\{\lambda_i\}_{i\in I}$ subordinate to that covering. Then the formula

(A.6.2) $$f = \sum_{i\in I} \lambda_i f_i$$

defines a smooth function on M. In fact, given $p \in M$, for each $i \in I$ it is true that either $p \in U_i$ and then f_i is defined at p, or $p \notin U_i$ and then $\lambda_i(p) = 0$. Moreover, since $\{\operatorname{supp} \lambda_i\}$ is locally finite, there exists an open neighborhood of p on which all but finitely many terms in the sum in (A.6.2) vanish, and this shows that f is well defined and smooth.

(b) Let C be closed in M and let U be open in M with $C \subset U$. Then there exists a smooth function $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda|_C = 1$ and $\operatorname{supp} \lambda \subset U$. Indeed, it suffices to consider a partition of unity subordinate to the open covering $\{U, M \setminus C\}$. ⋆

The following result is a related application. We note that the full Whitney embedding theorem does not require compactness of the manifold and it also provides an estimate on the dimension of the Euclidean space.

A.6.3 Theorem (Weak form of the Whitney embedding theorem) *Let M be a compact smooth manifold. Then there exists an embedding of M into $\mathbf{R}^n$ for n suffciently big.*

Proof. Since M is compact, there exists an open covering $\{(V_i, \varphi_i)\}_{i=1}^a$ such that for each i, $\bar{V_i} \subset U_i$ where (U_i, φ_i) is a local chart of M. For each i, we can find $\lambda_i \in C^\infty(M)$ such that $0 \leq \lambda_i \leq 1$, $\lambda_i|_{\bar{V_i}} = 1$ and $\operatorname{supp} \lambda_i \subset U_i$. Put

$$f_i(x) = \begin{cases} \lambda_i(x)\varphi_i(x), & \text{if } x \in U_i, \\ 0, & \text{if } x \in M \setminus U_i. \end{cases}$$

Then $f_i : M \to \mathbf{R}^m$ is smooth, where $m = \dim M$. Define also smooth functions

$$g_i = (f_i, \lambda_i) : M \to \mathbf{R}^{m+1} \quad \text{and} \quad g = (g_1, \ldots, g_a) : M \to \mathbf{R}^{a(m+1)}.$$

It is enough to check that g is an injective immersion. In fact, on the open set V_i, we have that $g_i = (\varphi_i, 1)$ is an immersion, so g is an immersion. Further, if $g(x) = g(y)$ for x, $y \in M$, then $\lambda_i(x) = \lambda_i(y)$ and $f_i(x) = f_i(y)$ for all i. Take an index j such that $\lambda_j(x) = \lambda_j(y) \neq 0$. Then x, $y \in U_j$ and $\varphi_j(x) = \varphi_j(y)$. Due to the injectivity of φ_j, we must have $x = y$. Hence g is injective. □

A.7 Vector fields

Let M be a smooth manifold. A *vector field* on M is a map $X : M \to TM$ such that $X(p) \in T_pM$ for $p \in M$. Sometimes, we also write X_p instead of $X(p)$. As we have seen, TM carries a canonical manifold structure, so it makes sense to call X is a smooth vector field if the map $X : M \to TM$ is smooth. Hence, a smooth vector field on M is a smooth assignment of tangent vectors at the various points of M. From another point of view, recall the natural projection $\pi : TM \to M$; the requirement that $X(p) \in T_pM$ for all p is equivalent to having $\pi \circ X = \mathrm{id}_M$.

More generally, let $f : M \to N$ be a smooth mapping. Then a *vector field along* f is a map $X : M \to TN$ such that $X(p) \in T_{f(p)}N$ for $p \in M$. The most important case is that in which f is a smooth curve $\gamma : [a, b] \to N$. A vector field along γ is a map $X : [a, b] \to TN$ such that $X(t) \in T_{\gamma(t)}N$ for $t \in [a, b]$. A typical example is the tangent vector field $\dot\gamma$.

Let X be a vector field on M. Given a smooth function $f \in C^\infty(U)$ where U is an open subset of M, the directional derivative $X(f) : U \to \mathbf{R}$ is defined to be the function $p \in U \mapsto X_p(f)$. Further, if $(x_1, \ldots, x_n)$ is a coordinate system on U, we have already seen that $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of T_pM for $p \in U$. It follows that there are functions $a_i : U \to \mathbf{R}$ such that

$$X|_U = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}. \tag{A.7.1}$$

A.7.2 Proposition *Let X be a vector field on M. Then the following assertions are equivalent:*

a. X is smooth.

b. For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, the functions a_i defined by (A.7.1) are smooth.

c. For every open set V of M and $f \in C^\infty(V)$, the function $X(f) \in C^\infty(V)$.

Proof. Suppose X is smooth and let $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ be a coordinate system on U. Then $X|_U$ is smooth and $a_i = dx_i \circ X|_U$ is also smooth.

Next, assume (b) and let $f \in C^\infty(V)$. Take a coordinate system $(U, (x_1, \ldots, x_n))$ with $V \subset U$. Then, by using (b) and the fact that $\frac{\partial f}{\partial x_i}$ is smooth,

$$X(f)|_U = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} \in C^\infty(U).$$

Since V can be covered by such U, this proves (c).

Finally, assume (c). For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, we have a corresponding coordinate system $(\pi^{-1}(U), x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$ of TM. Then

$$(x_i \circ \pi) \circ X|_U = x_i \quad \text{and} \quad dx_i \circ X|_U = X(x_i)$$

are smooth. This proves that X is smooth. □

In particular, the proposition shows that the coordinate vector fields $\frac{\partial}{\partial x_i}$ associated to a local chart are smooth. The arguments in the proof also show that if X is a vector field on M satisfying $X(f) = 0$ for every $f \in C^\infty(V)$ and every open $V \subset M$, then $X = 0$. This remark forms the basis of our next definition, and is explained by noting that in the local expression (A.7.1) for a coordinate system defined on $U \subset V$, the functions $a_i = dx_i \circ X|_U = X(x_i) = 0$.

Next, let X and Y be smoth vector fields on M. Their *Lie bracket* $[X, Y]$ is defined to be the unique vector field on M that satisfies

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{A.7.3}$$

for every $f \in C^\infty(M)$. By the remark in the previous paragraph, such a vector field is unique if it exists. In order to prove existence, consider a coordinate system $(U, (x_1, \ldots, x_n))$. Then we can write

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y|_U = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$$

for a_i, $b_j \in C^\infty(U)$. If $[X, Y]$ exists, we must have

$$[X,Y]|_U = \sum_{i=1}^n \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}, \tag{A.7.4}$$

because the coefficients of $[X, Y]|_U$ in the local frame $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ must be given by $[X, Y](x_j) = X(Y(x_j)) - Y(X(x_j))$. We can use formula A.7.4 as the definition of a vector field on U; note that such a vector field is smooth and satisfies property (A.7.3) for functions in $C^\infty(U)$. We finally define $[X, Y]$ globally by covering M with domains of local charts: on the overlap of two charts, the different definitions coming from the two charts must agree by the above uniqueness result; it follows that $[X, Y]$ is well defined.

A.7.5 Proposition *Let X, Y and Z be smooth vector fields on M. Then*
 a. $[Y, X] = -[X, Y]$.
 b. If f, $g \in C^\infty(M)$, then

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

 c. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. *(Jacobi identity)*

We omit the proof of this propostion which is simple and only uses (A.7.3). Note that (A.7.3) combined with the commutation of mixed second partial derivatives of a smooth function implies that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for coordinate vector fields associated to a local chart.

Let $f : M \to N$ be a diffeomorphism. For every smooth vector field X on M, the formula $df \circ X \circ f^{-1}$ defines a smooth vector field on N which we denote by f_*X. More generally, if $f : M \to N$ is a smooth map which needs not be a diffeomorphism, smooth vector fields X on M and Y on N are called *f-related* if $df \circ X = Y \circ f$. The proof of the next propostion is an easy application of (A.7.3).

A.7.6 Proposition *Let $f : M \to M'$ be smooth. Let X, Y be smooth vector fields on M, and let X', Y' be smooth vector fields on M'. If X and X' are f-related and Y and Y' are f-related, then also $[X, Y]$ and $[X', Y']$ are f-related.*

Flow of a vector field

Next, we discuss how to "integrate" vector fields. Let X be a smooth vector field on M. An *integral curve* of X is a smooth curve γ in M such that

$$\dot{\gamma}(t) = X(\gamma(t))$$

for all t in the domain of γ.

In order to study existence and uniqueness questions for integral curves, we consider local coordinates. So suppose $\gamma : (a, b) \to M$ is a smooth curve in M, $0 \in (a, b)$, $\gamma(0) = p$, $(U, \varphi =$

$(x_1,\dots,x_n))$ is a local chart around p, X is a smooth vector field in M and $X|_U = \sum_i a_i \frac{\partial}{\partial x_i}$ for $a_i \in C^\infty(U)$. Then γ is an integral curve of X on $\gamma^{-1}(U)$ if and only if

$$\frac{d\gamma_i}{dr}\Big|_t = (a_i \circ \varphi^{-1})(\gamma_1(t),\dots,\gamma_n(t)) \tag{A.7.7}$$

for $i = 1,\dots,n$ and $t \in \gamma^{-1}(U)$, where $\gamma_i = x_i \circ \gamma$. Equation (A.7.7) is a system of first order ordinary differential equations for which existence and uniqueness theorems are known. These, translated into manifold terminology yield the following proposition.

A.7.8 Proposition *Let X be a smooth vector field on M. For each $p \in M$, there exists a (possibly infinite) interval $(a(p), b(p)) \subset \mathbf{R}$ and a smooth curve $\gamma_p : (a(p), b(p)) \to M$ such that:*

a. *$0 \in (a(p), b(p))$ and $\gamma_p(0) = p$.*
b. *γ_p is an integral curve of X.*
c. *γ_p is maximal in the sense that if $\mu : (c, d) \to M$ is a smooth curve satisfying (a) and (b), then $(c, d) \subset (a(p), b(p))$ and $\mu = \gamma_p|_{(c,d)}$.*

Let X be a smooth vector field on M. Put

$$\mathcal{D}_t = \{\, p \in M \mid t \in (a(p), b(p)) \,\}$$

and define $X_t : \mathcal{D}_t \to M$ by setting

$$X_t(p) = \gamma_p(t).$$

Note that we have somehow reversed the rôles of p and t with this definition. The collection of X_t for all t is called the *flow* of X.

A.7.9 Example Take $M = \mathbf{R}^2$ and $X = \frac{\partial}{\partial r_1}$. Then $\mathcal{D}_t = \mathbf{R}^2$ for all $t \in \mathbf{R}$ and $X_t(a_1, a_2) = (a_1 + t, a_2)$ for $(a_1, a_2) \in \mathbf{R}^2$. Note that if we replace $\mathbf{R}^2$ by the punctured plane $\mathbf{R}^2 \setminus \{(0, 0)\}$, the sets $\mathcal{D}_t$ become proper subsets of M. ⋆

A.7.10 Theorem a. *For each $p \in M$, there exists an open neighborhood V of p and $\epsilon > 0$ such that the map*

$$(-\epsilon, \epsilon) \times V \to M, \quad (t, p) \mapsto X_t(p)$$

is well defined and smooth.

b. *The domain $\mathrm{dom}(X_s \circ X_t) \subset \mathcal{D}_{s+t}$ and $X_{s+t}|_{\mathrm{dom}(X_s \circ X_t)} = X_s \circ X_t$. Further, $\mathrm{dom}(X_s \circ X_t) = X_{s+t}$ if $st > 0$.*
c. *$\mathcal{D}_t$ is open for all t, $\cup_{t>0}\mathcal{D}_t = M$ and $X_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse X_{-t}.*

Proof. Part (a) is a local result and is simply the smooth dependence of solutions of ordinary differential equations on the intial conditions. We prove part (b). First, we remark the obvious fact that, if $p \in \mathcal{D}_t$, then $s \mapsto \gamma_p(s + t)$ is an integral curve of X with initial condition $\gamma_p(t)$ and maximal domain $(a(p) - t, b(p) - t)$; therefore $(a(p) - t, b(p) - t) = (a(\gamma_p(t)), b(\gamma_p(t)))$. Next, let $p \in \mathrm{dom}(X_s \circ X_t)$. This means that $p \in \mathrm{dom}(X_t) = \mathcal{D}_t$ and $\gamma_p(t) = X_t(p) \in \mathrm{dom}(X_s) = \mathcal{D}_s$. Then $s \in (a(\gamma_p(t)), b(\gamma_p(t)))$, so $s+t \in (a(\gamma_p(t))+t, b(\gamma_p(t))+t) = (a(p), b(p))$, that is $p \in \mathcal{D}_{s+t}$. Further, $X_{s+t}(p) = \gamma_p(s + t) = \gamma_{\gamma_p(t)}(s) = X_s(X_t(p))$ and we have already proved the first two assertions. Next, assume that $s, t > 0$ (the case $s, t \leq 0$ is similar); we need to show that $\mathcal{D}_{s+t} \subset \mathrm{dom}(X_s \circ X_t)$. But this follows from reversing the argument above as $p \in \mathcal{D}_{s+t}$ implies that $s + t \in (a(p), b(p))$, and this implies that $t \in (a(p), b(p))$ and $s = (s + t) - t \in (a(p) - t, b(p) - t) = (a(\gamma_p(t)), b(\gamma_p(t)))$. Finally, we prove part (c). The statement about the union follows from part (a). Note that

$\mathcal{D}_0 = M$. Fix $t > 0$ and $p \in \mathcal{D}_t$; we prove that p is an interior point of $\mathcal{D}_t$ and X_t is smooth on a neighborhood of p (the case $t < 0$ is analogous). Indeed, since $\gamma_p([0,t])$ is compact, part (a) yields an open neighborhood W_0 of this set and $\epsilon > 0$ such that $(s,q) \in (-\epsilon,\epsilon) \times W_0 \mapsto X_s(q) \in M$ is well defined and smooth. Take an integer $n > 0$ such that $t/n < \epsilon$ and put $\alpha_1 = X_{\frac{t}{n}}|_{W_0}$. Then, define inductively $W_i = \alpha_i^{-1}(W_{i-1}) \subset W_{i-1}$ and $\alpha_i = X_{\frac{t}{n}}|_{W_i-1}$ for $i = 2,\ldots,n$. It is clear that α_i is smooth and W_i is an open neighborhood of $\gamma_p(\frac{n-i}{n}t)$ for all i. In particular, W_n is an open neighborhood of p in W. Moreover,

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n|_{W_n} = (X_{\frac{t}{n}})^n|_{W_n} = X_t|_{W_n}$$

by the last assertion of part (b), so X_t is smooth on W_n. Now $\mathcal{D}_t$ is open and X_t is smooth on $\mathcal{D}_t$. It is obvious that the image of X_t is $\mathcal{D}_{-t}$. Since X_{-t} is also smooth on $\mathcal{D}_{-t}$, it follows from part (b) that X_t and X_{-t} are inverses one of the other and this completes the proof of the theorem. □

The Frobenius theorem

Let M be a smooth manifold of dimension n. A *distribution* $\mathcal{D}$ *of rank* (or *dimension*) k is a choice of k-dimensional subspace $\mathcal{D}_p \subset T_pM$ for each $p \in M$. A distribution $\mathcal{D}$ of rank k is called *smooth* if for every $p \in M$ there exists an open neighborhood U of p and k smooth vector fields $X_1,\ldots,X_k$ on U such that $\mathcal{D}_q$ equals the span of $X_1(q),\ldots,X_k(q)$ for every $q \in U$. A vector field X is said to *belong to* (or *lie in*) the distribution $\mathcal{D}$ (and we write $X \in \mathcal{D}$) if $X(p) \in \mathcal{D}_p$ for $p \in M$. A distribution $\mathcal{D}$ is called *involutive* if X, $Y \in \mathcal{D}$ implies that $[X,Y] \in \mathcal{D}$. A submanifold N of M is called an *integral manifold* of a distribution $\mathcal{D}$ if $T_pN = \mathcal{D}_p$ for $p \in N$.

If X is a nowhere zero smooth vector field on M, then of course the line spanned by X_p in T_pM for $p \in M$ defines a smooth distribution on M. In this special case, Proposition A.7.8 guarantees the existence and uniqueness of maximal integral submanifolds. Our next intent is to generalize this result to arbitrary smooth distributions. A necessary condition is given in the following proposition. The contents of the Frobenius theorem is that the condition is also sufficient.

A.7.11 Proposition *A smooth distribution $\mathcal{D}$ on M admitting integral manifolds through any point of M must be involutive.*

Proof. Given smooth vector fields X, $Y \in \mathcal{D}$ and $p \in M$, we need to show that $[X,Y]_p \in \mathcal{D}_p$. By assumption, there exists an integral manifold N passing thorugh p. By shrinking N, we may further assume that N is embedded. Denote by ι the inclusion of N into M. Then $d\iota_{\iota^{-1}(p)} : T_{\iota^{-1}(p)}N \to T_pM$ is an isomorphism onto $\mathcal{D}_p$. Therefore there exist vector fields $\tilde{X}$ and $\tilde{Y}$ on N which are ι-related to resp. X and Y. Due Theorem A.4.6, $\tilde{X}$ and $\tilde{Y}$ are smooth, so by using Proposition A.7.6 we finally get that $[X,Y]_p = d\iota([\tilde{X},\tilde{Y}]_{\iota^{-1}(p)}) \in \mathcal{D}_p$. □

It is convenient to use the following terminology in the statement of the Frobenius theorem. A coordinate system $(U, \varphi = (x_1,\ldots,x_n))$ of a smooth manifold M of dimension m will be called *cubic* if $\varphi(U)$ is an open cube centered at the origin of $\mathbf{R}^m$, and it will be called *centered at a point* $p \in U$ if $\varphi(p) = 0$.

A.7.12 Theorem (Frobenius, local version) *Let $\mathcal{D}$ be a smooth distribution of rank k on a smooth manifold of dimension n. Suppose that $\mathcal{D}$ is involutive. Then, given $p \in M$, there exists an integral manifold of $\mathcal{D}$ containing p. More precisely, there exists a cubic coordinate system $(U, \varphi = (x_1,\ldots,x_n))$ centered at p such that the "slices"*

$$x_i = \text{constant} \qquad \textit{for } i = k+1,\ldots,n$$

are integral manifolds of $\mathcal{D}$. Further, if $N \subset U$ is a connected *integral manifold of $\mathcal{D}$, then N is an open submanifold of one of these slices.*

Proof. We proceed by induction on k. Suppose first that $k = 1$. Choose a smooth vector field $X \in \mathcal{D}$ defined on a neighborhood of p such that $X_p \neq 0$. It suffices to construct a coordinate system $(U, \varphi = (x_1, \ldots, x_n))$ around p such that $X|_U = \frac{\partial}{\partial x_1}|_U$. Indeed, it is easy to get a coordinate system $(V, \psi = (y_1, \ldots, y_n))$ centered at p such that $\frac{\partial}{\partial y_1} = Y_p$. The map

$$\sigma(t, a_2, \ldots, a_n) = X_t(\psi^{-1}(0, a_2, \ldots, a_n))$$

is well defined and smooth on $(-\epsilon, \epsilon) \times W$ for some $\epsilon > 0$ and some neighborhood W of the origin in $\mathbf{R}^{n-1}$. We immediately see that

$$d\sigma\left(\frac{\partial}{\partial r_1}\Big|_0\right) = X_p = \frac{\partial}{\partial y_1}\Big|_p \quad \text{and} \quad d\sigma\left(\frac{\partial}{\partial r_i}\Big|_0\right) = \frac{\partial}{\partial y_i}\Big|_p \quad \text{for } i = 2, \ldots, n.$$

By the inverse function theorem, σ is a local diffeomorphism at 0, so its local inverse yields the desired local chart φ.

We next assume the theorem is true for distributions of rank $k-1$ and prove it for a given distribution $\mathcal{D}$ of rank k. Choose smooth vector fields $X_1, \ldots, X_n$ spanning $\mathcal{D}$ on a neighborhood $\tilde{V}$ of p. The result in case $k = 1$ yields a coordinate system $(V, y_1, \ldots, y_n)$ centered at p such that $V \subset \tilde{V}$ and $X_1|_V = \frac{\partial}{\partial y_1}|_V$. Define the following smooth vector fields on V:

$$\begin{aligned} Y_1 &= X_1 \\ Y_i &= X_i - X_i(y_1)X_1 \qquad \text{for } i = 2, \ldots, k \end{aligned}$$

Plainly, $Y_1, \ldots, Y_k$ span $\mathcal{D}$ on V. Let $S \subset V$ be the slice $y_1 = 0$ and put

$$Z_i = Y_i|_S \qquad \text{for } i = 2, \ldots, k.$$

Since

(A.7.13)
$$Y_i(y_1) = X_i(y_1) - X_i(y_1)\underbrace{X_1(y_1)}_{=1} = 0 \qquad \text{for } i = 2, \ldots, k,$$

we have $Z_i(q) \in T_qS$ for $q \in S$, so $Z_2, \ldots, Z_k$ span a smooth distribution $\mathcal{D}'$ of rank $k-1$ on S; we next check that $\mathcal{D}'$ is involutive. Since Z_i and Y_i are related under the inclusion $S \subset V$, also $[Z_i, Z_j]$ and $[Y_i, Y_j]$ are so related. Eqn. (A.7.13) gives that $[Y_i, Y_j](y_1) = 0$, so, on V

$$[Y_i, Y_j] = \sum_{\ell=1}^{k} c_{ijk} Y_\ell$$

for $c_{ijk} \in C^\infty(V)$. Hence

$$[Z_i, Z_j] = \sum_{\ell=1}^{k} c_{ijk}|_S Z_\ell,$$

as we wished. By the inductive hypothesis, there exists a coordinate system $(w_2, \ldots, w_n)$ on some neighborhood of p in S such that the slices $w_i =$ constant for $i = k+1, \ldots, n$ define integral manifolds of $\mathcal{D}'$.

Let $\pi : V \to S$ be the linear projection relative to $(y_1, \ldots, y_n)$. Set

$$\begin{aligned} x_1 &= y_1, \\ x_i &= w_i \circ \pi \qquad \text{for } i = 2, \ldots, n. \end{aligned}$$

It is clear that there exists an open neighborhood U of p in V such that $(U, \varphi = (x_1, \ldots, x_n))$ is a cubic coordinate system of M centered at p. Now the first assertion in the statement of the theorem follows if we prove that $Y_i(x_j) = 0$ on U for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$, for this will imply that $\frac{\partial}{\partial x_1}|_q, \ldots, \frac{\partial}{\partial x_k}|_q$ is a basis of $\mathcal{D}_q$ for every $q \in U$. In order to do that, note that

$$\frac{\partial x_j}{\partial y_1} = \begin{cases} 1, & \text{if } j = 1, \\ 0, & \text{if } j = 2, \ldots, n \end{cases}$$

on U. Hence

$$Y_1 = X_1 = \frac{\partial}{\partial y_1} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_1} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_1},$$

so $Y_1(x_j) = 0$ or $j > k$. Next, take $i \leq k$ and $j > k$. Owing to the involutivity of $\mathcal{D}$,

$$[Y_1, Y_i] = \sum_{\ell=1}^{k} c_{i\ell} Y_\ell$$

for some $c_{ik} \in C^\infty(U)$. Therefore

$$\frac{\partial}{\partial x_1}(Y_i(x_j)) = Y_1(Y_i(x_j)) - Y_i(Y_1(x_j)) = Y_1(Y_i(x_j)) = \sum_{\ell=2}^{k} c_{i\ell} Y_\ell(x_j),$$

which, for fixed $x_2, \ldots, x_n$, is a system of $k-1$ homogeneous linear ordinary differential equations in the functions $Y_\ell(x_j)$ of the variable x_1. Of course, the initial condition $x_1 = 0$ corresponds to $S \cap U$ along which we have

$$Y_i(x_j) = Z_i(x_j) = Z_i(w_j) = 0,$$

where the latter equatility follows from the fact that Z_i lies in $\mathcal{D}'$ and $w_j =$ constant for $j > k$ define integral manifolds of $\mathcal{D}'$. By the uniqueness theorem of solutions of ordinary differential equations, $Y_i(x_j) = 0$ on U.

Finally, suppose that $N \subset U$ is a connected integral manifold of $\mathcal{D}$. Let ι denote the inclusion of N into U let and $\pi : \mathbf{R}^n \to \mathbf{R}^k \times \mathbf{R}^{n-k}$ be the projection. Then $d(\pi \circ \varphi \circ \iota)_q(T_q N) = d(\pi \circ \pi)_q(\mathcal{D}_q) = 0$ for $q \in U$. By connectedness of N, $\pi \circ \varphi \circ \iota$ is a constant function on N. Thus N is contained in one of the slices $x_i =$ constant for $i = k+1, \ldots, n$, say S. The inclusion of N into M is continuous (since N is a submanifold of M) and has image contained in S; since S is embedded in M, the inclusion of N into S is continuous and thus smooth by Theorem A.4.6. Since N is a submanifold of M, this inclusion is also an immersion. Of course, $\dim N = \dim S$, so this inclusion is indeed a local diffeomorphism. Hence N is an open submanifold of S. □

A.7.14 Theorem *Integral manifolds of involutive distributions are quasi-embedded submanifolds. More precisely, suppose that $f : M \to N$ is smooth, P is an integral manifold of an involutive distribution $\mathcal{D}$ on M, and $f(M) \subset P$. Consider the induced map $f_0 : M \to P$ that satisfies $\iota \circ f_0 = f$, where $\iota : P \to N$ is the inclusion. Then f_0 is continuous (and hence smooth by Theorem A.4.6).*

Proof. Let U be an open subset of P, $q \in U$ and $p \in f_0^{-1}(q)$. It suffices to prove that p is an interior point of $f_0^{-1}(U)$. By the local version of the Frobenius theorem (A.7.12), there exists a cubic coordinate system $(V, \psi = (x_1, \dots, x_n))$ of N centered at q such that the slices

$$x_i = \text{constant} \qquad \text{for } i = k+1, \dots, n \tag{A.7.15}$$

are the integral manifolds of $\mathcal{D}$ in V. Also, we can shrink V so that $V \cap U$ is exactly the slice

$$x_{k+1} = \cdots = x_n = 0. \tag{A.7.16}$$

We have that $f^{-1}(V)$ an open neighborhood of q in M; let W be its connected component containing p. Of course, W is open. It is enough to show that $f_0(W) \subset V \cap U$, or which is the same, $f(W)$ is contained in (A.7.16). Since $f(W)$ is connected, it is contained in a component of $V \cap P$ with respect to the relative topology. Since $f(W)$ meets (A.7.16) at least at the point q, it suffices to show that the components of $V \cap P$ in the relative topology are contained in the slices of the form (A.7.15). Let C be a component of $V \cap P$ with respect to the relative topology; note that C need not be connected in the topology of P, but, by second-countability of P, C is a countable union of connected integral manifolds of $\mathcal{D}$ in V, each of which is contained in a slice of the form (A.7.15). Let $\pi : V \to \mathbf{R}^{n-k}$ be given by $\pi(r) = (x_{k+1}(r), \dots, x_n(r))$. It follows that $\pi(C)$ is a countable connected subset of $\mathbf{R}^{n-k}$; hence, it is a single point. □

A *maximal integral manifold* of a distribution $\mathcal{D}$ on a manifold M is a connected integral manifold N of $\mathcal{D}$ such that every connected integral manifold of $\mathcal{D}$ which intersects N is an open submanifold of N.

A.7.17 Theorem (Frobenius, global version) *Let $\mathcal{D}$ be a smooth distribution on M. Suppose that $\mathcal{D}$ is involutive. Then through any given point of M there passes a unique integral manifold of $\mathcal{D}$.*

Proof. Let $\dim M = n$ and $\dim \mathcal{D} = k$. Given $p \in M$, define N to be the set of all points of M reachable from p by following piecewise smooth curves whose smooth arcs are everywhere tangent to $\mathcal{D}$. By the local version A.7.12 and the σ-compactness of its topology, M can be covered by countably many cubic coordinate systems $(U_i, x_1^i, \dots, x_n^i)$ such that the integral manifolds of $\mathcal{D}$ in U_i are exactly the slices

$$x_j^i = \text{constant} \qquad \text{for } j = k+1, \dots, n. \tag{A.7.18}$$

Note that a slice of the form (A.7.18) that meets N must be contained in N, and that N is covered by such slices. We equip N with the finest topology with respect to which the inclusions of all such slices are continuous. We can also put a differentiable structure on N by declaring that such slices are open submanifolds of N. We claim that this turns N into a connected smooth manifold of dimension k. N is clearly connected since it is path-connected by construction. N is also Hausdorff, because M is Hausdorff and the inclusion of N into M is continuous. It only remains to prove that N is second-countable. It suffices to prove that only countably many slices of U_i meet N. For this, we need to show that a single slice S of U_i can only meet countably many slices of U_j. For this purpose, note that $S \cap U_j$ is an open submanifold of S and therefore consists of at most countably many components, each of which being a connected integral manifold of $\mathcal{D}$ and hence lying entirely in a slice of U_j. Now it is clear that N is an integral manifold of $\mathcal{D}$ through p.

Next, let N' be another connected integral manifold of $\mathcal{D}$ meeting N at a point q. Given $q' \in N'$, there exists a piecewise smooth curve integral curve of $\mathcal{D}$ joining q to q' since connected manifolds are path-connected. Due to $q \in N$, this curve can be juxtaposed to a piecewise smooth curve integral curve of $\mathcal{D}$ joining p to q. We get $q' \in N$ and this proves that $N' \subset N$. Since N' is a submanifold of M, the inclusion of N' into M is continuous. By Theorem A.7.14, the inclusion of N' into N is smooth. Hence N' is an open submanifold of N.

Finally, suppose that N' is in addition to the above a maximal integral manifold. The above argument shows that $N \subset N'$ and N is an open submanifold of N'. It follows that the identity map $N \to N'$ is a diffeomorphism and this proves the uniqueness of N. □