CHAPTER 2

Connections

2.1 Introduction

Contemplate $\mathbf{R}^n$. Of course, the presence of the identity map as a global chart allows one to canonically identify the tangent spaces of $\mathbf{R}^n$ at its various points with $\mathbf{R}^n$ itself. Therefore, a smooth vector field X in $\mathbf{R}^n$ can be viewed simply as a smooth map $X : \mathbf{R}^n \to \mathbf{R}^n$. Thus, one has a canonical way of differentiating vector fields in $\mathbf{R}^n$, namely, if X, $Y : \mathbf{R}^n \to \mathbf{R}^n$ are two vector fields, then the derivative of Y along X is the directional derivative $dY(X) = X(Y)$.

Whereas a smooth manifold M comes already equipped with a notion of derivative of smooth maps, there is no canonical way to differentiate vector fields on M. We solve this problem by considering all possible ways of defining derivatives of vector fields. Any such choice is called a connection. The name originates from the fact that, at least along a given curve, a connection provides a way to identify ("connect") tangent spaces of M at different points; this is the idea of parallel transport along the curve. A geodesic is then a curve whose velocity vector is constant in this sense.

The main consequence of the theory of connections for Riemannian geometry is that a Riemannian metric on M uniquely specifies a connection on M, called the Levi-Cività connection. In the case in which M is a surface in $\mathbf{R}^3$, for the Levi-Cività connection on M we recover the derivative in $\mathbf{R}^3$ projected back to M.

Connections can be defined in a variety of ways. We will use the Koszul formalism.

2.2 Connections

Let M be a smooth manifold. A *(Koszul) connection* in M is a bilinear map $\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$, where we write $\nabla_X Y$ instead of $\nabla(X, Y)$, such that

a. $\nabla_{fX} Y = f\nabla_X Y$, and

b. $\nabla_X (fY) = X(f)Y + f\nabla_X Y$ (Leibniz rule)

for every X, $Y \in \Gamma(TM)$ and $f \in C^\infty(M)$.

Let ∇ be a connection in a smooth manifold M. We want to analyse of the dependence of ∇ on its arguments. To begin with, we claim that, for a given open set U in M, $(\nabla_X Y)|_U$ depends only on $X|_U$ and $Y|_U$. Indeed, let X', $Y' \in \Gamma(TM)$ be vector fields satisfying $X'|_U = X|_U$ and $Y'|_U = Y|_U$. Fix $p \in U$. Construct a smooth function f on M with support contained in U and such that $f \equiv 1$ on some neighborhood V of p with $V \subset \bar{V} \subset U$. Then, using part (a) in the definition of connection and the fact that $fX = fX'$ on M,

$$(\nabla_X Y)_p = f(p)(\nabla_X Y)_p = (f\nabla_X Y)_p = (\nabla_{fX} Y)_p = (\nabla_{fX'} Y)_p = f(p)(\nabla_{X'} Y)_p = (\nabla_{X'} Y)_p$$

This shows that $\nabla_X Y = \nabla_{X'} Y$ on U. Next, note that $fY = fY'$ on M implies that $\nabla_X(fY) = \nabla_X(fY')$, so the Leibniz rule and the facts that $f(p) = 1$, $X_p(f) = 0$ imply that $(\nabla_X Y)_p = (\nabla_X Y')_p$. Since p was taken to be an arbitrary point in U, $\nabla_X Y = \nabla_X Y'$ on U, and this completes the check of the claim.

2.2.1 Remark In a moment, we will refine the above discussion and show that, for a given point $p \in M$, the value of $(\nabla_X Y)_p$ depends only on X_p and the restriction of Y along a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = X_p$. Indeed, this is a consequence of the expression of the connection (2.2.4).

Choose a chart $(U, \varphi = (x^1, \ldots, x^n))$ of M around p. We know from the above that $\nabla_X Y|_U = \nabla_{X|_U}(Y|_U)$. Write

$$X|_U = \sum_j a^j \frac{\partial}{\partial x^j} \quad \text{and} \quad Y|_U = \sum_k b^k \frac{\partial}{\partial x^k}$$

for a^i, $b^j \in C^\infty(U)$. Then, using the defining properties of a connection, in the open set U,

$$\begin{aligned}
\nabla_X Y &= \nabla_X \left(\sum_k b^k \frac{\partial}{\partial x^k} \right) \\
&= \sum_k X(b^k) \frac{\partial}{\partial x^k} + b^k \nabla_X \frac{\partial}{\partial x^k} \\
&= \sum_{j,k} a^j \frac{\partial b^k}{\partial x^j} \frac{\partial}{\partial x^k} + \sum_{j,k} a^j b^k \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \\
&= \sum_{i,j} a^j \frac{\partial b^i}{\partial x^j} \frac{\partial}{\partial x^i} + \sum_{i,j,k} a^j b^k \Gamma^i_{jk} \frac{\partial}{\partial x^i},
\end{aligned}$$

where we have set

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_k \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

It follows that the local representation of $\nabla_X Y$ in the chart (U, φ) is

$$\nabla_X Y = \sum_i \left(\sum_j a^j \frac{\partial b^i}{\partial x^j} + \sum_{j,k} \Gamma^i_{jk} a^j b^k \right) \frac{\partial}{\partial x^i}. \tag{2.2.2}$$

In particular,

$$(\nabla_X Y)_p = \sum_i \left(\sum_j a^j(p) \frac{\partial b^i}{\partial x^j}(p) + \sum_{j,k} \Gamma^i_{jk}(p) a^j(p) b^k(p) \right) \frac{\partial}{\partial x^i}\Big|_p. \tag{2.2.3}$$

It is also convenient to rewrite the preceding formula in the following form

$$(\nabla_X Y)_p = \sum_i \left(X_p(b^i) + \sum_{j,k} \Gamma^i_{jk}(p) a^j(p) b^k(p) \right) \frac{\partial}{\partial x^i}\Big|_p. \tag{2.2.4}$$

Note that this formula involves only the values of the a^j, b^k at p, and the directional derivatives of the b^i in the direction of X_p, so the claim in Remark 2.2.1 is checked.

The smooth functions Γ_{jk}^i are called the *Christoffel symbols* of ∇ with respect to the chosen chart. The Christoffel symbols of a connection satisfy a complicated rule of change upon change of coordinates, which will be used in the proof of Proposition 2.3.1. For the moment, we just want to remark that the Christoffel symbols can be used to specify a connection locally. For instance, one could set Γ_{jk}^i identically zero in a given chart (U, φ) and then define a connection for vector fields on U. Doing this for a family of charts whose domains cover the manifold, and noting that a convex linear combination of connections is still a connection, a smooth partition of unity can be thus used to define a global connection in M in analogy with the argument in tbe proof of Proposition 1.2.3. This proves that connections exist in any given manifold.

Rather than insisting in the argument of the preceding paragraph, it is better to use Proposition 2.2.5 below in order to construct a connection in a given manifold. Indeed, in an n-dimensional smooth manifold, we need n^3 smooth functions Γ_{jk}^i to specify a connection locally, and we need n^2 smooth functions g_{ij} to specify a Riemannian metric locally, recall (1.2.1). Even taking into account equivalence classes of such objects, it is apparent that there exist "more" connections in a given smooth manifold than the already large amount of available Riemannian metrics. The point is that, as shown by the next proposition, a Riemannian manifold admits a preferred connection.

2.2.5 Proposition *Let (M, g) be a Riemannian manifold. Then there exists a unique connection ∇ in M, called the* Levi-Cività connection*, such that:*
a. $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$*, and*
b. $\nabla_X Y - \nabla_Y X - [X, Y] = 0$
for every vector fields X, Y, $Z \in \Gamma(TM)$.

Proof. The strategy of the proof is to first use the two conditions in the statement to deduce a formula for ∇. This formula is called the *Koszul formula*, and this proves uniqueness. The next steps, which are easy but tedious and will be skipped, are to use the Koszul formula to define the connection, and to check the defined object indeed satisfies the defining conditions of a connection and the conditions in the statement of this theorem.

Let X, Y and Z be vector fields in M. The so-called permutation trick is to use condition (a) to write

$$\begin{array}{rcl} Xg(Y,Z) & = & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,Z) & = & g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ -Zg(X,Z) & = & -g(\nabla_Z X, Y) - g(X, \nabla_Z Y), \end{array}$$

add up these equations, and use condition (b) to arrive at the Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\big(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X)\big) \tag{2.2.6}$$

Note that this formula uniquely defines $\nabla_X Y$, since Z is arbitrary and g is nondegenerate. $\square$

The condition (a) in Proposition 2.2.5 is usually refered to as saying that the connection ∇ is *compatible with the metric g*, or that ∇ is a *metric connection.* The condition (b) expresses the fact that the *torsion* of ∇, which is defined as the left-hand-side therein, is nul.

Henceforth, in this book, for a given Riemannian manifold, we will always use the Levi-Cività connection in order to differentiate tangent vectors.

2.2.7 Example Consider the upper half-plane $\mathbf{R}^2_+ = \{ (x,y) \in \mathbf{R}^2 \mid y > 0 \}$ endowed with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$. In this example, we show a practical method to compute the Levi-Cività connection of $(\mathbf{R}^2_+, g)$. Start with $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{y^2}$: differentiate it with respect to y and use Proposition 2.2.5(a) to write

$$2g\left(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y^2}\right) = -2\frac{1}{y^3},$$

so

$$g\left(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -\frac{1}{y^3}; \tag{2.2.8}$$

similarly, differentiate it with respect to x to get

$$g\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 0.$$

Next, consider $g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \frac{1}{y^2}$; differentiation with respect to x and y yields respectively

$$g\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0, \qquad g\left(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = -\frac{1}{y^3}. \tag{2.2.9}$$

We use Proposition 2.2.5(b) in the form of

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} - \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0,$$

where the last equality holds because $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are coordinate vector fields. Now differentiation of $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$ gives that

$$g\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -g\left(\frac{\partial}{\partial x}, \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\right) = -g\left(\frac{\partial}{\partial x}, \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}\right) = \frac{1}{y^3},$$

where we have used (2.2.8) in the last equality, and it also gives

$$g\left(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) = -g\left(\frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}\right) = 0,$$

where we have used the first formula of (2.2.9) in the last equality. Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are orthogonal everywhere, it easily follows from the above formulas that

$$\begin{cases} \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} & = & & \frac{1}{y}\frac{\partial}{\partial y} \\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} & = & -\frac{1}{y}\frac{\partial}{\partial x} & \\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} & = & & -\frac{1}{y}\frac{\partial}{\partial y} \end{cases}$$

⋆

2.3 Parallel transport along a curve

Let (M, g) be a Riemannian manifold, and denote by ∇ its Levi-Cività connection.

A *vector field along a curve* $\gamma : I \to M$, $I \subset \mathbf{R}$ an interval, is a map $X : I \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for all t. If γ is a smooth curve, the most obvious example of a vector field aong γ is its tangent vector field $\gamma'(t)$. In general, if γ is an embedding, then any vector field along γ can be extended to a smooth vector field in M defined on a neighborhood of the image of γ. On the other hand, if γ is not an embedding, then there are vector fields along γ that do not come from vector fields defined on open subsets of M. An example is given by taking γ to be a curve with self-intersections, or even a constant curve.

The set of smooth vector fields along a curve $\gamma : I \to M$ will be denoted $\Gamma(\gamma^*TM)$. The connection ∇ in M induces a derivative of vector fields along γ as follows.

2.3.1 Proposition *Let $\gamma : I \to M$ be a smooth curve. Then there exists a unique linear map $\frac{\nabla}{dt} : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$, called the* covariant derivative along γ, *satisfying the following conditions:*

a. $\frac{\nabla}{dt}(fX) = \frac{df}{dt}X + f\frac{\nabla}{dt}X$ for every smooth function $f : I \to \mathbf{R}$.

b. If X admits an extension to a vector field $\bar{X}$ defined on a open subset U of M, then

$$\left(\frac{\nabla}{dt}X\right)(t) = (\nabla_{\gamma'(t)}\bar{X})_{\gamma(t)}$$

for every t satisfying $\gamma(t) \in U$.

Proof. We first prove the uniqueness result. Suppose first that the image of γ lies in the domain of one chart $(U, \varphi = (x^1, \ldots, x^n))$. Then we can write $\gamma(t) = (x^1(t), \ldots, x^n(t))$, so

$$\gamma'(t) = \sum_j (x^j)'(t)\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}.$$

If X is a vector field along γ, we can also write

$$X(t) = \sum_k a^k(t)\frac{\partial}{\partial x^k}\Big|_{\gamma(t)}.$$

Note that, although in general X cannot be extended to a vector field defined on an open set of M, X is written as a linear combination of vector fields that admit such extensions. So, if we have a linear map as in the statement, then

$$\begin{aligned}
\frac{\nabla}{dt}X &= \sum_k (a^k)'\frac{\partial}{\partial x^k} + a^k\nabla_{\gamma'(t)}\frac{\partial}{\partial x^k} \\
&= \sum_i (a^i)'\frac{\partial}{\partial x^i} + \sum_{j,k} a^k(x^j)'\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} \\
&= \sum_i (a^i)'\frac{\partial}{\partial x^i} + \sum_{i,j,k} a^k(x^j)'\Gamma^i_{jk}\frac{\partial}{\partial x^i} \\
&= \sum_i \left((a^i)' + \sum_{j,k}\Gamma^i_{jk}(x^j)'a^k\right)\frac{\partial}{\partial x^i} \qquad (2.3.2)
\end{aligned}$$

In general, one sees by a argument analogous to that used in section 2.2 that $(\frac{\nabla}{dt}X)|_J$ depends only on $X|_J$ for any open subinterval J of I, and the image of γ can be covered by finitely many domains of charts, so the local expressions show that $\frac{\nabla}{dt}$ is uniquely defined, if it exists.

In order to prove existence, one uses the local expression to define $\frac{\nabla}{dt}$ in the domain of a chart. Then, one needs to show that the definition is independent of the choice of chart. Here it is necessary to use the rule of change for the Christoffel symbols (cf. Exercise 3). Finally, one easily checks that the defined map satisfies the two conditions in the statement. $\square$

A vector field X along a smooth curve $\gamma : I \to \mathbf{R}$ is called *parallel* if $\frac{\nabla}{dt}X = 0$ on I. This definition can be obviously extended to include curves that are only piecewise smooth.

2.3.3 Proposition *Let $\gamma : I \to M$ be a piecewise smooth curve, and let $t_0 \in I$. Given a vector $v \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field X along γ such that $X(t_0) = v$.*

Proof. Suppose first that I is bounded. The image of γ can be covered by finitely many domains of charts of M. Thus, without loss of generality, we may assume that the image of γ lies in the domain of one chart $(U, \varphi = (x^1, \ldots, x^n))$. Write $\gamma(t) = (x^1(t), \ldots, x^n(t))$ and

$$X(t) = \sum_k a^k(t) \frac{\partial}{\partial x^k}\Big|_{\gamma(t)}.$$

Then, equation (2.3.2) implies that $\frac{\nabla}{dt}X = 0$ is equivalent to

$$(a^i)' + \sum_{j,k} \Gamma^i_{jk}(x^j)' a^k = 0 \qquad (2.3.4)$$

for all i. This is a system of ordinary linear differential equations of first order in the unknowns $a^1, \ldots, a^n$, which is known to have unique solutions defined on all of I for given initial conditions. In our case, the initial conditions are given by $a_k(t_0) = dx^k(v)$.

In the general case, we can cover I by the union of a chain of increasing bounded intervals, construct X along each bounded interval, and use the uniqueness result to see that so constructed vector fields piece together to yield a global solution. $\square$

It follows from the proof of the preceding proposition that the map that assigns to a vector $v \in T_{\gamma(t_0)}M$ a parallel vector field $X \in \Gamma(\gamma^*TM)$ with $X(t_0) = v$ is linear. Evaluating X at another time t_1 gives thus a linear map $P^\gamma_{t_1,t_0} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ which will be called the *parallel translation map along γ from t_0 to t_1.*

2.3.5 Proposition *Let $\gamma : I \to M$ be a piecewise smooth curve. Then the parallel translation maps along γ enjoys the following properties:*

a. $P^\gamma_{t_0,t_0}$ is the identity map of $T_{\gamma(t_0)}M$;
b. $P^\gamma_{t_2,t_1} \circ P^\gamma_{t_1,t_0} = P^\gamma_{t_2,t_0}$ (chain rule);
c. $P^\gamma_{t_0,t_1} = (P^\gamma_{t_1,t_0})^{-1}$;
d. $P^\gamma_{t_1,t_0} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ is an isometry;

for every t_0, t_1, $t_2 \in I$.

Proof. Assertions (a), (b) and (c) are immediate. We show that assertion (d) is a consequence of condition (a) in the definition of a connection (in fact, it is equivalent to that condition) as follows. If X is a parallel vector field along γ, then $\frac{\nabla X}{dt} = 0$ along γ, so

$$\frac{d}{dt} g(X(t), X(t)) = 2g(\left(\frac{\nabla}{dt}X\right)(t), X(t)) = 0,$$

and the norm of X is constant along γ. $\square$

2.3.6 Example We now use the result of Example 2.2.7 to describe the parallel transport map along the curve $\gamma(t) = (t, y_0)$ in $(\mathbf{R}^2_+, g)$, where $y_0 > 0$. Denote by $X(t) = a(t)\frac{\partial}{\partial x} + b(t)\frac{\partial}{\partial y}$ a smooth vector field along γ, where a, $b : \mathbf{R} \to \mathbf{R}$ are smooth functions. Then

$$\begin{aligned} \frac{\nabla}{dt}X &= a'\frac{\partial}{\partial x} + a\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + b\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} \\ &= \left(a' - \frac{b}{y_0}\right)\frac{\partial}{\partial x} + \left(b' + \frac{a}{y_0}\right)\frac{\partial}{\partial y}, \end{aligned}$$

so the condition that X be parallel is that

$$\begin{cases} a' = \omega b \\ b' = -\omega a \end{cases}$$

where $\omega = y_0^{-1}$. The general solution of this system of first-order ordinary differential equations is

$$\begin{aligned} a(t) &= a_0 \cos\omega t + b_0 \sin\omega t \\ b(t) &= -a_0 \sin\omega t + b_0 \cos\omega t \end{aligned}$$

where $(a(0), b(0)) = (a_0, b_0)$. It follows that

$$P^\gamma_{t,0}\left(a_0\frac{\partial}{\partial x} + b_0\frac{\partial}{\partial y}\right) = (a_0\cos\omega t + b_0\sin\omega t)\frac{\partial}{\partial x} + (-a_0\sin\omega t + b_0\cos\omega t)\frac{\partial}{\partial y}$$

which is merely rotation in the Euclidean sense at a constant rate; note that the rate $\omega \to \infty$ as $y_0 \to 0$. ⋆

2.4 Geodesics

Let (M, g) be a Riemannian manifold, and denote by ∇ its Levi-Cività connection.

A smooth curve $\gamma : I \to M$, $I \subset M$ an interval, is called a *geodesic* if and only if $\frac{\nabla}{dt}\gamma' = 0$ on I. Thus we require that the tangent vector field γ' be parallel along γ. According to 2.3.5(d), this implies that the length of γ' must be constant. We also refer to the latter property as saying that γ is a curve *parametrized with constant speed* or γ is a curve p*arametrized proportional to arc-length*. Observe that constant curves are geodesics.

We can get the local expression of the geodesic equation immediately from (2.3.4). Let $\gamma : I \to M$ be a smooth curve whose image lies in the domain of a chart $(U, \varphi = (x^1, \ldots, x^n))$ of M. Writing $\gamma(t) = (x^1(t), \ldots, x^n(t))$, we have that $\frac{\nabla}{dt}\gamma' = 0$ if and only if

$$(x^i)'' + \sum_{j,k}\Gamma^i_{jk}(x^j)'(x^k)' = 0 \qquad (2.4.1)$$

for all i. Note that this is a second order system of non-linear ordinary differential equations in the unknowns $x^1, \ldots, x^n$, for which we have a local existence and uniqueness result. Indeed, we quote the following theorem from [Spi70].

2.4.2 Theorem *Consider the second order system of ordinary differential equations*

$$\sigma'' = F(\sigma, \sigma'),$$

where $F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a smooth map, in the unknown $\sigma : I \to \mathbf{R}^n$, $I \subset \mathbf{R}$ an open interval. Then, given $(x_0, a_0) \in \mathbf{R}^n \times \mathbf{R}^n$, there exists a neighborhood $U \times V$ of (x_0, a_0) and $\delta > 0$

such that, for any $(x,a) \in U \times V$, there is a unique solution $\sigma_{x,a} : (-\delta, \delta) \to \mathbf{R}^n$ with initial conditions $\sigma_{x,a}(0) = x$ and $\sigma'_{x,a}(0) = a$. Moreover, the map $\Sigma : U \times V \times (-\delta,\delta) \to M$, defined by $\Sigma(x,a,t) = \sigma_{x,a}(t)$, is smooth.

It also follows from the theory of ordinary differential equations that any solution of the geodesic equation (2.4.1) is automatically smooth. Equation (2.4.1) has a particular homogeneity feature that we explore now. Namely, if $\gamma : (a,b) \to M$ is a solution of (2.4.1), then it is immediate to check that for every $k \in \mathbf{R} \setminus \{0\}$ the curve $\eta : (\frac{a}{k}, \frac{b}{k}) \to \mathbf{R}$ defined by $\eta(t) = \gamma(kt)$ is also a solution.

2.4.3 Proposition *Given $p \in M$, there exists a neighborhood U of p and $\epsilon > 0$ such that, for any $q \in U$ and $v \in T_qM$ with $g_q(v,v)^{1/2} \leq \epsilon$, there is a unique geodesic $\gamma_v : (-2,2) \to M$ such that $\gamma_v(0) = q$ and $\gamma'_v(0) = v$. Moreover, the map $\Gamma : \cup_{q\in U} B(0_q, \epsilon) \times (-2,2) \to M$ defined by $\Gamma(v,t) = \gamma_v(t)$ is smooth.*

Proof. Let (V, φ) be a local chart of M around p, and consider the map $d\varphi : TM|_V \to \varphi(V) \times \mathbf{R}^n$. The geodesic equation in M corresponds via $d\varphi$ to a second order differential equation for curves on $\varphi(V) \times \mathbf{R}^n$, to which we apply Theorem 2.4.2. We deduce that there exists an open neighborhood of 0_p in TM such that for every $v \in W$ there exists a unique geodesic $\gamma_v : (-\delta, \delta) \to M$ such that $\gamma_v(0) = \pi(v)$ and $\gamma'_v(0) = v$, where $\pi : TM \to M$ is the projection, and $\gamma_v(t)$ is smooth on $(v,t) \in W \times (-\delta, \delta)$. By continuity of g, we may shrink W and assume that it is of the form

$$W = \{\, v \in TM|_U \; : \; g_{\pi(v)}(v,v)^{1/2} < \epsilon' \,\}$$

for some open neighborhood U of p in M and some $\epsilon' > 0$ (cf. Exercise 1). The homogeneity of the geodesic equation refered to above yields that multipliying the length of v by $\delta/2$ makes the interval of definition of γ_v to be multiplied by $2/\delta$. Therefore we can take $\epsilon = \epsilon'\delta/2$ and we are done. $\square$

Henceforth, in this book, for $p \in M$ and $v \in T_pM$, we will denote by γ_v the unique geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. Note that the homogeneity of the geodesic equation yields that $\gamma_{kv}(t) = \gamma_v(kt)$. It follows from Proposition 2.4.3 that there exists open neighborhood Ω of the zero section in TM consisting of vectors v such that $\gamma_v(1)$ is defined. The *exponential map*

$$\exp : \Omega \to M$$

is defined by setting $\exp(v) = \gamma_v(1)$. It follows from the last assertion in Proposition 2.4.3 that the exponential map is smooth. Sometimes we will also write $\exp_p = \exp|_{T_pM}$ for $p \in M$. Now $\gamma_v(t) = \gamma_{tv}(1) = \exp_p(tv)$ for $v \in T_pM$ and sufficiently small t.

2.4.4 Proposition *Let $p \in M$. Then:*

a. The exponential map exp_p maps an open neighborhood of $0_p \in T_pM$ diffeomorphically onto an open neighborhood of p in M.

b. There exists an open neighborhood U of p and $\epsilon > 0$ such that, for any $q \in U$, there exists a unique $v \in T_pM$ with $g_p(v,v)^{1/2} < \epsilon$ such that $\exp_p v = q$.

Proof. We compute the differential $d(\exp_p)_{0_p} : T_{0_p}(T_pM) \to T_pM$. Recall that $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$ for $v \in T_pM$. Differentiating this equation with respect to t at $t = 0$ yields that

$$d(\exp_p)_{0_p}(v) = \gamma'_v(0) = v. \tag{2.4.5}$$

Hence $d(\exp_p)_{0_p}$ is the identity, where as usual we have identified $T_{0_p}(T_pM)$ with T_pM. It follows from the inverse function theorem that $\exp_p$ maps an open neighborhood of 0_p in T_pM, which can be taken of the form $B(0_p, \epsilon)$ for some $\epsilon > 0$, diffeomorphically onto an open neighborhood of p in M. Parts (a) and (b) follow. $\square$

The neighborhood of p given in the previous proposition is usually called a *normal neighborhood* of p. Hence we have that any point in a normal neighborhood of p can be joined to p by a unique geodesic in that neighborhood. Next, we want to improve this result in the sense of connecting two movable points in a neighborhood of p by a geodesic. We need a lemma.

2.4.6 Lemma *Let $\pi : TM \to M$ be the projection. Then, given $p \in M$, the map*

$$\Phi : \Omega \to M \times M, \qquad \Phi(v) = (\pi(v), \exp(v))$$

is a local diffeomorphism from an open neighborhood W of 0_p in TM onto an open neighborhood of (p, p) in $M \times M$.

Proof. The result follows from the inverse function theorem if we can show that $d\Phi_{0_p} : T_{0_p}(TM) \to T_pM \oplus T_pM$ is an isomorphism. Each vector in the tangent space $T_{0_p}(TM)$ is the tangent vector at $t = 0$ to a curve c in TM passing through 0_p at $t = 0$. First, let $c(t) = tv \in TM$ where $v \in T_pM$. Then $d\Phi_{0_p}(c'(0)) = \frac{d}{dt}\big|_{t=0}\Phi(c(t)) = \frac{d}{dt}\big|_{t=0}(p, \exp_p(tv)) = (0, v)$ by equation (2.4.5). Next, let $c(t) = 0_{\gamma(t)} \in T_{\gamma(t)}M \subset TM$, where γ is a curve in M with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$. Then $d\Phi_{0_p}(c'(0)) = \frac{d}{dt}\big|_{t=0}\Phi(0_{\gamma(t)}) = \frac{d}{dt}\big|_{t=0}(\gamma(t), \gamma(t)) = (v, v)$. The two calculations together imply that $d\Phi_{0_p}$ is surjective and hence, by dimensional reasons, an isomorphism. $\square$

2.4.7 Proposition *Given $p \in M$, there exists an open neighborhood U of p and $\epsilon > 0$ such that:*

a. For any x, $y \in U$, there exists a unique $v \in T_xM$ with $g_x(v, v)^{1/2} < \epsilon$ such that $\exp_x v = y$. Set $\gamma_v(t) = \exp_x(tv)$.
b. The map $\Psi : U \times U \times [0, 1]$ defined by $\Psi(x, y, t) = \gamma_v(t)$ is smooth.
c. For all $x \in U$, the map $\exp_x$ is a diffeomorphism from $B(0_x, \epsilon)$ onto a normal neighborhood of x containing U.

Proof. (a) Let W be a neighborhood of 0_p in TM such that $\Phi(v) = (\pi(v), \exp(v))$ is a diffeomorphism of W onto a neighborhood of (p, p) in $M \times M$ as in Lemma 2.4.6. By shrinking W, if necessary, we may assume that $W = \cup_{x \in V} B(0_x, \epsilon)$ for some open neighborhood V of p and some $\epsilon > 0$. Let U be a neighborhood of p in M such that $U \times U \subset \Phi(W)$. Then, for any $(x, y) \in U \times U$, there is a unique $v \in W$ such that $\Phi(v) = (x, y)$, meaning that there is a unique $v \in B(0_x, \epsilon)$ such that $\exp_x v = y$.

(b) This follows immediately from the fact that $\Psi(x, y, t) = \exp(t\Phi^{-1}(x, y))$.

(c) Since $B(0_x, \epsilon) \subset W$, the map Φ is a diffeomorphism from $B(0_x, \epsilon)$ onto its image. But, for fixed $x \in U$, $\Phi(v) = (x, \exp_x(v))$ for $v \in B(0_x, \epsilon)$. $\square$

The set U in the preceding proposition is a normal neighborhood of each of its points; we will call such a set U an *ϵ-totally normal neighborhood* of p. Note that it is not claimed that the geodesic γ_v is that proposition is entirely contained in U. However, it is possible to work a bit harder and find a possibly smaller totally normal neighborhood of p with that property. [1]

2.4.8 Example In order to complete our analysis of the Riemannian manifold $(\mathbf{R}^2_+, g)$ of Examples 2.2.7 and 2.3.6, we now determine its geodesics. So let $\gamma(t) = (x(t), y(t))$ be a smooth curve in $\mathbf{R}^2_+$. Then $\gamma' = x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}$ and

$$\frac{\nabla}{dt}\gamma' = x'' \frac{\partial}{\partial x} + x' \frac{\nabla}{dt}\frac{\partial}{\partial x} + y'' \frac{\partial}{\partial y} + x' \frac{\nabla}{dt}\frac{\partial}{\partial y}.$$

[1] Ref?

We also have

$$\frac{\nabla}{dt}\frac{\partial}{\partial x} = x'\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + y'\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = -\frac{y'}{y}\frac{\partial}{\partial x} + \frac{x'}{y}\frac{\partial}{\partial y},$$

and

$$\frac{\nabla}{dt}\frac{\partial}{\partial y} = x'\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} + y'\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -\frac{x'}{y}\frac{\partial}{\partial x} - \frac{y'}{y}\frac{\partial}{\partial y},$$

so

$$\frac{\nabla}{dt}\gamma' = \Big(x'' - 2\frac{x'y'}{y}\Big)\frac{\partial}{\partial x} + \Big(y'' + \frac{x'^2 - y'^2}{y}\Big)\frac{\partial}{\partial y}.$$

Therefore the geodesic equations are

$$\left\{ \begin{array}{l} x'' - 2\frac{x'y'}{y} = 0 \\ y'' + \frac{x'^2 - y'^2}{y} = 0 \end{array} \right. \tag{2.4.9}$$

Note that $x(t) = x_0$ is a solution of (2.4.9); indeed, the second equation gives that

$$\left(\frac{y'}{y}\right)' = \frac{y''y - y'^2}{y^2} = 0,$$

so $y(t) = y_0e^{kt}$ where $y_0 > 0$ and $k \in \mathbf{R}$. This shows that the vertical lines are geodesics. Note that in the parametrization that we obtained, they are defined on all of $\mathbf{R}$.

Next, suppose that γ is a geodesic which is not a vertical line. By the uniqueness result for geodesics, it follows that $x'(t) \neq 0$ for all t in the domain of γ. The first equation of (2.4.9) then gives

$$\frac{x''}{x'} = 2\frac{y'}{y}$$

from where we get that

$$(\log(x'))' = (2\log y)'$$

and hence that

$$x' = cy^2 \tag{2.4.10}$$

for some real constant c which may be assumed to be positive by reversing the orientation of γ, if necessary. Of course γ is parametrized with constant speed, which for simplicity we assume it is 1; then $\frac{1}{y^2}(x'^2 + y'^2) = 1$; substituing (2.4.10) gives that

$$\frac{dy}{y\sqrt{1 - c^2y^2}} = \pm dt$$

Direct integration then yields

$$\operatorname{arcsech}(cy) = \pm t - t_0,$$

and changing the initial point we may assume that $t_0 = 0$. Then

$$y(t) = R\operatorname{sech} t \tag{2.4.11}$$

where $R = c^{-1} > 0$. Finally, equation (2.4.10) implies that

$$x(t) = x_0 + R\tanh t \tag{2.4.12}$$

for some $x_0 \in \mathbf{R}$. Note that equations (2.4.12) and (2.4.11) are defined on all of $\mathbf{R}$, and they parametrize the semi-circle of center $(x_0, 0)$ and radius R in $\mathbf{R}^2_+$.

Any geodesic of $(\mathbf{R}^+_2, g)$ is of one of the above types. Indeed, given initial conditions for a geodesic, it is readily seen that there exists a (unique) vertical line or semi-circle as above satisfying the given initial conditions. $\star$

2.5 Isometries and Killing fields

It is more or less clear that isometries should preserve any object canonically associated to a Riemannian manifold. Let (M,g) and (M',g') be Riemannian manifolds, denote by ∇ and ∇' the corresponding Levi-Cività connections, and let $f : M \to M'$ be an isometry. It follows from the Koszul formula (2.2.6) that f maps ∇ to ∇' is the sense that

$$\nabla'_{f_*X} f_*Y = f_*(\nabla_X Y)$$

where X, $Y \in \Gamma(TM)$. In particular, if $\gamma : I \to M$ is a geodesic of (M,g) then $f \circ \gamma : I \to M'$ is a geodesic of (M',g').

It is interesting to rephrase the last assertion in terms the exponential map. Namely, if f is an isometry of (M,g), $p \in M$ and $v \in T_pM$ lies in the domain of $\exp_p$, then $df_p(v)$ lies in the domain of $\exp_{f(p)}$ and

$$f(\exp_p(v)) = \exp_{f(p)}(df_p(v)).$$

In particular, if p is a fixed point of f then, on a normal neighborhood of p, we can write

$$f = \exp_p \circ\, df_p \circ \exp_p^{-1};$$

namely, $\exp_p^{-1}$ defines a local chart on a normal neighborhood of p that linearizes f.

A *Killing vector field* (sometimes, simply a *Killing field*) on a Riemannian manifold (M,g) is a smooth vector field X on M whose local flow $\{\varphi_t\}$ consists of local isometries of M, namely, $\varphi_t^* g = g$ wherever defined. By differentiation with respect to t, we immediately see that this condition is equivalent to the vanishing of Lie derivative of g with respect to X,

$$L_X g = 0,$$

or equivalently,

$$Xg(Y,Z) = g([X,Y],Z) + g(Y,[X,Z]) \tag{2.5.1}$$

for every Y, $Z \in \Gamma(TM)$.

2.5.2 Proposition *Let (M,g) be a Riemannian manifold.*

a. *The set of Killing fields on M form a Lie subalgebra of the Lie algebra of smooth vector fields on M.*
b. *A smoothy vector field $X \in \Gamma(TM)$ is a Killing field if and only if*

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for every Y, $Z \in \Gamma(TM)$, i. e. $(\nabla X)_p$ is skew-symmetric as a linear operator on T_pM for all $p \in M$.

Proof. (a) The set of Killing fields on M is a subspace of $\Gamma(TM)$ because $L_X g = 0$ is linear in X, and closed under the Lie bracket because $L_{[X,Y]} = [L_X, L_Y]$ for all X, $Y \in \Gamma(TM)$.

(b) Using that the Levi-Cività connection is compatible with the metric and has no torsion (Proposition 2.2.5(a) and (b)), equation (2.5.1) is seen to be equivalent to

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_X Y - \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X),$$

which implies the result. $\square$

2.5.3 Remark In chapter 3 we will see that Killing fields are complete if M is e. g. compact, and in chapter 5 we will bound the dimension of the Lie algebra of Killing fields on M by $\frac{1}{2}n(n+1)$, where $n = \dim M$.

Recall that the set $\mathrm{Isom}(M, g)$ of all isometries of a Riemannian manifold (M, g) forms a subgroup of the group of all diffeomorphisms of M, which has the structure of a Lie group with respect to the compact-open topology; moreover, the map $\mathrm{Isom}(M, g) \times M \to M$ is smooth [KN96]. In particular, if all Killing fields are complete, then the Lie algebra of $\mathrm{Isom}(M, g)$ is naturally identified with the Lie algebra of Killing fields of M.

2.6 Induced connections

At this juncture, it is convenient to introduce the following extension of Proposition 2.3.1. We will be using it especially in the case $\dim N = 2$.

2.6.1 Proposition *Let N be a smooth manifold, and let $\varphi : N \to M$ be a smooth map. Then there exists a unique bilinear map $\nabla^\varphi : \Gamma(TN) \times \Gamma(\varphi^*TM) \to \Gamma(\varphi^*TM)$, called the* induced connection along *φ, satisfying the following conditions:*
a. $\nabla^\varphi_{fX} Y = f\nabla^\varphi_X Y$;
b. $\nabla^\varphi(fY) = X(f)Y + f\nabla^\varphi_X Y$;
c. If Y admits an extension to a vector field $\hat{Y}$ defined on a open subset U of M, then

$$\left(\nabla^\varphi_X Y\right)_p = \left(\nabla_{d\varphi(X_p)}\hat{Y}\right)_{\varphi(p)}$$

for every $p \in \varphi^{-1}(U)$;
where $X \in \Gamma(TN)$, $Y \in \Gamma(\varphi^*TM)$ and $f : N \to \mathbf{R}$ is a smooth function.

2.6.2 Proposition *Let $\varphi : N \to M$ be a smooth map, let X, $Y \in \Gamma(TN)$ be vector fields in N and let U, $V \in \Gamma(\varphi^*TM)$ be vector fiels along φ. Then the following identities hold:*

$$\begin{gathered} \nabla^\varphi_X(\varphi_* Y) - \nabla^\varphi_Y(\varphi_* X) - \varphi_*[X, Y] = 0, \quad \textit{and} \\ X\, g(U, V) = g(\nabla^\varphi_X U, V) + g(\nabla^\varphi_X V, U). \end{gathered}$$

2.7 Examples

The Euclidean space

We claim that the Levi-Cività connection ∇ in $\mathbf{R}^n$ coincides with the usual derivative. In fact, let $(x^1, \ldots, x^n)$ denote the standard global coordinates in $\mathbf{R}^n$. We have that

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} \quad \text{and} \quad \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

for all i, j. Plugging these relations into the Koszul formula (2.2.6) gives that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ for all i, j, namely, all the Christoffel symbols $\Gamma^i_{jk} = 0$. If

$$X = \sum_j a^j \frac{\partial}{\partial x^j} \quad \text{and} \quad Y = \sum_k b^k \frac{\partial}{\partial x^k},$$

for a_i, $b_j \in C^\infty(\mathbf{R}^n)$, then, using formula (2.2.2),

$$\nabla_X Y = \sum_i \left(\sum_j a^j \frac{\partial b^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} = X(Y) = dY(X),$$

proving the claim. We also get, from equation (2.3.4), that a vector field X along a curve $\gamma : [a,b] \to M$, given as

$$X(t) = \sum_k a^k(t) \frac{\partial}{\partial x^k}\Big|_{\gamma(t)},$$

is parallel if and only the a_k are constant functions, namely, the parallel vector fields in $\mathbf{R}^n$ are the constant vector fields. It follows that the parallel transport map along γ from a to b is given by the differential of the translation map, that is,

$$P^\gamma_{b,a} = d(\tau_v)_{\gamma(a)},$$

where τ_v is the translation in $\mathbf{R}^n$ by the vector $v = \gamma(b) - \gamma(a)$, and, in particular, is independent of the curve γ joining $\gamma(a)$ and $\gamma(b)$. Finally, the geodesic equation (2.4.1) in $\mathbf{R}^n$ is

$$(x^i)'' = 0$$

for all i, so the geodesics are the lines. Hence

$$\exp_p(v) = p + v$$

for $p \in \mathbf{R}^n$ and $v \in T_p\mathbf{R}^n = \mathbf{R}^n$.

Product Riemannian manifolds

Let (M_i, g_i), where $i = 1$, 2, denote two Riemannian manifols and consider the product Riemannian manifold $(M, g) = (M_1, g_1) \times (M_2, g_2)$. Let $U_i \in \Gamma(TM_i)$, where $i = 1$, 2, be arbitrary vector fields. Of course, U_1 and U_2 can be identified with vector fields on M, and it follows from the construction of (M, g) that $[U_1, U_2] = 0$ and $g(U_1, U_2) = 0$ in M.

Now, suppose that X, Y, $Z \in \Gamma(TM)$ can be decomposed as $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, $Z = Z_1 + Z_2$, where X_i, Y_i, $Z_i \in \Gamma(TM_i)$ for $i = 1$, 2 (not every vector field on M admits such a decomposition!). Note that

$$Xg(Y, Z) = X_1 g_1(Y_1, Z_1) + X_2 g_2(Y_2, Z_2)$$

and

$$g([X, Y], Z) = g_1([X_1, Y_1], Z_1) + g_2([X_2, Y_2], Z_2).$$

It then follows from the Koszul formula (2.2.6) applied three times that

$$\begin{aligned} g(\nabla_X Y, Z) &= g_1(\nabla^1_{X_1} Y_1, Z_1) + g_2(\nabla^2_{X_2} Y_2, Z_2) \\ &= g(\nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2, Z), \end{aligned}$$

where ∇ denotes the Levi-Cività connection of M and ∇^i denotes the Levi-Cività connection of M_i for $i = 1$, 2. Since g is nondegenerate and any tangent vector to M can be extended to a vector field

Z which decomposes as $Z_1 + Z_2$, this calculation yields the following formula for the Levi-Cività connection of a Riemannian product:

$$\nabla_X Y = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2. \tag{2.7.1}$$

It follows from this formula that the Christoffel symbol Γ^i_{jk} of ∇ is zero unless all the three indices i, j, k correspond to coordinates of the same factor M_ℓ, where $\ell = 1$ or 2, in which case Γ^i_{jk} is a function on M_ℓ and a Christofell symbol of ∇^ℓ. Therefore if γ is a curve in M with components γ_1 in M_1 and γ_2 in M_2, and X is a vector field along γ, then we can decompose $X = X_1 + X_2$ where X_i is a vector field along γ_i, and equation (2.3.2) gives $\frac{\nabla X}{dt} = \frac{\nabla X_1}{dt} + \frac{\nabla X_2}{dt}$. In particular, X is parallel along γ if and only if X_i is parallel along M_i for $i = 1$, 2. As $\gamma'(t) = \gamma_1'(t) + \gamma_2'(t)$, in particular yet, γ is a geodesic if and only if γ_i is a geodesic of M_i for $i = 1$, 2.

Riemannian submanifolds and isometric immersions

Let (M, g), $(\overline{M}, \overline{g})$ be Riemannian manifolds, and suppose that $\iota : M \to \overline{M}$ is an isometric immersion. We would like to relate the Levi-Cività connections ∇ of M and $\overline{\nabla}$ of $\overline{M}$. Since this is a local problem, we can work in a neighborhood a point $p \in M$ and assume that ι is the inclusion map. Now the tangent bundle TM is a subbundle of $T\overline{M}$, the metric g is the restriction of $\overline{g}$, and every vector field on M admits an extension to a vector field on $\overline{M}$.

Let X, Y and Z be vector fields on M, and let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ be extensions of those vector fields to vector fields on $\overline{M}$. Note that $[\overline{X}, \overline{Y}]$ is an extension of $[X, Y]$ to a vector field on $\overline{M}$. It follows from two applications of the Koszul formula (2.2.6) that

$$\begin{aligned}
2\overline{g}((\nabla_X Y)_p, Z_p) &= 2g((\nabla_X Y)_p, Z_p) \\
&= \mathfrak{S} \ \pm X_p\, g(Y, Z) \pm g([X, Y]_p, Z_p) \\
&= \mathfrak{S} \ \pm \overline{X}_p\, \overline{g}(\overline{Y}, \overline{Z}) \pm \overline{g}([\overline{X}, \overline{Y}]_p, \overline{Z}_p) \\
&= 2\overline{g}((\overline{\nabla}_{\overline{X}} \overline{Y})_p, \overline{Z}_p) \\
&= 2\overline{g}((\overline{\nabla}_{\overline{X}} \overline{Y})_p, Z_p),
\end{aligned}$$

where $\mathfrak{S}$ denotes cyclic summation in X, Y, Z. Since $(\nabla_X Y)_p \in T_pM$ and Z_p can be any element of T_pM, it follows that

$$(\nabla_X Y)_p = \Pi_p \left((\overline{\nabla}_{\overline{X}} \overline{Y})_p \right), \tag{2.7.2}$$

where $\Pi_p : T_p\overline{M} \to T_pM$ is the orthogonal projection.

The most important case is that of Riemannian submanifolds of Euclidean space. If M is a Riemannian submanifold of $\mathbf{R}^n$, then formula (2.7.2) implies that a smooth curve γ in M is a geodesic of M if and only if its second derivative γ'' in $\mathbf{R}^n$ is everywhere normal to M; in other words, the geodesics of M are the "curves with normal acceleration".

The sphere S^n

Let $p \in S^n$ and $v \in T_pS^n$. We now determine the unique geodesic γ of S^n with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$. If $v = 0$, then γ is a constant curve, so we may assume that $v \neq 0$. Since p and v are orthogonal vectors in $\mathbf{R}^{n+1}$, they span a 2-dimensional subspace which we denote by E. Let $f : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be the linear reflection on E. Then f is an orthogonal transformation of $\mathbf{R}^{n+1}$ and leaves S^{n+1} invariant. Now every orthogonal transformation of $\mathbf{R}^{n+1}$ is an isometry. Since S^{n+1} has the induced metric from $\mathbf{R}^{n+1}$, f restricts to an isometry of S^n which we denote by the same letter. Owing to the fact that an isometry maps geodesics to geodesics, the curve $\tilde{\gamma} = f \circ \gamma$ is a geodesic of S^n. Since f leaves E pointwise fixed, the initial conditions of $\tilde{\gamma}$ are $\tilde{\gamma}(0) = f(\gamma(0)) = f(p) = p$ and $\tilde{\gamma}'(0) = f(\gamma'(0)) = f(v) = v$, namely, the same as those of γ. By the uniqueness of geodesics with given initial conditions, we have that $\tilde{\gamma} = \gamma$, or, what is the same, $f(\gamma(t)) = \gamma(t)$ for all t in the domain of γ. It follows that γ is contained in E and thus must coincide with the great circle $S^n \cap E$ parametrized with constant speed on its domain of definition. This argument shows that the great circles are locally geodesics; but then, the great circles are geodesics.

In particular, the geodesics of S^n parametrized by arc-length are periodic of period 2π. Finally, we have the formula

$$\exp_p(v) = \cos(||v||)p + \sin(||v||)\frac{v}{||v||}$$

for $v \neq 0$.

Riemannian coverings

Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering.

2.7.3 Proposition *The geodesics of (M, g) are the projections of the geodesics of $(\tilde{M}, \tilde{g})$, and the geodesics of $(\tilde{M}, \tilde{g})$ are the liftings of the geodesics of (M, g).*

Proof. Suppose $\tilde{\gamma}$ and γ are *continuous* curves in $\tilde{M}$, M such that $\pi \circ \tilde{\gamma} = \gamma$. Since π is a local isometry, it maps a sufficiently small arc of $\tilde{\gamma}$ isometrically onto a small arc of γ. It follows that $\tilde{\gamma}$ is a geodesic if and only if γ is a geodesic. This shows that the classes of curves described in the statement of the proposition are indeed geodesics. Now we need only to remark that every continuous curve in M is the projection of any of its continuous liftings in $\tilde{M}$, and every continuous curve in $\tilde{M}$ is the continuous lifting of its projection to M. $\square$

The real projective space

We apply Proposition 2.7.3 to the Riemannian covering map $\pi : S^n \to \mathbf{R}P^n$. The geodesics of S^n have already been determined as being the great circles parametrized with constant speed, so the geodesics of $\mathbf{R}P^n$ are the projections of those. In particular, since π identifies antipodal points of S^n, the geodesics of $\mathbf{R}P^n$ paramerized by arc-length are periodic of period π.

Flat tori

Let Γ be a lattice in $\mathbf{R}^n$ and consider the induced Riemannian metric g_Γ on T^n. We apply Proposition 2.7.3 to the Riemannian covering map $\pi : \mathbf{R}^n \to (T^n, g_\Gamma)$ to deduce that the geodesics of (T^n, g_Γ) are simply the projections of the straight lines in $\mathbf{R}^n$. In this way, we see that some geodesics of (T^n, g_Γ) are periodic and some are dense in T^n.

Next, let Γ' be another lattice in $\mathbf{R}^n$. We have already remarked that (T^n, g_Γ) and $(T^n, g_{\Gamma'})$ are generally non-isometric. Nevertheless, there exists a unique affine transformation f of $\mathbf{R}^n$ that maps Γ to Γ', and hence induces a diffeomorphism $\bar{f} : \mathbf{R}^n/\Gamma \to \mathbf{R}^n/\Gamma'$ such that the diagram

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{f} & \mathbf{R}^n \\ \downarrow & & \downarrow \\ \mathbf{R}^n/\Gamma & \xrightarrow{\bar{f}} & \mathbf{R}^n/\Gamma' \end{array}$$

is commutative. In general, $\bar{f}$ is not an isometry, but since f maps straight lines to straight lines, $\bar{f}$ maps the geodesics of (T^n, g_Γ) to the geodesics of $(T^n, g_{\Gamma'})$. Hence we get an example of two non-isometric metrics on the same smooth manifold with the same geodesics.

Lie groups ★

Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. In this example, we will describe the Levi-Cività connection associated to a bi-invariant metric on G. We start with a definition and a proposition.

We say that an inner product $\langle , \rangle$ on $\mathfrak{g}$ is ad-*invariant* if the identity

$$\langle \mathrm{ad}_Z X, Y \rangle + \langle X, \mathrm{ad}_Z Y \rangle = 0 \tag{2.7.4}$$

holds for every X, Y, $Z \in \mathfrak{g}$.

2.7.5 Proposition *Every* Ad-*invariant inner product on* $\mathfrak{g}$ *is* ad-*invariant, and the converse holds if* G *is connected.*

Proof. Let $\langle , \rangle$ be an inner product on $\mathfrak{g}$. It being Ad-invariant means that

$$\langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle = \langle X, Y \rangle \tag{2.7.6}$$

for every $g \in G$ and X, $Y \in \mathfrak{g}$. In particular, taking $g = \exp tZ$ for $Z \in \mathfrak{g}$ and differentiating at $t = 0$ yields identity (2.7.4).

Assume now that G is connected and $\langle , \rangle$ is ad-invariant. Then (2.7.4) holds; note that what it is really saying is that $f'_{X,Y}(0) = 0$ for all X, $Y \in \mathfrak{g}$, where

$$f_{X,Y}(t) = \langle \mathrm{Ad}_{\exp tZ} X, \mathrm{Ad}_{\exp tZ} Y \rangle,$$

and from this information we will show that $f_{X,Y}(t) = f_{X,Y}(0)$. Indeed, since $t \mapsto \mathrm{Ad}_{\exp tZ}$ is a homomorphism,

$$f_{X,Y}(t+s) = f_{X',Y'}(t)$$

where $X' = \mathrm{Ad}_{\exp sZ} X$ and $Y' = \mathrm{Ad}_{\exp sZ} Y$. Differentiating this identity at $t = 0$ gives that $f'_{X,Y}(s) = f'_{X',Y'}(0) = 0$. Since $s \in \mathbf{R}$ is arbitrary, this implies that $f_{X,Y}$ is constant, as desired.

So far we have shown that (2.7.6) holds if g lies in the image of exp. But there exists an open neighborhood U of the identity of G contained in the image of exp, and it is known that U generates G as a group due to the connectedness of G. Since $g \mapsto \mathrm{Ad}_g$ is a homomorphism, this finally implies that (2.7.6) holds for every $g \in G$. □

Let g be a bi-invariant metric on G. Now we are ready to use the Koszul formula (2.2.6) to compute the Levi-Cività connection on left-invariant vector fields. Let X, Y, $Z \in \mathfrak{g}$. Since X and Y are left-invariant vector fields and g is a left-invariant metric, $g(X,Y)$ is a constant function on G. Therefore $Zg(X,Y) = 0$. Similarly, $Yg(Z,X) = Zg(X,Y) = 0$. Regarding the other terms of (2.2.6), the preceding proposition shows that g_1 is an ad-invariant inner product on $\mathfrak{g}$, so

$$g([Z,X],Y) + g(X,[Z,Y]) = g_1(\mathrm{ad}_Z X, Y) + g_1(X, \mathrm{ad}_Z, Y) = 0. \tag{2.7.7}$$

We deduce that

$$\nabla_X Y = \frac{1}{2}[X,Y] \tag{2.7.8}$$

for all X, $Y \in \mathfrak{g}$ (this formula shows in particular that $\nabla_X Y$ is also a left-invariant vector field, but this fact of course also follows from general properties of isometries, cf. section 2.5). An important application of this formula is that $\nabla_X X = 0$ for all $\in \mathfrak{g}$, and this means that every one-parameter subgroup of G thorough the identity is a geodesic. This is also equivalent to saying that the exponential map of G as a Lie group and the exponential map of G as the Riemannian manifold (G, g) coincide. Of course, the geodesics of G through an arbitrary point are left-translates of one-parameter subgroups, namely, of the form $t \mapsto g \exp tX$ for $g \in G$ and $X \in \mathfrak{g}$.

2.8 Exercises

1 Let (M, g) be a Riemannian manifold, consider its tangent bundle TM, and fix a point $p \in M$. Prove that any open neighborhood W of 0_p in TM contains a neighborhood of the form

$$\bigcup_{x \in U} B(0_x, \epsilon) = \{\, v \in TM|_U \ :\ g_{\pi(v)}(v, v)^{1/2} < \epsilon \,\}$$

for some open neighborhood U of p in M and some $\epsilon > 0$.

2 Let A, B be nowhere zero smooth functions on $\mathbf{R}^2$ and consider the Riemannian metric $g = A^2\, dx^2 + B^2\, dy^2$, where x, y are the standard coordinates on $\mathbf{R}^2$.
a. Compute the Christoffel symbols of g.
b. Write down the geodesic equations of g.

3 Let (x^i) be a system of local coordinates on a smooth manifold M which is equipped with a connection ∇, and consider the Christoffel symbols Γ^k_{ij} which are defined by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$. If $(x^{i'})$ is another system of local coordinates on M, prove that the following transformation law holds:

$$\Gamma^{k'}_{i'j'} = \sum_{i,j,k} \Gamma^k_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} + \sum_k \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k}.$$

Use this law to check that formula (2.3.2) defines $\frac{\nabla X}{dt}$ independently of choice of local chart.

4 Let M be a Riemannian manifold of dimension n. Given $p \in M$, prove that there exists an open neighborhood U of p, and n smooth vector fields $E_1, \ldots, E_n$ defined on U which are orthonormal at each point of U and such that $(\nabla_{E_i} E_j)_p = 0$ for all i, j.

5 Let M be a Riemannian manifold. Suppose X is a smooth vector field along a smooth curve $\gamma : I \to M$. If $\phi : J \to I$ is a diffeomorphism, define the reparametrizations $\eta = \gamma \circ \phi$ and $Y = X \circ \phi$.
a. Show that Y is a smooth vector field along η.
b. Denote by t, s the parameters along γ, η, resp., where $t = \phi(s)$, and prove that

$$\left(\frac{\nabla}{ds} Y \right)(s) = \left(\frac{\nabla}{dt} X \right)(\phi(s))\, \phi'(s)$$

for $s \in J$.
c. Deduce that the parallelism of a vector field along a curve does not depend on the parametrization.

6 Let M be a Riemannian manifold. The goal of this exercise is to characterize the curves on M that are geodesics up to a reparametrization.

a. Assume $\gamma : \mathbf{R} \to M$ is a geodesic, $\phi : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism and $\eta : \mathbf{R} \to M$ is given by $\eta = \gamma \circ \phi$. Show that there exists a smooth function $f : \mathbf{R} \to \mathbf{R}$ such that $\nabla_{\eta'}\eta' = f\eta'$.

b. Conversely, suppose that $\eta : \mathbf{R} \to M$ satisfies $\nabla_{\eta'}\eta' = f\eta'$ for some smooth function $f : \mathbf{R} \to \mathbf{R}$, and show that there exists a diffeomorphism $\phi : \mathbf{R} \to \mathbf{R}$ such that $\gamma = \eta \circ \phi^{-1}$ is a geodesic.

7 In this exercise, we describe the geodesics of the real hyperbolic space.

a. Describe the geodesics of $\mathbf{R}H^n$ in the hyperboloid model using an argument very similar to the one which was used in the case of S^n.

b. Use the result of (a) to describe the geodesics of $\mathbf{R}H^n$ in Poincaré's disk and upper half-space models (cf. exercises 3 and 4 of chapter 1).

c. Check that in the case in which $n = 2$, the result of (b) coincides with he result of Example 2.4.8.

8 Consider the Poincaré upper half-plane model $\mathbf{R}^2_+ = \{ (x, y) \in \mathbf{R}^2 \mid y > 0 \}$ with the metric $g = \frac{1}{y^2}\left(dx^2 + dy^2\right)$.

a. Prove that any geodesic of $\mathbf{R}^2_+$ is the fixed point set of some isometry. (Hint: Use Example 2.4.8 and Exercise 5 of chapter 1; conjugate R by appropriate isometries of the form τ_a, h_r.) Such isometries deserve to be called *reflections*. Show that the differential of a reflection at a fixed point p is a reflection of $T_p\mathbf{R}^2_+$ on a straight line.

b. Show that the composition of reflections on two geodesics through the point $p = (0, 1)$ yields an isometry that fixes that point and induces a rotation on the tangent space. Show also that any rotation of $T_p\mathbf{R}^2_+$ arises in this way. Deduce that the isometry group of $\mathbf{R}_2^+$ acts transitively on the unit tangent bundle (namely, the set of unit tangent vectors).

A Riemannian manifold with the property that its isometry group acts transitively on its unit tangent bundle is called *isotropic*.

9 Let M be a smooth manifold equipped with a connection ∇. If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve and X is a smooth vector field along γ, prove the following formula:

$$\left(\frac{\nabla}{dt}X\right)_0 = \lim_{t\to 0} \frac{P^\gamma_{0,t}X(t) - X(0)}{t}.$$

(Hint: Write X as a linear combination of the vectors in a parallel frame along γ.)

10 Let M be a Riemannian manifold and consider its Levi-Cività connection ∇. If X is a smooth vector field on M and $\{\varphi_t\}$ denotes its local flow, and $v \in TM$, prove the following formula:

$$\nabla_v X = \frac{\nabla}{dt}\Big|_{t=0} d(\varphi_t)_p v.$$

(Hint: Use the first identity in Proposition 2.6.1 in order to commute two different derivatives.)

2.9 Additional notes

§1 The development of the idea of connection presented here, usually called an *affine connection*■2■, took some time to evolve to that form. Starting around 1868, Elwin Christoffel became interested in the theory of invariants and wrote six papers on that topic. In these, he introduced

■2■?

the Christoffel symbols and solved the local equivalence problem for quadratic differential forms by essentially introducing the Riemann-Christoffel curvature tensor. These results influenced Gregorio Ricci-Curbastro in Padua to begin his investigations in 1884 on quadratic differential forms. In four papers between 1888 and 1892, Ricci-Curbastro exposed the technique of absolute differential calculus, a new invariant formalism originally constructed to deal with the transformation theory of partial differential equations, which he used to study the transformation theory of quadratic differential forms. A pupil of him, Tulio Levi-Civita, wrote a dissertation, published in 1893, where he developed the calculus of tensors including covariant differentiation, bulding on ideas from Ricci-Curbastro and Lie's then recently appeared theory of transformation groups. In 1900, Ricci (using this name for the first time instead of his full name) jointly with Levi-Civita published a fundamental paper [RL00] in which preface they state:

> "The algorithm of absolute differential calculus, the instrument matériel of the methods . . . can be found complete in a remark due to Christoffel. But the methods themselves and the advantages they offer have their raison d'être and their source in the intimate relationships that join them to the notion of an n-dimensional variety, which we owe to the brilliant minds of Gauss and Riemann. . . . Being thus associated in an essential way with V^n, it is the natural instrument of all those studies that have as their subject, such a variety, or in which one encounters as a characteristic element a positive quadratic form of the differentials of n variables or of their derivatives."

When in 1915 Albert Einstein used tensor calculus to explain theory of relativity, Levi-Cività initiated and kept mathematical correspondence with him until 1917. In that year, inspired by Einstein's general theory of relativity, Levi-Cività made what is probably his most important contribution to mathematics: the introduction of the concept of parallel displacement. His book [Lev05] on absolute diferential calculus, originally a collection of lecture notes in Italian, also contains applications to general relativity.

Soon it was realized that connections existed independent of the Riemannian metric. Between the years of 1918 and 1923, Hermann Weyl's efforts towards the unification of electromagnetism and gravitation brought in new ideas and placed the concept of parallel displacement of a tangent vector at the base of the definition of an affine connection on a smooth manifold. Tensor calculus was systematized by Jan Schouten (who discovered the idea of parallel displacement independently in 1918) in his book *Ricci-Kalkül* in 1924 (entirely rewritten in 1954). At the same time, Élie Cartan introduced in the 1920's projective and conformal connections and, more generally, a new concept of a connection on a manifold. However, at that time, Cartan faced difficulty trying to express notions for which there was no truly suitable language. In [Ehr51], Charles Ehresmann gave a rigorous global definition of a Cartan connection as a special case of a more general notion of connection on a principal bundle, today called an *Ehresmann connection* or simply a *connection*, which is mostly considered to be the definitive one. The axiomatic approach to affine connections that we use in this book is due to Jean-Louis Koszul (cf. [Nom54]). For more details on the history of connections, see the introduction of [Str34]. For the general theory of connections on principal bundles, see [KN96].

§2 The idea of parallel displacement is a simple though deep notion in geometry. Consider a 2-sphere Σ touching a 2-plane π at a point p. Now let Σ roll over π so that the touching point traces a curve γ in Σ, and let q be the endpoint of γ. Suppose v is a vector tangent to π at p. Of course, there is a unique vector v' which is tangent to π at q and parallel to v in the plane. *The parallelism of Levi-Cività says that v', regarded as vector tangent to Σ at q, is the parallel displacement of v, regarded as a vector tangent to Σ at p, along γ.* More generally, one can replace Σ by a 2-surface at let it roll over π to define the parallel displacement of vectors on Σ.