Submanifold geometry

7.1 Introduction

In this chapter, we study the extrinsic geometry of Riemannian manifolds. Historically speaking, the field of Differential Geometry started with the study of curves and surfaces in $\mathbf{R}^3$, which originated as a development of the invention of infinitesimal calculus. Later investigations considered arbitrary dimensions and codimensions. Our discussion in this chapter is centered in submanifolds of spaces forms.

A most fundamental problem in submanifold geometry is to discover simple (sharp) relationships between intrinsic and extrinsic invariants of a submanifold. We begin this chapter by presenting the standard related results for submanifolds of space forms, with some basic applications. Then we turn to the Morse index theorem for submanifolds. This is a very important theorem that can be used to deduce information about the topology of the submanifold. Finally, we present a brief account of the theory of isoparametric submanifolds of space forms, which in some sense are the submanifolds with the simplest local invariants, and we refer to [BCO16, ch. 4] for a fuller account.

7.2 The fundamental equations of the theory of isometric immersions

The first goal is to introduce a number of invariants of the isometric immersion. From the point of view of submanifold geometry, it does not make sense to distinguish between two isometric immersions of M into $\overline{M}$ that differ by an ambient isometry. We call two isometric immersions $f:(M,g)\to(\overline{M},\overline{g})$ and $f':(M,g')\to(\overline{M},\overline{g})$ *congruent* if there exists an isometry φ of $\overline{M}$ such that $f'=\varphi\circ f$. In this case,

$$g'=f'^{*}\overline{g}=f^{*}\varphi^{*}\overline{g}=f^{*}\overline{g}=g.$$

Because of this, the induced metric is considered to be one of the basic invariants of an isometric immersion, and it is sometimes referred to as the *first fundamental form* of the immersion.

Due to the fact that our first considerations are local, we may assume that f is an embedding; for simplicity, we assume that f is the inclusion. In this case, for every point $p\in M$, the tangent space T_pM is a subspace of $T_p\overline{M}$ and the metric g_p is the restriction of $\overline{g}_p$. Consider the $\overline{g}$-ortohognal bundle decomposition

$$T\overline{M}=TM\oplus TM^{\perp},$$

and denote by $(\cdot)^{\top}$ and $(\cdot)^{\perp}$ the respective projections. According to (2.8.2), the Levi-Civita connections ∇ and $\overline{\nabla}$ of M and $\overline{M}$, respectively, are related by

$$\nabla_XY=(\overline{\nabla}_{\overline{X}}\overline{Y})^{\top},$$

where X and Y are vector fields on M and $\overline{X}$, $\overline{Y}$ are arbitrary extensions to vector fields on $\overline{M}$. The *second fundamental form* of the immersion f is the bilinear form $B : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ given by

$$\begin{aligned} B(X,Y) &= \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y \\ &= (\overline{\nabla}_{\overline{X}}\overline{Y})^\perp, \end{aligned}$$

where X, $Y \in \Gamma(TM)$ and $\overline{X}$, $\overline{Y} \in \Gamma(T\overline{M})$ are arbitrary local extensions of X, Y. In order to check that the definition of $B(X,Y)$ does not depend on the choice of local extensions, choose other ones $\overline{X}'$, $\overline{Y}' \in T\overline{M}$. Then

$$(\overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y) - (\overline{\nabla}_{\overline{X}'}\overline{Y}' - \nabla_X Y) = \overline{\nabla}_{\overline{X}-\overline{X}'}\overline{Y} - \nabla_{\overline{X}'}(\overline{Y}' - \overline{Y}).$$

Note that the right hand side vanishes at a point $p \in M$. Indeed, the first term is zero because $\overline{X}'_p = X_p = \overline{X}_p$, and the second term is zero because $\overline{Y}' = Y = \overline{Y}$ along a curve in M tangent to $\overline{X}'_p = X_p$ (cf. Remark 2.2.1). The second fundamental form is another one of the basic invariants of an isometric immersion. The orthogonal decomposition

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{7.2.1}$$

is called the *Gauss formula*.

We agree to retain the above notation and make some remarks about B. First, observe that $B(X,Y) = B(Y,X)$. This is because

$$\begin{aligned} B(X,Y) - B(Y,X) &= \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y - \overline{\nabla}_{\overline{Y}}\overline{X} + \nabla_Y X \\ &= [\overline{X},\overline{Y}] - [X,Y] \\ &= 0 \end{aligned}$$

on M, where we have used the fact that $[\overline{X},\overline{Y}]$ is a local extension of $[X,Y]$. Next, note that it follows from the first defining condition of ∇ that B is $C^\infty(M)$-linear in the first argument; now it is a consequence of its symmetry that B is $C^\infty(M)$-linear also in the second argument. Therefore, for $p \in M$, $B(X,Y)_p$ depends only on X_p and Y_p. So there is a bilinear symmetric form

$$B_p : T_pM \times T_pM \to T_pM^\perp$$

given by $B_p(u,v) = B(U,V)_p$ where u, $v \in T_pM$, and U, V are arbitrary extensions of u, v to local vector fields on M. If $\xi \in T_pM^\perp$, the *Weingarten operator*, also called *shape operator* of the immersion f at ξ, is the self-adjoint linear endomorphism

$$A_\xi : T_pM \to T_pM,$$

given by

$$\langle A_\xi(u), v\rangle = \langle B(u,v), \xi\rangle,$$

where u, $v \in T_pM$. The eigenvalues of the Weingarten operator at ξ are called *principal curvatures* at ξ. Now the following lemma is proved by a simple computation.

7.2.2 Lemma *Let $\hat{\xi} \in \Gamma(TM^\perp)$ be a local extension of ξ to a normal vector field. Then*

$$A_\xi(u) = -(\overline{\nabla}_u\hat{\xi})^\top.$$

Proof. We have that

$$\langle A_\xi(u), v\rangle = \langle B(u,v),\xi\rangle = \langle \overline{\nabla}_{\overline{U}}\overline{V},\hat{\xi}\rangle_p = -\langle \overline{V},\overline{\nabla}_{\overline{U}}\hat{\xi}\rangle_p = -\langle v,\overline{\nabla}_u\hat{\xi}\rangle.$$

The result follows. □

The normalized trace of the second fundamental form

$$H = \frac{1}{n}\mathrm{tr}(B),$$

where $n = \dim M$, is called the *mean curvature vector* of M. Note that $n\langle \xi, H\rangle$ is the sum of the principal curvatures of M along ξ. A *minimal submanifold* is a submanifold with vanishing mean curvature. Minimal submanifolds are exactly the critical points of the volume functional (cf. exercise 12 of chapter 4) with respect to compactly supported variations. There is a vast literature devoted to them, especially in the case of minimal surfaces, which can be traced back at least to Euler and Lagrange. The minimal surface equation translates in coordinates to perhaps the best of all studied quasi-linear elliptic PDE, in terms of qualitative properties and explicit global solutions. For good introductions to minimal submanifolds, see [Law80, Sim83]. A classical reference to minimal surfaces is [Oss86]; a more recent one is [CM11].

Let us now turn to the last important invariant of an isometric immersion. Consider again the $\overline{g}$-orthogonal splitting $T\overline{M} = TM \oplus TM^\perp$. The bundle $TM^\perp \to TM$ is called the *normal bundle* of the isometric immersion. The connection in $T\overline{M}$ defines a connection $\nabla^\perp$ in $TM^\perp$, called the *normal connection* of the immersion, via the following formula,

$$\nabla^\perp_X\xi = (\overline{\nabla}_X\xi)^\perp,$$

where $\xi \in \Gamma(TM^\perp)$ and $X \in \Gamma(TM)$. It is a simple matter now to derive the *Weingarten formula*

$$\overline{\nabla}_X\xi = -A_\xi(X) + \nabla^\perp_X\xi; \tag{7.2.3}$$

indeed, we have

$$\nabla^\perp_X\xi = (\overline{\nabla}_X\xi)^\perp = \overline{\nabla}_X\xi - (\overline{\nabla}_X\xi)^\top = \overline{\nabla}_X\xi + A_\xi(X),$$

by Lemma 7.2.2, checking the equation. The normal connection is the third and last basic invariant of an isometric immersion that we wanted to mention.

Next, we want to state the fundamental equations involving the basic invariants of an isometric immersion. These are respectively called the *Gauss, Codazzi-Mainardi and Ricci equations.* At this juncture, we recall that the covariant derivative of the second fundamental form is given by

$$(\nabla^\perp_X B)(Y,Z) = \nabla^\perp_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y,\nabla_X Z)$$

(cf. section 4.4), and the *normal curvature* of the immersion is given by

$$R^\perp(X,Y)\xi = \nabla^\perp_X\nabla^\perp_Y\xi - \nabla^\perp_Y\nabla^\perp_X\xi - \nabla^\perp_{[X,Y]}\xi,$$

where X, Y, $Z \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$.

7.2.4 Proposition (Fundamental equations of an isometric immersion) *The first and second fundamental forms and the normal connection of an isometric immersion $f : (M,g) \to (\overline{M},\overline{g})$ satisfy the following equations:*

$$\begin{aligned} \langle \overline{R}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad + \langle B(X,Z),B(Y,W)\rangle - \langle B(X,W),B(Y,Z)\rangle \quad \textit{(Gauss equation)} \\ (\overline{R}(X,Y)Z)^\perp &= (\nabla^\perp_X B)(Y,Z) - (\nabla^\perp_Y B)(X,Z) \quad \textit{(Codazzi-Mainardi equation)} \\ \langle \overline{R}(X,Y)\xi,\eta\rangle &= \langle R^\perp(X,Y)\xi,\eta\rangle - \langle [A_\xi,A_\eta]X,Y\rangle \quad \textit{(Ricci equation)} \end{aligned}$$

where X, Y, Z, $W \in \Gamma(TM)$ and ξ, $\eta \in \Gamma(TM^\perp)$.

Proof. We first use the Gauss and Weingarten formulae (7.2.1), (7.2.3) to write

$$\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_Y Z &= \overline{\nabla}_X\nabla_Y Z + \overline{\nabla}_X B(Y,Z) \\
&= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z) + \nabla^{\perp}_X(B(Y,Z)) - A_{B(Y,Z)}X.
\end{aligned}$$

Then

$$\begin{aligned}
\overline{R}(X,Y)Z &= \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z \\
&= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z) + \nabla^{\perp}_X(B(Y,Z)) - A_{B(Y,Z)}X \\
&\quad -\nabla_Y\nabla_X Z - B(Y,\nabla_X Z) - \nabla^{\perp}_Y(B(X,Z)) + A_{B(X,Z)}Y \\
&\quad -\nabla_{[X,Y]}Z - B(\nabla_X Y, Z) + B(\nabla_Y X, Z) \\
&= R(X,Y)Z + A_{B(X,Z)}Y - A_{B(Y,Z)}X \\
&\quad +\nabla^{\perp}_X B(Y,Z) - \nabla^{\perp}_Y B(X,Z).
\end{aligned}$$

The tangential component of this equation is

$$(\overline{R}(X,Y)Z)^{\top} = R(X,Y)Z + A_{B(X,Z)}Y - A_{B(Y,Z)}X,$$

which is equivalent to the Gauss equation; the normal component is exactly the Codazzi-Mainardi equation.

Next, we use again the Gauss and Weingarten formulae to write

$$\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_Y\xi &= \overline{\nabla}_X\nabla^{\perp}_Y\xi - \overline{\nabla}_X A_\xi Y \\
&= \nabla^{\perp}_X\nabla^{\perp}_Y\xi - A_{\nabla^{\perp}_Y\xi}X - \nabla_X A_\xi Y - B(X, A_\xi Y).
\end{aligned}$$

Then

$$\begin{aligned}
\overline{R}(X,Y)\xi &= \overline{\nabla}_X\overline{\nabla}_Y\xi - \overline{\nabla}_Y\overline{\nabla}_X\xi - \overline{\nabla}_{[X,Y]}\xi \\
&= \nabla^{\perp}_X\nabla^{\perp}_Y\xi - A_{\nabla^{\perp}_Y\xi}X - \nabla_X A_\xi Y - B(X, A_\xi Y) \\
&\quad -\nabla^{\perp}_Y\nabla^{\perp}_X\xi + A_{\nabla^{\perp}_X\xi}Y + \nabla_Y A_\xi X - B(Y, A_\xi X) \\
&\quad -\nabla^{\perp}_{[X,Y]}\xi - A_\xi\nabla_X Y - A_\xi\nabla_Y X \\
&= R^{\perp}(X,Y)\xi + B(A_\xi X, Y) - B(X, A_\xi Y) \\
&\quad -(\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X.
\end{aligned}$$

It is easy to see that the tangential component of this equation yields again the Codazzi-Mainardi equation; we claim that the normal component is equivalent to the Ricci equation. In fact, it gives

$$\langle\overline{R}(X,Y)\xi,\eta\rangle = \langle R^{\perp}(X,Y)\xi,\eta\rangle + \langle B(A_\xi X,Y),\eta\rangle - \langle B(X,A_\xi Y),\eta\rangle,$$

where

$$\begin{aligned}
\langle B(A_\xi X,Y),\eta\rangle - \langle B(X,A_\xi Y),\eta\rangle &= \langle A_\xi X, A_\eta Y\rangle - \langle A_\xi Y, A_\eta X\rangle \\
&= \langle A_\eta A_\xi X, Y\rangle - \langle A_\xi A_\eta X, Y\rangle \\
&= -\langle [A_\xi, A_\eta]X, Y\rangle.
\end{aligned}$$

This completes the proof of the proposition. □

7.2.5 Corollary *If $(\overline{M}, \overline{g})$ is a space form of curvature κ, then the fundamental equations for an isometric immersion $f : (M, g) \to (\overline{M}, \overline{g})$ are given by:*

$$\begin{aligned} \langle R(X,Y)Z,W\rangle = -\kappa(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle) \\ - \langle B(X,Z),B(Y,W)\rangle + \langle B(X,W),B(Y,Z)\rangle \end{aligned} \quad \textit{(Gauss equation)} \tag{7.2.6}$$

$$(\nabla^\perp_X B)(Y,Z) = (\nabla^\perp_Y B)(X,Z) \quad \textit{(Codazzi-Mainardi equation)} \tag{7.2.7}$$

$$\langle R^\perp(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X,Y\rangle, \quad \textit{(Ricci equation)} \tag{7.2.8}$$

where X, Y, Z, $W \in \Gamma(TM)$ and ξ, $\eta \in \Gamma(TM^\perp)$.

Proof. The equations follow from the fact that the curvature tensor of $(\overline{M}, \overline{g})$ is given by

$$\overline{R}(X,Y)Z = -\kappa(\langle X,Z\rangle Y - \langle Y,Z\rangle X),$$

see the end of section 4.2. □

7.2.9 Theorem (Fundamental theorem of submanifold geometry) *Let M be an n-dimensional manifold. Assume that we are given a Riemannian metric g on M, a rank k vector bundle E over M endowed with a Riemannian metric and a compatible connection ∇', and a symmetric E-valued tensor field B' on TM such that they satisfy the Gauss, Codazzi-Mainardi and Ricci equations for some real number κ. Then, for each point $p \in M$, there exists an open neighbourhood U of p in M and an isometric immersion f from U into a the simply-connected space form of constant curvature κ and dimension $n+k$ such that g is the induced metric on U, $E|_U$ is isomorphic to the normal bundle of f, and B' and ∇' correspond respectively to the second fundamental form and the normal connection of f. Moreover, the isometric immersion f is locally uniquely defined up to congruence. If, in addition, M is assumed to be simply-connected, then the open set U can be taken to be all of M and f is uniquely defined up to congruence (however, in this case, f needs not be a global embedding).*

Proof. For simplicity, we prove the result for the case $\kappa = 0$ only. Define

$$A' : \Gamma(E) \to \Gamma(\mathrm{End}(TM)), \qquad \langle A'_\xi X, Y\rangle = \langle B'(X,Y), \xi\rangle$$

where X, $Y \in \Gamma(TM)$, $\xi \in \Gamma(E)$. Consider the rank $n+k$ Riemannian vector bundle $\bar{E} = TM \oplus E$, and define a connection $\bar{\nabla}$ on $\bar{E}$ as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + B'(X,Y) \quad \text{and} \quad \bar{\nabla}_X \xi = -A'_\xi X + \nabla'_X \xi$$

for all X, $Y \in \Gamma(TM)$, $\xi \in \Gamma(E)$, where ∇ denotes the Levi-Cività connection of (M, g). It is easy to see that $\bar{\nabla}$ is compatible with the Riemannian metric on $\bar{E}$. One laboriously checks that the Gauss, Codazzi-Mainardi and Ricci compatibility equations precisely express the fact that $\bar{\nabla}$ is flat, namely, its curvature $\bar{R}$ vanishes everywhere. Therefore we can find a parallel orthonormal frame $\xi_1, \ldots, \xi_{n+k}$ of $\bar{E}$ defined on an open neighborhood U of p in M (compare exercise 6 of chapter 4). Consider the 1-forms $\theta_1, \ldots, \theta_{n+k}$ on M defined by $\theta_i(X) = \langle \xi_i, X\rangle$ for all $X \in \Gamma(TM)$, where the inner product is taken in $\bar{E}$. We compute for X, $Y \in \Gamma(TM)$:

$$\begin{aligned} d\theta_i(X,Y) &= X(\theta_i(Y)) - Y(\theta_i(X)) - \theta_i([X,Y]) \\ &= \langle \bar{\nabla}_X \xi_i, Y\rangle + \langle \xi_i, \bar{\nabla}_X Y\rangle - \langle \bar{\nabla}_Y \xi_i, X\rangle - \langle \xi_i, \bar{\nabla}_X Y\rangle - \langle \xi_i, [X,Y]\rangle \\ &= \langle \xi_i, \nabla_X Y\rangle + \langle \xi_i, B(X,Y)\rangle - \langle \xi_i, \nabla_X Y\rangle - \langle \xi_i, B(Y,X)\rangle - \langle \xi_i, [X,Y]\rangle \\ &= 0, \end{aligned}$$

where we have used that each ξ_i is $\bar{\nabla}$-parallel, B is symmetric and ∇ is torsionless. By shrinking U, if necessary, we can find smooth functions $f_1, \ldots, f_{n+k}$ on U such that $df_i = \theta_i$ for all i. We claim that $f = (f_1, \ldots, f_{n+k}) : U \to \mathbf{R}^{n+k}$ has the required properties.

For all $X \in \Gamma(TE)$, we have

$$\begin{aligned} \langle df(X), df(X) \rangle &= \sum_{i=1}^{n+k} df_i(X)^2 \\ &= \theta_i(X)^2 \\ &= \langle \xi_i, X \rangle^2 \\ &= g(X, X), \end{aligned}$$

since $\xi_1, \ldots, \xi_{n+k}$ is orthonormal, which shows that f is an isometric immersion. By shrinking U further, if necessary, we may assume that f is an embedding. Next, define a bundle isomorphism $F : \bar{E}|_U \to T\mathbf{R}^{n+k}|_{f(U)}$ by sending ξ_i to the ith element e_i of the canonical frame of $\mathbf{R}^{n+k}$. Note that

$$df(X) = \sum_{i=1}^{n+k} \langle \xi_i, X \rangle e_i = \sum_{i=1}^{n+k} \langle \xi_i, X \rangle F(\xi_i) = F\left(\sum_{i=1}^{n+k} \langle \xi_i, X \rangle \xi_i \right) = F(X)$$

for all $X \in \Gamma(TM)$, namely, F maps TM onto $df(TM)$. By construction, F maps a parallel orthonormal frame to a parallel orthonormal frame, so it preserves the metric and the connection. It follows that F maps E to the normal bundle νM and

$$F(\bar{\nabla}_X Y) = D_{F(X)} F(Y), \quad \text{and} \quad F(\bar{\nabla}_X \xi) = D_{F(X)} F(\xi)$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(E)$, where D denotes the Levi-Cività connection of $\mathbf{R}^{n+k}$; taking the tangent and normal components yields that F maps B' and ∇' respectively to the second fundamental form and normal connection of f. This finishes the proof of the local existence result.

Suppose next $f = (f_1, \ldots, f_{n+k}) : U \to \mathbf{R}^{n+k}$ is a given isometric immersion defined on a neighborhood U of p in M such that $E|_U$ is isomorphic to the normal bundle of f, and B' and ∇' correspond respectively to the second fundamental form and the normal connection of f. We first claim f is necessarily obtained from the above construction. Indeed, let $\bar{E} := f^* T\mathbf{R}^{n+k}$ be the vector bundle over M which is induced along f, namely, whose sections are exactly the vector fields aong f. Then $\{\xi_i := f^*(e_i)\}$ is a parallel orthonormal frame in $\bar{E}$ and the induced connection on $\bar{E}$ is flat. The assumptions on f and $E|_U$ imply that there is a bundle isomorphism $\bar{E} \cong TM \oplus E$ preserving metrics and connections. Finally, $df_i(X) = \langle df(X), e_i \rangle = \langle X, \xi_i \rangle$ for all i and all $X \in \Gamma(TM)$. Now for the uniqueness, note that if U is connected, the frame $\{\xi_i\}$ is uniquely determined up to an orthogonal transformation of $\bar{E}_p \cong \mathbf{R}^{n+k}$, and the functions f_i are uniquely determined up to an additive constant by the condition $df_i(\cdot) = \langle \xi_i, \cdot \rangle$, so f is uniquely determined up to a rigid motion of $\mathbf{R}^{n+k}$. Note that this result can be rephrased as saying that the immersion $f : U \to \mathbf{R}^{n+k}$ for U connected is uniquely specified by the initial values at p, that is $f(p) \in \mathbf{R}^{n+k}$ and $df_p \in \mathrm{Hom}(T_pM, \mathbf{R}^{n+k})$.

Finally, assume M is simply-connected. Given $q \in M$, connect p to q by a smooth curve γ and cover its image by finitely many connected open sets $U_1, \ldots, U_r$ such that there is an isometric immersion $U_i \to \mathbf{R}^{n+k}$ as above for each i, where $p \in U_1$, $q \in U_r$ and $U_i \cap U_{i+1}$ is non-empty and connected for $i = 1, \ldots, r-1$. Then f exists and is uniquely defined on a neighborhood of q and, in fact, on $\cup_{i=1}^r U_i$ by its initial values at p. Moreover, f is unchanged by a smooth homotopy of γ fixing the endpoints. Since M is assumed simply-connected, this shows that the value of f on q is independent of the choice of γ. Hence f is globally defined on M. $\square$

7.3 The hypersurface case

Suppose that the codimension of M in $\overline{M}$ is one, and that both of these manifolds are oriented. Then a globally defined unit normal vector field ν can be defined on M; fix such ν. Then the second fundamental form can be viewed as real valued. Let $p \in M$. As the Weingarten operator $A_p = A_{\nu_p} : T_pM \to T_pM$ is self-adjoint, there exists a basis of T_pM consisting of eigenvectors of A_p with corresponding eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$, where $n = \dim M$. This defines functions $\lambda_1, \ldots, \lambda_n$ on M which are called the *principal curvatures* of M. The *multiplicity* of a principal curvature is the dimension of the corresponding eigenspace of A_p. The symmetric functions on the principal curvatures are invariants of the isometric immersion, up to sign in case of the symmetric functions of odd order (since the unit normal is unique up to sign only). Two significant instances of this invariants are the *mean curvature*

$$H := \frac{1}{n}\,\mathrm{trace}\,(A) = \frac{1}{n}(\lambda_1 + \cdots + \lambda_n),$$

and the *Gauss-Kronecker curvature*

$$K := \det(A) = \lambda_1 \cdots \lambda_n.$$

We specialize even more to the case in which $\overline{M} = \mathbf{R}^{n+1}$. Then the tangent spaces of $\mathbf{R}^{n+1}$ at its various points are canonically identified with $\mathbf{R}^{n+1}$ itself. The *Gauss map* of the immersion is the smooth map

$$g : M \to S^n,$$

where $g(p)$ is the unit vector $\nu_p \in S^n \subset \mathbf{R}^{n+1}$. Under the identifications, we can write

$$T_{\nu_p}S^n = (\mathbf{R}\nu_p)^\perp = T_pM.$$

It follows that the derivative of the Gauss map can be considered as a map $dg_p : T_pM \to T_pM$. Let $u \in T_pM$, and choose a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = u$. Then

$$\begin{aligned} dg_p(u) &= \frac{d}{dt}\Big|_{t=0}(\nu \circ \gamma)(t) \\ &= \overline{\nabla}_u \nu \\ &= (\overline{\nabla}_u \nu)^\top \\ &= -A_p(u), \end{aligned} \tag{7.3.1}$$

where we have used that $\langle \nu, \overline{\nabla}_u \nu \rangle = 0$ and Lemma 7.2.2.

Let u_i, u_j be eigenvectors of A_p of unit length associated to principal curvatures $\lambda_i(p)$, $\lambda_j(p)$, respectively; we may assume u_i and u_j are orthogonal (this is automatic if $\lambda_i(p) \neq \lambda_j(p)$). Then the Gauss equation (7.2.6) yields that the sectional curvature of the plane spanned by u_i, u_j is

$$-\langle R_p(u_i, u_j)u_i, u_j \rangle = \langle B_p(u_i, u_i), B_p(u_j, u_j) \rangle = \lambda_i \lambda_j. \tag{7.3.2}$$

In the case $n = 2$, we recover (compare Proposition 4.6.1):

7.3.3 Theorem (Theorema Egregium of Gauss) *The Gaussian curvature of a surface in $\mathbf{R}^3$ is an intrinsic invariant; namely, it depends only on the first fundamental form (Riemannian metric).*

We close this section with an application that will be later generalied.

7.3.4 Theorem (Hadamard's convexity theorem) *Let $f : M \to \mathbf{R}^{n+1}$ be an immersion of a compact connected smooth manifold of dimension $n \geq 2$. Assume that the induced Riemannian metric on M has positive sectional curvature. Then M is diffeomorphic to S^n, f is an embedding, and $f(M)$ is a convex hypersurface (namely, the (smooth) boundary of a convex body) in $\mathbf{R}^{n+1}$.*

Proof. The positivity of the sectional curvature implies via the Gauss formula (7.3.2) that the product of any two principal curvatures has always the same sign. It follows that all principal curvatures have the same sign, namely, the second fundamental form is definite as a symmetric bilinear form, for any choice of normal vector at any point. This shows that we can continuously choose a unit normal vector field ξ along f such that A_ξ is definite positive at all points (in particular, M is already orientable). Owing to equation (7.3.1), this implies that the Gauss map $g : M \to S^n$ is a local diffeomorphism. Since M is compact and S^n is simply-connected, indeed g is a global diffeomorphism.

Next, for a fixed unit vector $v \in S^n \subset \mathbf{R}^{n+1}$ we consider the *height function* $h_v : M \to \mathbf{R}$ defined by $h_v(x) = \langle f(x), v\rangle$ for $x \in M$. It is clear that $(\operatorname{grad} h_v)_p$ is the orthogonal projection of v into $df(T_pM)$, so p is a critical point of h_v if and only if $v \in \nu_pM := df(T_pM)^\perp$ if and only if $v = \pm\xi_p$; this proves that h_v has exacly two critical points, as g is a diffeomorphism. Moreover, for a critical point p and X, $Y \in \Gamma(TM)$, we have (cf. exercise 13 of chapter 4):

$$\begin{aligned} \operatorname{Hess}(h_v)(X,Y)_p &= X_p(Y(h_v)) \\ &= X_p\langle df(Y), v\rangle \\ &= \langle \nabla^f_{X_p} df(Y),\ v\rangle \\ &= \langle A_vX, Y\rangle_p. \end{aligned}$$

Since A_v is definite, any critical point is isolated and a local maximum or local minimum. Since h_v must have a global maximum and a global minimum by compacteness of M, we deduce that for every $v \in S^n$ the height function h_v has exactly two critical points and

$$\min h_v \ \leq h_v(x) \leq \max\ h_v$$

where the first (resp. second) equality occurs if and only if x is the point of global minimum (resp. maximum). We deduce that f is injective and $f(M)$ is the boundary of a convex body. □

7.4 Totally geodesic and totally umbilic submanifolds

A submanifold M of a Riemannian manifold $\bar{M}$ is called *totally geodesic* at a point $p \in M$ if the second fundamental form B vanishes at p, and it is called simply *totally geodesic* if B vanishes everywhere.

7.4.1 Proposition *For a submanifold M of $\bar{M}$, the following assertions are equivalent:*

a. M is totally geodesic in $\bar{M}$;
b. every geodesic of M is a geodesic of $\bar{M}$;
c. the geodesic γ_v of $\bar{M}$ with initial velocity $v \in T_pM$ is contained in M for small time (and hence is a geodesic in M).

Proof. Since B is symmetric, M is totally geodesic in $\bar{M}$ if and only $B(X,X) = 0$ for all $X \in \Gamma(TM)$ if and only if $B(v,v) = 0$ for all $v \in TM$. Gauss's formula (7.2.1) says that this is the case if and only if $\bar{\nabla}_X X = \nabla_X X$ for all $X \in \Gamma(TM)$. If this equation is true, plainly every geodesic in M will be a geodesic in $\bar{M}$. Conversely, assume every geodesic in M is a geodesic in

$\bar{M}$. Given $0 \neq v \in T_pM$, let γ_v be the geodesic in M with $\gamma_v'(0) = v$. Extend γ_v' to a smooth vector field $X \in \Gamma(TM)$ defined on a neighborhood of p. Since γ_v is also a geodesic of $\bar{M}$, we have $\bar{\nabla}_X X = 0 = \nabla_X X$. This proves the equivalence between (a) and (b). Next, if (b) holds, then the uniqueness of geodesics for given initial conditions says that all geodesics of $\bar{M}$ initially tangent to M come from geodesics of M, which implies (c). Finally, $\nabla_X X$ is the tangential component of $\bar{\nabla}_X X$ for $X \in \Gamma(TM)$, so a geodesic of $\bar{M}$ which is contained in M is a geodesic of M, which finishes the proof of the equivalence between (b) and (c). Note that the geodesic γ_v as in (c) is entirely contained in M, if M is complete. □

7.4.2 Corollary *A connected complete totally geodesic submanifold M of a Riemannian manifold $\bar{M}$ is completely characterized by T_pM for any given $p \in M$.*

Proof. In fact, it follows from the Hopf-Rinow theorem and Proposition 7.4.1 that $M = \overline{\exp}_p(T_pM)$, where $\overline{\exp}$ denotes the exponential map of $\bar{M}$. □

7.4.3 Proposition (Totally geodesic submanifolds of space forms) *The connected complete totally geodesic submanifolds of:*

a. $\mathbf{R}^n$ are the affine subspaces;
b. S^n are the great subspheres, namely, intersections of S^n with linear subspaces of $\mathbf{R}^{n+1}$;
c. $\mathbf{R}H^n$ are the intersections of hyperboloid model with linear subspaces of $\mathbf{R}^{1,n}$.

Proof. (a) Affine subspaces are clearly totally geodesic in $\mathbf{R}^n$. Since a totally geodesic submanifold is completely determined by its tangent space at a point, there can be no other examples. (b) Great circles of the subsphere are great circles of S^n, so this is a totally geodesic submanifold. The rest follows as in (a). The proof of (c) is similar. □

A submanifold M of a Riemannian manifold $\bar{M}$ is called *umbilic* in the direction of a normal vector ξ if the Weingarten operator A_ξ is a multiple of the identity operator, and it is called *totally umbilic* if every normal vector is umbilic; the latter property is equivalent to having

$$B(X,Y) = g(X,Y)\,H \qquad (7.4.4)$$

for all X, $Y \in \Gamma(TM)$, where H is the mean curvature vector. This equation is equivalent to

$$\langle A_\xi X, Y\rangle = \langle X, Y\rangle\langle \xi, H\rangle$$

for all $\xi \in \Gamma(\nu M)$, X, $Y \in \Gamma(TM)$. Note that the minimal totally umbilic submanifolds are precisely the totally geodesic submanifolds. A totally umbilic submanifold with non-zero parallel mean curvature is called an *extrinsic sphere*.

7.4.5 Proposition *A totally umbilic submanifold of dimension at least two in a space form is an extrinsic sphere.*

Proof. Differentiate (7.4.4) with respect to $Z \in \Gamma(TM)$ and use $\nabla g = 0$ to get to get $(\nabla^\perp_Z B)(X,Y) = g(X,Y)\,\nabla^\perp_Z H$. Now the Codazzi equation 7.2.4 says that

$$g(X,Y)\,\nabla^\perp_Z H = g(Z,Y)\,\nabla^\perp_X H.$$

Since $\dim M \geq 2$, we can choose $Y \perp Z$ and $X = Y$ to deduce $\nabla^\perp_Z H = 0$. Since Z is arbitrary, H is parallel. □

7.4.6 Proposition (Totally umbilic submanifolds of space forms) *The connected complete non-totally geodesic totally umbilic submanifolds of dimension at least two in:*

a. $\mathbf{R}^n$ *are the round spheres;*

b. S^n *are the small subspheres, namely, intersections of* S^n *with non-linear affine subspaces of* $\mathbf{R}^{n+1}$*;*

c. $\mathbf{R}H^n$ *are the intersections of hyperboloid model with non-linear affine subspaces of* $\mathbf{R}^{1,n}$*.*

Proof. (a) Let $\iota : M \to \mathbf{R}^n$ be a connected non-totally geodesic totally umbilic submanifold of Euclidean space with $\dim M \geq 2$. Then $B = g\,H$ and $\nabla^\perp H = 0$. For $X \in \Gamma(TM)$, we compute

$$\bar{\nabla}_X \left(\iota + \frac{H}{||H||^2} \right) = X - \frac{1}{||H||^2} A_H X = 0.$$

Connectedness of M implies that it is contained in the hypersphere of radius $1/||H||$ and center $p + \frac{1}{||H||^2} H(p)$ for any $p \in M$. If M has codimension one and is complete, it must coincide with that hypersphere. If M has higher codimension, note that $A_\xi = 0$ for $\xi \perp H$. This implies that a parallel normal vector field which is orthogonal to H at one point must be constant. It follows that M is contained in the affine subspace containing p and parallel to the linear subspace spanned by T_pM and $H(p)$, for all $p \in M$. Now if M is complete then it coincides with the intersection of the above hypersphere with this affine subspace.

(b) Let $\iota : M \to S^n$ be a connected non-totally geodesic totally umbilic submanifold of the sphere with $\dim M \geq 2$. Let $\theta = \operatorname{arccot} ||H|| \in (0, \pi/2]$. For $X \in \Gamma(TM)$, denoting the Levi-Cività connection of S^n by $\bar{\nabla}$, we compute

$$\bar{\nabla}_X \left(\cos\theta \iota + \sin\theta \frac{H}{||H||} \right) = (\cos\theta - ||H|| \sin\theta) X = 0.$$

If M is connected, this proves that M lies in the geodesic hypersphere of S^n of radius θ and center $n := \cos\theta p + \sin\theta \frac{H(p)}{||H||}$, for any $p \in M$. If M has codimension one and is complete, then it must coincide with this hypersphere, which is also the intersection of S^n with the affine hyperplane of $\mathbf{R}^{n+1}$ with normal n and distance $\cos\theta$ from the origin.

If M has higher codimension, let $\xi \perp H(p)$ and extend it to a parallel normal vector field $\hat{\xi}$ along M in S^n. Then, for all $X \in \Gamma(TM)$,

$$X(\hat{\xi}) = \bar{\nabla}_X \hat{\xi} + \langle X(\hat{\xi}), \iota \rangle \iota = -A_\xi X - \langle \hat{\xi}, X \rangle \iota = 0,$$

since $A_\xi = 0$, showing that $\hat{\xi}$ is constant in $\mathbf{R}^{n+1}$. This implies that M is contained in the hyperplane $\xi^\perp$. Another way to argue: note that for all $X \in \Gamma(TM)$,

$$X \left(\sin\theta \iota - \cos\theta \frac{H}{||H||} \right) = (\sin\theta + \cos\theta ||H||) X \in \Gamma(TM)$$

and

$$X(Y) = \bar{\nabla}_X Y + \langle X(Y), \iota \rangle \iota = \nabla_X Y + \langle X, Y \rangle (H - \iota)$$

for $Y \in \Gamma(TM)$. This implies that the span of T_pM and $\sin\theta\, p - \cos\theta\, \frac{H(p)}{||H||} = \sin\theta (p - H(p))$ is a constant subspace E of $\mathbf{R}^{n+1}$ along M. Either way, we deduce from the completeness of M that it coincides with the intersection of S^n with the affine subspace $p + E$ of $\mathbf{R}^{n+1}$.

(c) Similar to case (b), but replacing trigonometric functions by their hyperbolic brothers and Euclidean space by Lorentzian space. □

Osculating spaces

Let M be a submanifold of a Riemannian manifold $\bar{M}$. The submanifold M is called *full* or *substantial* in $\bar{M}$ if M is not contained in a totally geodesic submanifold of $\bar{M}$ of dimension smaller than $\dim \bar{M}$; otherwise we say that the codimension can be reduced, or that there is a *reduction of the codimension* of M. The smallest number to which the codimension can be reduced is called the *substantial codimension* of M in $\bar{M}$.

In order to study the substantial codimension of submanifolds, we discuss a bit about osculating spaces. The *k-th osculating space* to M at $p \in M$, for $k = 1, 2, \ldots$, is the subspace $\mathcal{O}_p^k(M)$ of $T_p\bar{M}$ spanned by the first k derivatives at 0 of all smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. Here the higher derivatives of γ are defined by

$$\gamma'' = \frac{\bar{\nabla}}{dt}\gamma', \ \gamma''' = \frac{\bar{\nabla}}{dt}\gamma'', \text{ etc.}$$

Clearly, $\mathcal{O}_p^1(M) = T_pM$ and there is an increasing chain of subspaces

$$\mathcal{O}_p^1(M) \subset \mathcal{O}_p^2(M) \subset \cdots \subset T_p\bar{M} \tag{7.4.7}$$

for all $p \in M$. It follows from the Gauss equation that

$$\frac{\bar{\nabla}}{dt}\gamma' = \frac{\nabla}{dt}\gamma' + B(\gamma', \gamma')$$

where B denotes the second fundamental form of M in $\bar{M}$. Since B is symmetric, the subspace of ν_pM spanned by the image of B_p coincides with the subspace spanned by the image of B_p restricted to the diagonal of T_pM. We deduce that $\mathcal{O}_p^2(M)$ is spanned by $\mathcal{O}_p^1(M)$ and the image of B_p. Similarly, one sees that $\mathcal{O}_p^k(M)$ is spanned by all vectors of the form

$$X_1|_p, \ \bar{\nabla}_{X_1}X_2|_p, \ldots, \bar{\nabla}_{X_1} \cdots \bar{\nabla}_{X_{k-1}}X_k|_p$$

for $X_1, \ldots, X_k \in \Gamma(TM)$. The *k-th normal space* $\mathcal{N}_p^k(M)$ of M in $\bar{M}$ is the orthogonal complement of $\mathcal{O}_p^k(M)$ in $\mathcal{O}_p^{k+1}(M)$, so that

$$\mathcal{O}_p^k(M) \overset{\perp}{\oplus} \mathcal{N}_p^k(M) = \mathcal{O}_p^{k+1}(M).$$

Note that $v \perp \mathcal{N}_p^1(M)$ if and only if the Weingarten operator $A_v = 0$.

If $\dim \mathcal{O}_p^k(M)$ is independent of $p \in M$, then the collection of k-th osculating spaces to M at all points can be made into a vector subbundle $\mathcal{O}^k(M)$ of the vector bundle $T\bar{M}|_M$; if this is true for all k, then also the collection of k-th normal spaces to M at all points can be made into a vector subbundle $\mathcal{N}^k(M)$ of the normal bundle νM.

7.4.8 Lemma *For each $p \in M$, the chain (7.4.7) stabilizes at some $k_0 \geq 2$ (which may depend on p), namely,*

$$\mathcal{O}_p^{k_0-1}(M) \subsetneq \mathcal{O}_p^{k_0}(M) = \mathcal{O}_p^{k_0+1}(M) = \cdots$$

If k_0 does not depend on p, then $\mathcal{O}^{k_0}(M)$ is a parallel subbundle of $T\bar{M}|_M$ and $\mathcal{N}^1(M) \oplus \cdots \oplus \mathcal{N}^{k_0-1}(M)$ is a parallel subbundle of νM.

Proof. The chain stabilizes simply by dimensional reasons. An arbitrary smooth section ξ of $\mathcal{O}^k(M)$ is a sum of terms of the form $f\bar\nabla_{X_1}\cdots\bar\nabla_{X_{k-1}}X_k$, where $f \in C^\infty(M)$. Since

$$\bar\nabla_X(f\bar\nabla_{X_1}\cdots\bar\nabla_{X_{k-1}}X_k) = X(f)\bar\nabla_{X_1}\cdots\bar\nabla_{X_{k-1}}X_k + f\bar\nabla_X\bar\nabla_{X_1}\cdots\bar\nabla_{X_{k-1}}X_k,$$

we see that $\bar\nabla_X\xi \in \Gamma(\mathcal{O}^{k+1}(M))$ for all $X \in \Gamma(TM)$. Now if k_0 does not depend on p, then $\bar\nabla_X\Gamma(\mathcal{O}^{k_0}(M)) \subset \Gamma(\mathcal{O}^{k_0}(M))$ for all $X \in \Gamma(TM)$, which is to say that $\mathcal{O}^{k_0}(M)$ is invariant under parallel transport in $T\bar M|_M$. This is equivalent to having that $\mathcal{N}^1(M)\oplus\cdots\oplus\mathcal{N}^{k_0-1}(M)$ is invariant under parallel transport in νM, which means that it is a parallel subbundle. □

In the remainder of this section, we assume that $\bar M$ is a space form.

7.4.9 Theorem (Erbacher) *Let M be an m-dimensional connected Riemannian submanifold of a space form $\bar M$. If $\mathcal{L}$ is a $\nabla^\perp$-parallel subbundle of νM containing $\mathcal{N}^1(M)$, and ℓ is the rank of $\mathcal{L}$, then there is a totally geodesic submanifold N of $\bar M$ of dimension $m+\ell$ that contains M.*

Proof. We consider separately the instances of space forms. The first case is $\bar M = \mathbf{R}^n$. Fix $p \in M$. It suffices to show that M is contained in the affine subspace $p + T_pM \oplus \mathcal{L}_p$ for some $p \in M$. Let γ be any piecewise smooth curve in $\bar M$ emanating from p and take any parallel normal vector field ξ along γ such that $\xi(0) \perp \mathcal{L}_p$. Since $\mathcal{L}$ is parallel along γ, we have that $\xi(t) \perp \mathcal{L}_{\gamma(t)}$ for all t. In particular, $\xi(t) \perp \mathcal{N}^1_{\gamma(t)}(M)$, so the Weingarten equation says that $\frac{\bar\nabla}{dt}\xi \equiv 0$, namely, ξ is constant in $\mathbf{R}^n$ along γ. Since γ is arbitrary and M is connected, this means that M is contained in the orthogonal complement to the vector $\xi(0)$. Since $\xi(0)$ is an arbitrary vector in $\nu_pM \cap \mathcal{L}_p^\perp$, this case is done.

Consider next the case $\bar M = S^n(1)$; we reduce it to the previous case as follows. View M as a submanifold of $\mathbf{R}^{n+1}$ and consider the augmented vector bundle $\hat{\mathcal{L}}$ over M where $\hat{\mathcal{L}}_p = \mathcal{L}_p \oplus \mathbf{R}\,p$ for all $p \in M$. Note that $\hat{\mathcal{L}}$ is a subbundle of the normal bundle $\hat\nu M$ of M in $\mathbf{R}^{n+1}$ that contains the first normal bundle $\hat{\mathcal{N}}^1(M)$ of M in $\bar M$. Let $\hat\nabla^\perp$ be the normal connection on $\hat\nu M$. Both $\hat\nabla^\perp$ and $\nabla^\perp$ are induced by $\bar\nabla$, so they coincide on νM. Given that $\mathcal{L}$ is $\nabla^\perp$-parallel and the position vector $\mathbf{p}$ is $\hat\nabla^\perp$-parallel, we see that $\hat{\mathcal{L}}$ is $\hat\nabla$-parallel. By the previous case, M is contained in $(p + T_pM \oplus \hat{\mathcal{L}}_p) \cap S^n(1)$ for some $p \in M$, which is isometric to $S^{m+\ell}(1)$, as $p + \hat{\mathcal{L}}_p = \hat{\mathcal{L}}_p$ is a linear subspace of dimension $\ell + 1$ of $\mathbf{R}^{n+1}$.

In case $\bar M = \mathbf{R}H^n$, by using the hyperboloid model the proof follows arguments similar to those in the previous two cases, where we use the canonical connection of Lorentz space $\mathbf{R}^{1,n}$. Indeed, as in the second case we view M as a submanifold of $\mathbf{R}^{1,n}$ and extend $\mathcal{L}$ to a subbundle $\hat{\mathcal{L}}$ of the normal bundle $\hat\nu M$ of M in $\mathbf{R}^{1,n}$ by adding the position vector field $\mathbf{p}$. We then prove, as in the first case, that M is contained in an affine subspace of $\mathbf{R}^{1,n}$ whose linear part, owing to $\langle \mathbf{p}, \mathbf{p}\rangle = 1$, is a Lorentz subspace isometric to $\mathbf{R}^{1,\ell}$

7.5 Focal points and the Morse index theorem

In this subsection, we state and prove the Morse index theorem for submanifolds of Riemannian manifolds. The discussion herein extends that in chapter 5 and, specially, Theorem 7.5.4 generalizes Theorem 5.5.3.

Let M be a submanifold of a Riemannian manifold $\bar M$. The restriction of the exponential map of $\bar M$ to the normal bundle of M is called the *normal exponential map* of M:

$$\exp^\perp : \nu M \to \bar M.$$

A critical value of $\exp^\perp$ is called a *focal point* of M. Note that this concept reduces to that of a conjugate point, in case M is a point.

7.5.1 Remark By the Sard-Brown theorem (see [Mil97]), the set of focal points of a submanifold has Lebesgue measure zero and hence is a nowhere dense set.

7.5.2 Proposition *A point $q \in \bar{M}$ is a focal point of a submanifold M if and only if there exists a geodesic $\gamma : [0,1] \to \bar{M}$ with $\gamma(0) = p \in M$, $\gamma(1) = q$, $\gamma'(0) = \xi \in \nu_p M$ and a Jacobi field J along γ such that $J(0) = u \in T_p M$ and $J'(0) + A_\xi u \in \nu_p M$.*

Proof. The point q is a critical value of $\exp^\perp$ if and only if it is in the image, namely, $q = \exp^\perp \xi$ for some $\xi \in \nu_p M$ and some $p \in M$, and the kernel of $d(\exp^\perp)_\xi$ is non-zero. Consider the geodesic $\gamma(s) = \exp_p(s\xi)$ for $s \in [0,1]$. Take a non-zero vector in $\ker d(\exp^\perp)_\xi$ represented by a smooth curve $\hat{\xi} : (-\epsilon, \epsilon) \to \nu M$, where $\hat{\xi}(0) = \xi$. This defines a smooth variation of γ through geodesics orthogonal to M:

$$H(s,t) = \exp^\perp_{c(t)}(s\hat{\xi}(t))$$

where $c(t) \in M$ is the footpoint of $\hat{\xi}(t)$ and $(s,t) \in [0,1] \times (-\epsilon, \epsilon)$. The associated variational vector field is a Jacobi field J along γ. Its initial conditions are (compare the proof of Proposition 5.4.4):

$$J(0) = \frac{\bar{\partial}}{\partial t}\Big|_{\substack{t=0\\s=0}} = c'(0) =: u,$$

and

$$\frac{\bar{\partial}}{\partial s}\Big|_{s=0} = d(\exp_{c(t)})_{0_{c(t)}}(\hat{\xi}(t)) = \hat{\xi}(t),$$

so

$$J'(0) = \overline{\nabla}_{\frac{\partial}{\partial s}} \frac{\bar{\partial}}{\partial t}\Big|_{\substack{t=0\\s=0}} = \overline{\nabla}_{\frac{\partial}{\partial t}} \frac{\bar{\partial}}{\partial s}\Big|_{\substack{t=0\\s=0}} = \bar{\nabla}_u \hat{\xi} = -A_\xi u + \nabla^\perp_u \hat{\xi},$$

completing the proof. □

A geodesic γ in a Riemannian manifold $\bar{M}$ which is perperndicular to a submanifold M at a point $p \in M$ is called an *M-geodesic*. A variational vector field along an M-geodesic which is associated to a variation through M-geodesics is called an *M-Jacobi field*. It follows from the proof of Proposition 7.5.2 that the space of M-Jacobi fields along an M-geodesic γ is the space of Jacobi fields J along γ that satisfy the initial conditions

$$J(0) \in T_p M \quad \text{and} \quad J'(0) + A_\xi J(0) \in \nu_p M,$$

where $p = \gamma(0)$ and $\xi = \gamma'(0)$. The *multiplicity* of a focal point $q = \gamma(s_0)$ to M along γ is the dimension of the kernel of $d(\exp^\perp)_{s_0\xi}$, which is also the dimension of the space of M-Jacobi fields along γ that vanish at s_0.

The Morse index theorem

Let M be a submanifold in a Riemannian manifold $\bar{M}$. Fix a unit speed M-geodesic $\gamma : [0,\ell] \to \bar{M}$ with $\gamma(0) = p \in M$. Denote by $\mathcal{V}$ the space of piecewise smooth vector fields Y along γ that satisfy the boundary conditions:

$$Y(0) \in T_p M, \quad Y'(0) + A_\xi Y(0) \in \nu_p M \quad \text{and} \quad Y(\ell) = 0,$$

where $\xi = \gamma'(0)$. Consider the index form I on $\mathcal{V}$ given by

$$I(X,Y) = -\langle A_\xi X, Y\rangle_0 + \int_0^\ell \langle X', Y'\rangle + \langle R(\gamma', X)\gamma', Y\rangle \, ds.$$

Note that $\frac{d^2}{dt^2}|_{t=0}E(\gamma_t) = I(Y,Y)$ for a variation of γ whose associated variational vector field Y lies in $\mathcal{V}$. It is not difficult to see that the kernel of I precisely consists of the M-Jacobi fields along γ that vanish at ℓ.

7.5.3 Lemma *Choose any subdivision $0 = s_0 < s_1 < \cdots < s_n = \ell$ such that $\gamma|_{[s_{i-1},s_i]}$ is minimizing for $i = 1,\ldots,n$, $\gamma(s_1)$ is not focal to M along γ, and $\gamma(s_{i-1})$, $\gamma(s_i)$ are not conjugate along γ for $i = 2,\ldots,n$. Then there is a I-orthogonal, vector space direct sum*

$$\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$$

where:

- *$\mathcal{V}^+$ is the subspace of $\mathcal{V}$ consisting of vector fields vanishing at $s_0,\ldots,s_{n-1}$;*
- *$\mathcal{V}^-$ is the subspace of $\mathcal{V}$ consisting of vector fields X that restrict to Jacobi fields along $[s_{i-1},s_i]$ for all $i = 1,\ldots,n$.*

Moreover, I is positive definite on $\mathcal{V}^+$. It follows that the index (resp. nullity) of I on $\mathcal{V}$ is equal to the index (resp. nullity) of I on $\mathcal{V}^-$; in particular, it is finite.

Proof. Let $X \in \mathcal{V}$. Since $\gamma(s_1)$ is not focal to M and $\gamma(s_{i-1})$ and $\gamma(s_i)$ are not conjugate points for $i \geq 2$ along γ, we can find $Y \in \mathcal{V}^-$ such that $Y(s_i) = X(s_i)$ for all $i = 0,\ldots,n$ (exercise 5 of chapter 5). Then $X - Y \in \mathcal{V}^+$. Clearly, $\mathcal{V}^+ \cap \mathcal{V}^- = \{0\}$.

Also, for $Y \in \mathcal{V}^-$ and $Z \in \mathcal{V}^+$, we have that Y is a Jacobi field (hence smooth) along $\gamma|_{[s_{i-1},s_i]}$ for $i = 1,\ldots,n$, so integration by parts allows us to rewrite the index form on $\mathcal{V}$ as (compare 5.4.1)

$$I(Y,Z) = -\langle A_\xi Y, Z\rangle_{s_0} - \sum_{i=1}^{n-1}\langle Y'(s_i^+) - Y'(s_i^-), Z\rangle + \int_0^\ell \langle -Y'' + R(\gamma',Y)\gamma', Z\rangle\, ds.$$

Since $Z(s_i) = 0$ for $i = 0,\ldots,n-1$, this formula shows that $I(Y,Z) = 0$.

Next we prove that I is positive definite on $\mathcal{V}^+$. Let $Z \in \mathcal{V}^+$. Since $\gamma|_{[s_{i-1},s_i]}$ is a minimizing geodesic and Z is the variational vector field associated to a variation that keeps $\gamma(s_{i-1})$ and $\gamma(s_i)$ fixed for $i = 1,\ldots,n$, we get that $I(Z,Z) \geq 0$. Suppose now, in addition, that $I(Z,Z) = 0$. For all $\tilde{Z} \in \mathcal{V}^+$ we have

$$0 \leq I(Z + \alpha\tilde{Z}, Z + \alpha\tilde{Z}) = 2\alpha I(Z,\tilde{Z}) + \alpha^2 I(\tilde{Z},\tilde{Z})$$

for all $\alpha \in \mathbf{R}$, which implies that $I(Z,\tilde{Z}) = 0$. Therefore Z is I-orthogonal to $\mathcal{V}^+$, and since it was already I-orthogonal to $\mathcal{V}^-$, we deduce that Z is a Jacobi field along γ. It follows that $Z = 0$.

The remaining assertions follow from the fact that $\mathcal{V}^-$ is finite-dimensional. □

7.5.4 Theorem (Morse) *Let M be a submanifold in a Riemannian manifold $\bar{M}$. Fix a unit speed M-geodesic $\gamma : [0,\ell] \to \bar{M}$ with $\gamma(0) = p \in M$. Then the index of $I : \mathcal{V}\times\mathcal{V} \to \mathbf{R}$ is finite and equals the sum of the multiplicities of focal points to M along γ of the form $\gamma(s)$ for some $0 < s < \ell$.*

7.5.5 Corollary *The set of focal points along an M-geodesic is discrete.*

Consider the restriction $\gamma_s := \gamma|_{[0,\ell]}$ for $s \in [0,\ell]$, the corresponding decomposition $\mathcal{V}_s = \mathcal{V}_s^+ \oplus \mathcal{V}_s^-$ as in Lemma 7.5.3, the associated index form I_s, and its index $\lambda(s)$. The proof of Theorem 7.5.4 is a consequence of Lemmata 7.5.6, 7.5.7, 7.5.8 and 7.5.10, in which we prove that λ is a left-continuous function with jumps precisely at the focal points.

7.5.6 Lemma $\lambda(s) = 0$ *for sufficiently small $s > 0$.*

Proof. This follows from the fact that γ_s is minimizing for sufficiently small $s > 0$. □

7.5.7 Lemma *$\lambda(s)$ is a non-decreasing function of $s \in [0,\ell]$.*

Proof. For $s' < s''$, we can view $\mathcal{V}_{s'} \subset \mathcal{V}_{s''}$ via the embedding

$$W \mapsto \overline{W}, \quad \text{where} \quad \overline{W}(s) = \begin{cases} W(s), & \text{if } 0 \le s \le s'; \\ 0, & \text{if } s' \le s \le s''; \end{cases}$$

and then $I_{s'}$ is the restriction of $I_{s''}$ to $\mathcal{V}_{s'}$, implying the result. □

7.5.8 Lemma *Given $s \in (0,\ell]$, we have $\lambda(s-\epsilon) = \lambda(s)$ for sufficiently small $\epsilon > 0$.*

Proof. Choose a subdivision of $[0,\ell]$ as in Lemma 7.5.3 such that $s \in (s_{i-1}, s_i]$ for some $i = 1, \ldots, n$. Since $\lambda(s) = 0$ for $s \in (0, s_1]$, we may assume $i \ge 2$. Since $\gamma(s_1)$ is not focal to M along γ, $d(\exp^\perp)_{s_1\xi} : T_{s_1\xi}(\nu M) \to T_{\gamma(s_1)}\bar{M}$ is an isomorphism, where $\xi = \gamma'(0)$, so given $u \in T_{\gamma(s_1)}\bar{M}$, there exists a unique M-Jacobi field along γ_{s_1} whose value at s_1 is u. This, together with exercise 5 of chapter 5, shows that

$$\mathcal{V}^-_{s'} \cong T_{\gamma(s_1)}\bar{M} \oplus \cdots \oplus T_{\gamma(s_{i-1})}\bar{M} =: U \tag{7.5.9}$$

for all $s' \in (s_{i-1}, s]$. Note that $\lambda(s')$ is the index of $I_{s'}$ on U. Since $I_{s'}$, as a bilinear form on U, depends continuously on s', $I_{s'}$ is negative definite on any subspace of U on which I_s is negative definite, for sufficiently small $s - s' > 0$. This implies $\lambda(s') \ge \lambda(s)$ for sufficiently small $s - s' > 0$, and hence the desired result in view of Lemma 7.5.7. □

7.5.10 Lemma *Given $s \in (0,\ell)$, let $\nu(s)$ denote the nullity of I_s. Then $\lambda(s+\epsilon) = \lambda(s) + \nu(s)$ for sufficiently small $\epsilon > 0$.*

Proof. Choose a subdivision of $[0,\ell]$ as in Lemma 7.5.3 such that $s \in (s_{i-1}, s_i)$ for some $i = 1, \ldots, n$. Again, we may assume $i \ge 2$. Consider $I_{s'}$ as a bilinear form on U for $s' \in [s, s_i)$, where U is given as in (7.5.9). Note that I_s is positive definite on a subspace of dimension $\dim U - \lambda(s) - \nu(s)$. By continuity, also $I_{s'}$ is positive definite on that subspace for sufficiently small $s' - s > 0$. Therefore $\lambda(s+\epsilon) \le \lambda(s) + \nu(s)$ for sufficiently small $\epsilon > 0$.

To prove the reverse inequality, we start with linearly independent vector fields $X_1, \ldots, X_{\lambda(s)}$ in $\mathcal{V}_s$ spanning a subspace on which I_s is negative definite. Extend these vector fields over $\gamma_{s+\epsilon}$ by setting them equal to zero on $[s, s+\epsilon]$ as in Lemma 7.5.7. If $\epsilon > 0$ is sufficiently small, these extensions span a subspace of dimension $\lambda(s)$ of $\mathcal{V}_{s+\epsilon}$ on which $I_{s+\epsilon}$ is negative definite.

Next, by hypothesis we can find $\nu(s)$ linearly independent M-Jacobi fields $Y_1, \ldots, Y_{\nu(s)}$ along γ_s vanishing at s; extend them over $\gamma_{s+\epsilon}$ by zero. By making use of the technique of the theorem of Jacobi-Darboux 5.5.3, we can produce perturbations $\tilde{Y}_1, \ldots, \tilde{Y}_{\nu(s)} \in \mathcal{V}_{s+\epsilon}$ that span a subspace on which $I_{s+\epsilon}$ is negative definite. Since $X_1, \ldots, X_{\lambda(s)}, Y_1, \ldots, Y_{\nu(s)}$ were clearly linearly independent, we can also take the perturbations so that $X_1, \ldots, X_{\lambda(s)}, \tilde{Y}_1, \ldots, \tilde{Y}_{\nu(s)}$ are linearly independent. This completes the proof the lemma and of Theorem 7.5.4. □

7.6 Theory of isoparametric submanifolds

We say a Riemannian submanifold M of a space form $\bar{M}$ has *constant principal curvatures* if the principal curvatures along any locally defined parallel normal vector field are constant. If, in addition, M has flat normal bundle, then it is called an *isoparametric submanifold* of $\bar{M}$. Note that in case M has codimension one, both conditions reduce to simply requiring that M has constant principal curvatures. In view of the Fundamental Theorem 7.2.9, isoparametric submanifolds are sometimes said to be the submanifolds having the "simplest" local invariants.

Basic structure

Let M be an isoparametric submanifold of a space form $\bar{M}$. Since the normal bundle is flat ($R^{\perp} \equiv 0$), the Ricci equation (7.2.8) yields that, for every $p \in M$, $\{ A_\xi \mid \xi \in \nu_p M \}$ is a commutative family of symmetric endomorphisms of T_pM, hence simultaneously diagonalizable, say, with pairwise distinct eigenvalues $\lambda_1(\xi), \dots, \lambda_g(\xi)$ and common eigenspaces $E_1(p), \dots, E_g(p)$. Since the principal curvatures $\lambda_1(\xi), \dots, \lambda_g(\xi)$ are constant along any extension of ξ to a parallel normal vector field, we obtain g mutually orthogonal Frobenius distributions $E_1, \dots, E_g$ on M such that $TM = \bigoplus_{i=1}^g E_i$; each E_i is called a *curvature distribution*. The dimension of E_i is called a *multiplicity*. The restriction of λ_i to ν_pM is a linear functional, so there exists $v_i(p) \in \nu_pM$ such that $\lambda_i(\xi) = \langle \xi, v_i(p) \rangle$ for all $\xi \in \nu_pM$. This way we obtain g smooth normal vector fields $v_1, \dots, v_g$ along M, called *curvature normals*, which moreover are parallel, as:

$$\langle \nabla^{\perp}_X v_i, \xi \rangle = X \langle v_i, \xi \rangle - \langle v_i, \nabla^{\perp}_X \xi \rangle = X(\lambda_i(\xi)) = 0,$$

for every parallel normal vector field ξ and $X \in \Gamma(TM)$. Now for each $\xi \in \nu M$, the corresponding Weingarten operator satisfies

$$A_\xi|_{E_i} = \langle \xi, v_i \rangle \, \mathrm{id}_{E_i} \tag{7.6.1}$$

for $i = 1, \dots, g$; equivalently,

$$B(X_i, Y_j) = \langle X_i, Y_j \rangle \, v_i \tag{7.6.2}$$

for all $X_i \in E_i$, $Y_j \in E_j$ and $i, j = 1, \dots, g$. It follows from (7.6.1) that the case $g = 1$ precisely corresponds to the class of totally umbilic submanifolds of $\bar{M}$. Note that the substantial codimension of M equals the number of linearly independent curvature normals; this number is called the *rank* of M. We will always assume that M is full in $\bar{M}$, that is, not contained in a proper totally geodesic submanifold. It then follows that the curvature normals of M span the normal space at each point.

Another fundamental invariant of M is the covariant derivative of the second fundamental form. By taking derivatives and using the parallelism of the metric and the curvature normals, we obtain from (7.6.2) that

$$\nabla_{X_i} B(Y_j, Z_k) = \langle \nabla_{X_i} Y_j, Z_k \rangle \, (v_j - v_k) \tag{7.6.3}$$

for all $X_i \in E_i$, $Y_j \in E_j$, $Z_k \in E_k$ and $i, j, k = 1, \dots, g$. The Codazzi equation (7.2.7) is the symmetry of ∇B in all three arguments, which owing to (7.6.3), gives

$$\langle \nabla_{Z_k} X_i, Y_j \rangle \, (v_i - v_j) = \langle \nabla_{X_i} Y_j, Z_k \rangle \, (v_j - v_k) = \langle \nabla_{Y_j} Z_k, X_i \rangle \, (v_k - v_i).$$

Taking $i = j \neq k$ (in case $g \geq 2$) in the first equality shows that $\nabla_{X_i} Y_i \in \Gamma(E_i)$ for all X_i, $Y_i \in E_i$, namely, each curvature distribution E_i is *auto-parallel*; it follows that it is involutive and thus, by Frobenius theorem, integrable. Again, by auto-parallelism of E_i, its leaf through a point $p \in M$, denoted $S_i(p)$, is a totally geodesic submanifold of M. It follows from (7.6.1) that $S_i(p)$ is a totally umbilic submanifold of $\bar{M}$. In particular, in case $v_i = 0$ the distribution E_i is called the *nullity distribution* and its leaves are totally geodesic in $\bar{M}$.

We will be mostly concerned with the case of isoparametric submanifolds of Euclidean space. That this case contains the case of isoparametric submanifolds of spheres is the subject of the next proposition (see also Proposition 7.6.12).

7.6.4 Proposition *A Riemannian submanifold M of the unit sphere S^n of $\mathbf{R}^{n+1}$ is a rank k isoparametric submanifold of S^n if and only if it is a rank k isoparametric submanifold of $\mathbf{R}^{n+1}$.*

Proof. Let M be a submanifold of S^n. Every normal vector field ξ to M in S^n is also a normal vector field in $\mathbf{R}^{n+1}$. On the other hand, a normal vector field η to M in $\mathbf{R}^{n+1}$ can be written $\eta = \xi - f\mathbf{p}$, where ξ is the component of η tangential to S^n, $\mathbf{p}$ is the position vector, and f is a smooth function on M.

We will use the Weingarten formula. Denote by A, $\hat{A}$ the Weingarten operators of M, and by $\nabla^\perp$, $\hat{\nabla}^\perp$ the normal connections of M, viewed as a submanifold of S^n, $\mathbf{R}^{n+1}$, respectively. Denote by ∇ the Levi-Cività connection of S^n and take $X \in \Gamma(TM)$. Note that

$$\nabla_X \xi = X(\xi) - \langle X(\xi), \mathbf{p}\rangle\mathbf{p} = X(\xi),$$

since $\langle X(\xi), \mathbf{p}\rangle = -\langle \xi, X(\mathbf{p})\rangle = -\langle \xi, X\rangle = 0$. It follows that

$$\begin{aligned} -\hat{A}_\eta X + \hat{\nabla}^\perp_X \eta &= X(\eta) \\ &= X(\xi) - X(f)\mathbf{p} - fX \\ &= \nabla_X \xi - X(f)\mathbf{p} - fX \\ &= -A_\xi X + \nabla^\perp_X \xi - X(f)\mathbf{p} - fX. \end{aligned}$$

Comparing tangent and normal components, we obtain that

$$\hat{\nabla}^\perp_X \eta = \nabla^\perp_X \xi - X(f)\mathbf{p} \quad \text{and} \quad \hat{A}_\eta X = A_\xi X + fX.$$

We deduce from these equations that η is $\hat{\nabla}^\perp$-parallel if and only if ξ is $\nabla^\perp$-parallel and f is constant; and, in this case, the eigenvalues of $\hat{A}_\eta$ are of the form $\lambda_i(\xi) + \mu$, where $\lambda_i(\xi)$ is an eigenvalue of A_ξ and μ is a constant, with the same eigenspaces. Note that the curvature normals v_i, $\hat{v}_i$ of M as an isoparametric submanifold of S^n, $\mathbf{R}^{n+1}$, resp., are related by $\hat{v}_i = v_i - \mathbf{p}$. □

Parallel foliation

Let M be an isoparametric submanifold of a space form $\bar{M}$. For a fixed parallel normal vector field ξ, a fundamental construction is the *parallel map*

$$\pi_\xi : M \to \bar{M}, \qquad \pi_\xi(x) = \exp^\perp \xi(x),$$

namely, the restriction of the normal exponential map along ξ. For simplicity, in the sequel we assume $\bar{M} = \mathbf{R}^{n+k}$, where $n = \dim M$.

Now $\pi_\xi(x) = x + \xi(x)$ for $x \in M$. Using the canonical parallelism of $\mathbf{R}^{n+k}$, the differential of this map is $\mathrm{id} - A_\xi$, so its kernel is $\bigoplus\{E_i \mid \langle \xi, v_i\rangle = 1\}$. Since π_ξ has constant rank, its image M_ξ is a submanifold of $\bar{M}$ of dimension $n - \dim\ker d(\pi_\xi)_p$ for $p \in M$. The map $\pi_\xi : M \to M_\xi$ is a submersion, and M_ξ is called a *parallel manifold* in case $\dim M_\xi = n$, or a *focal manifold* in case $\dim M_\xi < n$. We thus see that the *focal set* of M, namely the subset of $\bar{M}$ consisting of all focal points of M along normal geodesics, decomposes into focal manifolds, and M_ξ is a focal manifold precisely if $\ker d\pi_\xi$ is non-zero, in which case π_ξ is called a *focal map*.

Since $\pi := \pi_\xi : M \to M_\xi$ is a submersion, there is an orthogonal decomposition $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}_p = \ker d\pi_p$ and $d\pi_p : \mathcal{H}_p M \to T_{\pi(p)}(M_\xi)$ is an isomorphism. Since $d\pi_p = \mathrm{id} - A_\xi$, we can view $T_pM = T_{\pi(p)}(M_\xi)$ and then $\nu_{\pi(p)}(M_\xi) = \nu_p M \oplus \mathcal{V}_p$.

7.6.5 Lemma *The connected components of the level sets of $\pi : M \to M_\xi$ are totally geodesic submanifolds of M. Moreover, the Weingarten operators of M preserve the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$.*

Proof. The first assertion is to be a consequence of the Codazzi equation (7.2.7); cf. third equality below. Let U, $V \in \Gamma(\mathcal{V})$, $X \in \Gamma(\mathcal{H})$. Since $\nabla^\perp \xi = 0$, $A_\xi U = U$, $A_\xi V = V$, and using that A_ξ is a self-adjoint operator, we have:

$$\begin{aligned}
\langle \nabla_U V, X\rangle &= \langle \nabla_U (A_\xi V), X\rangle \\
&= \langle (\nabla_U A_\xi)V + A_\xi(\nabla_U V), X\rangle \\
&= \langle (\nabla_X A_\xi)V, U\rangle + \langle \nabla_U V, A_\xi X\rangle \\
&= \langle \nabla_X (A_\xi V) - A_\xi(\nabla_X V), U\rangle + \langle \nabla_U V, A_\xi X\rangle \\
&= \langle \nabla_U V, A_\xi X\rangle,
\end{aligned}$$

proving that $\langle \nabla_U V, d\pi(X)\rangle = 0$ and hence that $\mathcal{V}$ is auto-parallel.

Further, since ξ is parallel, the Ricci equation (7.2.8) says that the Weingarten operators of M at p commute with A_{ξ_p}. Therefore they commute with $d\pi_p$ and thus preserve its kernel $\mathcal{V}_p$, for all $p \in M$. Since they are symmetric endomorphisms of T_pM, they also preserve $\mathcal{H}_p$. □

7.6.6 Remark In fact, it is not hard to show that any component of the level set $\pi^{-1}(\pi(p))$ in Lemma 7.6.5 is an isoparametric submanifold of $\nu_{\hat{p}}M_\xi$, where $\hat{p} = p + \xi(p)$ (cf. Exercise 15). Using as a tool the normal holonomy of focal manifolds, one can work harder and see that those level sets are connected and indeed *homogeneous* isoparametric submanifolds [BCO16, § 4.3.3]. These are called *slices* of the given isoparametric submanifold.

Any smooth curve in M_ξ admits a locally defined lifting to a horizontal smooth curve in M (cf. exercise 19 of chaper 3). Let $\gamma : [a, b] \to M$ be a horizontal smooth curve and put $\hat{\gamma} = \pi \circ \gamma$. Since $\nu_{\gamma(t)}M \subset \nu_{\hat{\gamma}(t)}M_\xi$ for all t, any normal vector field η to M along γ can be also considered as a normal vector field $\hat{\eta}$ to M_ξ aong $\hat{\gamma}$.

7.6.7 Proposition (Tube formula) *For $v \in \nu_pM \subset \nu_{\pi(p)}M_\xi$, let $\hat{A}_v$ denote the Weingarten operator of M_ξ. Then*

$$\hat{A}_v = A_v \circ ((\mathrm{id}_{T_pM} - A_{\xi_p})|_{\mathcal{H}_p})^{-1}.$$

Proof. Using the canonical parallelism of $\mathbf{R}^{n+k}$ and the Weingarten formula (7.2.3), we can write

$$-\hat{A}_{\hat{\eta}(t)}\hat{\gamma}'(t) + \hat{\nabla}^\perp_{\hat{\gamma}'(t)}\hat{\eta} = \hat{\eta}'(t) = \eta'(t) = -A_{\eta(t)}\gamma'(t) + \nabla^\perp_{\gamma'(t)}\eta,$$

where $\hat{\nabla}^\perp$ denotes the normal connection of M_ξ. Since γ is horizontal, we know from Lemma 7.6.5 that $A_{\eta(t)}\gamma'(t) \in \mathcal{H}_{\gamma(t)} = T_{\hat{\gamma}(t)}(M_\xi)$. It follows that $A_{\eta(t)}\gamma'(t) = \hat{A}_{\hat{\eta}(t)}\hat{\gamma}'(t)$. Now we need only remark that $\hat{\gamma}'(t) = d\pi_{\gamma(t)}(\gamma'(t)) = \gamma'(t) - A_{\xi(\gamma(t))}\gamma'(t)$. □

7.6.8 Corollary *The submanifold M_ξ has constant principal curvatures. Moreover, if M_ξ is a parallel manifold ($\dim M_\xi = \dim M$), then it is isoparametric.*

Proof. It follows from the tube formula that the principal curvatures of M_ξ at $v \in \nu_{\pi(p)}M_\xi$ are of the form

$$\frac{\lambda_i(v)}{1 - \lambda_i(\xi_p)}$$

for $i = 1, \dots, g$, where the λ_i are the principal curvatures of M. Therefore M_ξ has constant principal curvatures.

It also follows from the proof of Proposition 7.6.7 that $\nabla^\perp_{\hat\gamma'(t)}\hat\eta = \nabla^\perp_{\gamma'(t)}\eta$. In case $\dim M_\xi = \dim M$, we have $\nu_{\pi(p)}(M_\xi) = \nu_p M$ implying that $\hat R^\perp = R^\perp = 0$. □

For $p \in M$, denote by $F_M(p)$ the subset of $\bar M$ consisting of focal points of M relative to p. Note that $F_M(p)$ consists of the points $q \in p + \nu_p M$ such that $d\pi_{q-p} = \mathrm{id} - A_{q-p}$ has not full rank.

7.6.9 Corollary *If M_ξ is a parallel manifold, then $p + \nu_p M = \hat p + \nu_{\hat p} M_\xi$ and $F_M(p) = F_{M_\xi}(\hat p)$ for all $p \in M$, where $\hat p = p + \xi(p)$.*

Proof. The first assertion is a consequence of the facts that $\nu_p M = \nu_{\hat p} M_\xi$ and $\hat p \in p + \nu_p M$. For the second one, the tube formula (7.6.7) yields that

$$\begin{aligned} \mathrm{id} - \hat A_{q-\hat p} &= (\mathrm{id} - A_{\xi(p)} - A_{q-\hat p})(\mathrm{id} - A_{\xi(p)})^{-1} \\ &= (\mathrm{id} - A_{q-p})(\mathrm{id} - A_{\xi(p)})^{-1}, \end{aligned}$$

for all $q \in p + \nu_p M$, whence we see that $\mathrm{id} - \hat A_{q-\hat p}$ is invertible if and only if so is $\mathrm{id} - A_{q-p}$, as desired. □

The Coxeter group

We next describe the Coxeter group associated to a complete isoparametric submanifold M of Euclidean space.

The affine normal space $p + \nu_p M$ meets the focal set $F_M(p)$ along the union of the affine hyperplanes $H_i(p) = p + \{\, \xi \in \nu_p M \mid \langle \xi, v_i(p)\rangle = 1 \,\}$ corresponding to non-zero curvature normals, called *focal hyperplanes* with respect to p. For each focal hyperplane $H_i(p)$, the orthogonal reflection of $p + \nu_p M$ on the hyperplane $H_i(p)$ will be denoted by $\tilde r_i^p$. We will show that the group generated by all the $\tilde r_i^p$ is a finite Coxeter group.

Recall that, in general, an (abstract) *Coxeter group* is a finitely presented group

$$\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$, and the condition $m_{ij} = \infty$ means that no relation of the form $(r_i r_j)^m$ is imposed. The number n is called the *rank* of the Coxeter group. In 1934, H. S. M. Coxeter proved that every finite group generated by orthogonal reflections on hyperplanes in an Euclidean space is a Coxeter group, whereas in 1935 he proved that every finite Coxeter group admits a faithful representation as group generated by reflections on an Euclidean space and classified the finite Coxeter groups. They fall into: three families of increasing rank A_n, B_n, D_n; one family of rank two, $I_2(p)$; and six exceptional groups, E_6, E_7, E_8, F_4, H_3 and H_4.

Note that for each non-zero curvature normal v_i of M, the leaf $S_i(p)$ of E_i passing through p is the hypersphere in $p + E_i(p) + \mathbf{R}\, v_i(p)$ of center $c_i(p) = p + \frac{v_i(p)}{||v_i||^2}$ and radius $1/||v_i||$. Let $a_i : M \to M$ denote the map that restricts to the antipodal map of $S_i(x)$ for all $x \in M$. Namely, a_i is given by the parallel map π_{η_i} where $\eta_i = 2\frac{v_i}{||v_i||^2}$. This shows that a_i is an involutive diffeomorphism of M (but not an isometry).

From the fact that the normal bundle is globally flat[1], for p, $q \in M$ we have a well-defined parallel transport map $\tau_{p.q} : \nu_p M \to \nu_q M$. Let $\tilde\tau_{p,q} : p + \nu_p M \to q + \nu_q M$ the associated affine parallel transport map. The parallelism of the curvature normals in fact implies that

$$\tilde\tau_{p,q}(F_M(p)) = F_M(q). \tag{7.6.10}$$

[1]!

7.6.11 Lemma *For all $p \in M$ and $i = 1, \ldots, g$, it holds that*

$$\tilde{r}_i^p(p) = a_i(p) \quad \textit{and} \quad \tilde{r}_i^p = \tilde{\tau}_{p,a_i(p)}.$$

Proof. The first assertion follows from the fact that $v_i(p)$ points in the radial direction of $S_i(p)$. To prove the second one, first consider the parallel transport from p to $a_i(p)$ in the normal bundle to $S_i(p)$ in $p + \mathbf{R}\, v_i(p) \oplus E_i(p)$; this map clearly takes $v_i(p)$ to $v_i(a_i(p)) = -v_i(p)$. Since $p+\mathbf{R}\, v_i(p) \oplus E_i(p)$ is totally geodesic in $\bar{M}$ and $S_i(p)$ is invariant under the Weingarten operators of M, it follows from the Weingarten formula that the above map is the restriction of $\tilde{\tau}_{p,a_i(p)}$. Finally, if ξ is a normal vector field to M along a curve γ in $S_i(p)$, which is parallel and everywhere normal to $v_i(p)$, then, due to (7.6.1),

$$\frac{\bar{\nabla}\xi}{dt} = -A_{\xi(t)}\gamma'(t) + \frac{\bar{\nabla}^{\perp}\xi}{dt} = -\langle \xi(t), v_i(\gamma(t)) \rangle\, \gamma'(t) = 0,$$

namely, ξ is constant in $\bar{M}$. This shows that $\tilde{\tau}_{p,a_i(p)}$ is the identity on $v_i(p)^{\perp}$, and finishes the proof. □

In view of Corollary 7.6.9, equation (7.6.10), $a_i(p) = \pi_{\eta_i}(p)$, and $M = M_{\eta_i}$, we now have that

$$\begin{aligned} \tilde{\tau}_{p,a_i(p)}(F_M(p)) &= F_M(a_i(p)) \\ &= F_{M_{\eta_i}}(\pi_{\eta_i}(p)) \\ &= F_M(p) \end{aligned}$$

for all $p \in M$. Due to Lemma 7.6.11, this says that $\tilde{r}_i^p$ acts on $p + \nu_p M$ by permuting the focal hyperplanes $H_i(p)$. Since there are only finitely many focal hyperplanes, this implies that the group W^p generated by all the $\tilde{r}_i^p$ is finite. Owing to the above quoted result of Coxeter, we deduce that W^p is a Coxeter group, called the *Coxeter group of M at p*. Note that the dependence on the point $p \in M$ is not very important, since $\tilde{\tau}_{p,q}$ conjugates W^p to W^q, so the conjugation class is uniquely defined and denoted simply by W. It is also usual to see W as a Coxeter group acting on the linear space $\nu_p M$. Note that the rank of W as a Coxeter group is the same as the rank of M as an isoparametric submanifold.

Decomposition theorems

Let M be a connected complete isoparametric submanifold of Euclidean space. The following remark is very important for the results in this subsection. Since the Coxeter group W associated to M is a finite group of orthogonal transformations of $p+\nu_p M$, it must have a fixed point (namely, the center of mass of any orbit). This means that there is a non-zero vector in $\bigcap_i H_i(p)$, so a non-zero parallel normal vector field ζ such that $\langle \zeta, v_i \rangle = 0$ for all non-zero curvature normals v_i.

7.6.12 Proposition *A connected complete isoparametric submanifold M of $\bar{M} = \mathbf{R}^n$ admits a splitting $M = N \times E_0$ such that $N = M \cap E_0^{\perp}$ is an isoparametric submanifold (of the same rank as M) of a sphere of dimension $n - \dim E_0$, and E_0 is the nullity distribution of M. Moreover, M and N have the same Weyl group.*

Proof. We denote the zero curvature normal by v_0, if it is present. Let ζ be a parallel normal vector field such that $\langle \zeta, v_i \rangle = 1$ for all $i \neq 0$, as above. The differential of the parallel map π_ζ has kernel equal to $\mathcal{D} = \bigoplus_{i \neq 0} E_i$. By Lemma 7.6.5, this distribution is auto-parallel. Since

$TM = \mathcal{D} \oplus E_0$ is an orthogonal decomposition, we have $\nabla_X U \in \Gamma(E_0)$ for $X \in \Gamma(\mathcal{D})$ and $U \in \Gamma(E_0)$. As a curvature distribution, E_0 is auto-parallel, so the Gauss formula yields

$$\bar{\nabla}_X U = \nabla_X U \in \Gamma(E_0)$$

for $X \in \Gamma(TM)$ and $U \in \Gamma(E_0)$, which implies that the distribution E_0 is constant along M as a subspace of Euclidean space; since M is complete, the leaf of the distribution through $p \in M$ is the affine subspace $p + E_0$.

Since M is ruled by affine subspaces parallel to the constant subspace E_0, it immediately follows that $M = N \times E_0$ where $N = M \cap E_0^\perp$ and N is connected. It remains to be seen that N is isoparametric in $E_0^\perp$. Note that $TN = \mathcal{D}|_N$ and $\nu N = \nu M|_N$. Since $E_0^\perp$ is totally geodesic in $\bar{M}$, we see from the Weingarten formula that the normal connection of M in $\bar{M}$ restricts to the normal connection of N in $E_0^\perp$, and the Weingarten operators of M (leave N invariant and) restrict to the Weingarten operators of N. We deduce that N is isoparametric in $E_0^\perp$. Since $d\pi_\eta(TN) = 0$ and N is connected, the map π_ζ is a constant c, which gives that N is contained in the sphere of center c and radius $||\zeta||$ in $E_0^\perp$.

The last assertion is true because the nullity distribution does not contribute to the Weyl group. □

7.6.13 Corollary *For a connected complete isoparametric submanifold M of $\mathbf{R}^n$, the following are equivalent:*

a. All curvature normals are non-zero.
b. M is contained in a round sphere of $\mathbf{R}^n$.
c. M is compact.

Proof. In the notation of the proposition: if all curvature normals are non-zero, then $M = N$ is contained in a sphere; complete isoparametric submanifolds of Euclidean space are always closed, so they are compact if contained in a sphere; by the proposition, M can be compact only if E_0 is trivial. □

Let $\bar{M}_1$ and $\bar{M}_2$ be Riemannian manifolds, and let M_i be a submanifold of $\bar{M}_i$ for $i = 1, 2$. The *extrinsic product* of M_1 and M_2 is the product $M_1 \times M_2$ viewed as a submanifold of the Riemannian product $\bar{M}_1 \times \bar{M}_2$.

An isoparametric submanifold M of Euclidean space $\mathbf{R}^n$ is said to be *reducible* if M is the extrinsic product of isoparametric submanifolds $M_i \subset \mathbf{R}^{n_i}$ for $i = 1, 2$ ($n = n_1 + n_2$), where M_1, M_2 are not points; note that in this case the Coxeter group of M is the product of the Coxeter groups of M_1 and M_2. Otherwise, we say that M is *irreducible.*

Let W denote a Coxeter group faithfully represented as a group generated by reflections acting on an Euclidean space V. The group W is called *reducible* if there exists a non-trivial decomposition $V = V_1 \oplus V_2$ into W-invariant subspaces. Note that in this case W is isomorphic to a product $W_1 \times W_2$ where W_i is a Coxeter group acting on V_i, for $i = 1, 2$.

7.6.14 Proposition *Let M be a compact isoparametric submanifold of $\mathbf{R}^n$ with Coxeter group W. Then M is reducible if and only if W is reducible.*

Proof. Assume W is reducible, namely, $W = W_1 \times W_2$ where W_i acts on $\mathbf{R}^{n_i}$ and $n = n_1 + n_2$. We want to prove that M is reducible. By applying a translation, we may assume that M passes through the origin of $\mathbf{R}^n$. Owing to Corollary 7.6.13, we know that all curvature normals of M are non-zero.

The set of generators of W splits as a union of the set of generators of W_1 and those of W_2; there is a corresponding splitting of the set of curvature normals into two sets V_1 and V_2. Note that these two sets V_1 and V_2 of vectors are mutually orthogonal. Let ζ be a parallel normal vector field to M in $\mathbf{R}^n$ such that $\langle \zeta, v_i \rangle = 1$ for every curvature normal v_i. Decompose $\zeta = \zeta_1 + \zeta_2$ where ζ_i lies in the span of V_i. Then, for each $i = 1, 2$, the number $\langle \zeta_i, v_j \rangle$ equals 1 or 0 according to whether v_j lies in V_i or not.

Define the distributions $\mathcal{D}_i = \ker(\mathrm{id} - A_{\zeta_i})$ and $\nu_i M = \sum_{v_j \in \mathsf{V}_i} \mathbf{R} v_j$, and put $V_i = \mathcal{D}_i \oplus \nu_i M$ for $i = 1, 2$. Note that there is a g- and B-orthogonal decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, where each $\mathcal{D}_i$ is parallel in M and invariant under its Weingarten operators. We claim that V_1 is constant as a subspace of $\mathbf{R}^n$ along M. Let $X \in \Gamma(TM)$, $Y \in \Gamma(\mathcal{D}_1)$, $\xi \in \Gamma(\nu_1 M)$. We easily compute that

$$\bar{\nabla}_X (Y + \xi) = \nabla_X Y + B(X, Y) - A_\xi X + \nabla^\perp_X \xi$$

lies in $\Gamma(V_1)$, proving the claim. Similarly, V_2 is constant as a subspace of $\mathbf{R}^n$ along M. Let $\mathbf{R}^{n_i}$ be the linear subspace of $\mathbf{R}^n$ given by V_i at the origin $0 \in M$ for $i = 1, 2$. Then $\mathbf{R}^n = \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2} = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, and we put $M_i = M \cap \mathbf{R}^{n_i}$ for $i = 1, 2$; note that M_i is the integral manifold of $\mathcal{D}_i$ through the origin. Similarly to the above computation, we easily see that $\mathcal{D}_1$ is constant as a subspace of $\mathbf{R}^n$ along $M \cap (\{p_1\} \times \mathbf{R}^{n_2})$ for $p_1 \in M_1$, and $\mathcal{D}_2$ is constant as a subspace of $\mathbf{R}^n$ along $M \cap (\mathbf{R}^{n_1} \times \{p_2\})$ for $p_2 \in M_2$. It follows that the integral manifolds of $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) are all of the form $M_1 + p_2 = M \cap V_1$ (resp. $p_1 + M_2 = M \cap V_2$), which gives that $M = M_1 \times M_2$.

As in the proof of Proposition 7.6.12, one sees that M_i is (compact) isoparametric in $\mathbf{R}^{n_i}$. The submanifolds M_1 and M_2 are integral manifolds of the auto-parallel distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, therefore they are totally geodesic submanifolds of M. Note that the normal bundle of M_i in V_i is the restriction of $\nu_i M$ to M_i, so the Coxeter group of M_i is indeed W_i. □

We gather from Propositions 7.6.12 and 7.6.14 that every complete isoparametric submanifold of Euclidean space splits an extrinsic product of its Euclidean factor and a number of irreducible compact isoparametric submanifolds (with irreducible Coxeter groups).

7.7 Examples and classification of isoparametric submanifolds

Isoparametric hypersurfaces

From Proposition 7.6.12 we recover Levi-Cività [LC37] and B. Segre's[Seg38] result that the number g of principal curvatures of an isoparametric hypersurface of Euclidean space is at most two, and it is either a hyperplane, a hypersphere or the boundary of a tube of constant radius around an affine subspace. Cartan extended Segre's bound on g to hyperbolic spaces and obtained a similar classification. In fact, Cartan studied isoparametric hypersurfaces systematically in a remarkable series of four papers [Car38, Car39a, Car39b, Car40] during the years 1938-40, and pointed out that isoparametric hypersurfaces in spheres are much more interesting and difficult objects of study.

From Proposition 7.6.14 we see that an isoparametric hypersurface in $S^n(1)$ with $g = 2$ must be a product $S^{m_1}(r_1) \times S^{m_2}(r_2)$, where $r_1^2 + r_2^2 = 1$; the family $\{S^{m_1}(\cos t) \times S^{m_2}(\sin t)\}_{t \in [0, \pi/2]}$ comprises an isoparametric family in the unit sphere S^n, where $n = m_1 + m_2 + 1$. The principal curvatures are easily seen to be

$$\lambda_1 = \cot t, \ \lambda_2 = -\tan t = \cot(t + \frac{\pi}{2}),$$

with (arbitrary) multiplicities m_1 and m_2. In particular, the focal hypersurfaces are points, corresponding $t = 0$ and $t = \pi/2$, and $t = \pi/4$ is the only parameter value corresponding to a minimal hypersurface; this hypersurface was found by W. K. Clifford in 1873 and is today known as the *Clifford torus*. Note that this foliation is the orbital foliation obtained from the standard action of $\mathbf{SO}(m_1+1)\times\mathbf{SO}(m_2+1)$ on $\mathbf{R}^{m_1+1}\times\mathbf{R}^{m_2+1}$.

Cartan constructed four examples of isoparametric foliations in spheres with $g=3$, all with uniform multiplicity equal to 1, 2, 4 or 8, and then proved that there are no other examples with $g=3$. Those hypersurfaces are all homogeneous. The simplest example lives in the unit sphere in the 5-dimensional Euclidean vector space V of traceless real symmetric 3×3 matrices equipped with the inner product $\langle X,Y\rangle=\mathrm{tr}(XY)$, which we describe as follows. There is an action of the Lie group $G=\mathbf{SO}(3)$ on V given by conjugation, namely, $g\cdot X=gXg^{-1}$ for $g\in G$ and $X\in V$. The isoparametric foliation of the unit sphere S^4 of V consists of conjugation classes of matrices of norm 1. The conjugation class of $X\in S^4$ is a compact submanifold; indeed it is the image of the immersion

$$g\in\mathbf{SO}(3)\mapsto gXg^{-1}\in S^4$$

which becomes injective after factoring $\mathbf{SO}(3)$ by the centralizer $Z_G(X)$ of X, which is a closed subgroup. Each symmetric matrix is conjugate to a diagonal matrix, so we can parametrize such classes by diagonal matrices. The centralizer of a diagonal (resp. arbitrary) matrix is discrete if and only if the matrix has pairwise distinct entries (resp. eigenvalues). Consider the following orthonormal basis of V:

$$e_1=\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},\quad e_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$e_3=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad e_4=\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad e_5=\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Then the diagonal matrices in S^4 can be parametrized by the geodesic

$$\gamma(t)=\cos te_1+\sin te_2.$$

The matrix $\gamma(t)$ has distinct eigenvalues if and only if $t\neq k\pi/3$ for $k\in\mathbf{Z}$; for such a value of t, the conjugation class (orbit) M_t is 3-dimensional, and the tangent space $T_{\gamma(t)}M_t$ is spanned by e_3, e_4, e_5. In fact, denote by E_{ij} the matrix with coefficient 1 at position (i,j) and 0 elsewhere, and put $X_{ij}=E_{ij}-E_{ji}$. The one-parameter subgroup

$$g_s=\begin{pmatrix}\cos s&\sin s&0\\-\sin s&\cos s&0\\0&0&1\end{pmatrix}$$

of $\mathbf{SO}(3)$ yields the following tangent vector at $p=\gamma(t)$:

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}g_spg_s^{-1}&=X_{12}p-pX_{12}\\&=[X_{12},\gamma(t)]\\&=\frac{1}{\sqrt{6}}\cos t[X_{12},E_{11}+E_{22}-2E_{33}]+\frac{1}{\sqrt{2}}\sin t[X_{12},E_{11}-E_{22}]\\&=(-2\sin t)e_3;\end{aligned}$$

we call this vector $X_{12}p$. Similarly,

$$X_{23}p=(-2\sin t)e_4,\ X_{13}p=(-2\sin t)e_5.$$

Now

$$\xi = -\gamma'(t) = \sin t e_1 - \cos t e_2$$

is a unit normal vector to M_t in S^4. We extend ξ to a normal vector field along $s \mapsto g_s p$ by putting $\xi(s) = dg_s(\xi) = g_s\xi$ and then

$$\begin{aligned} A_\xi(e_3) &= \frac{-1}{2\sin t} A_\xi(X_{12}p) \\ &= \frac{1}{2\sin t}(\bar{\nabla}_{X_{12}p}\xi)^\perp \\ &= \frac{1}{2\sin t}\left(\frac{d}{ds}\Big|_{s=0} g_s \xi g_s^{-1}\right)^\top \\ &= \frac{1}{2\sin t}\left(+\sin t[X_{12}, e_1] - \cos t[X_{12}, e_2]\right) \\ &= \cot t\, e_3. \end{aligned}$$

Similarly,

$$A_\xi(e_4) = \cot(t + \frac{\pi}{3})e_4 \quad \text{and} \quad A_\xi(e_5) = \cot(t + \frac{2\pi}{3})e_5.$$

Therefore the principal curvatures are

$$\lambda_1 = \cot t,\ \lambda_2 = \cot(t + \frac{\pi}{3}),\ \lambda_3 = \cot(t + \frac{2\pi}{3}),$$

with corresponding curvature distributions spanned by e_3, e_4 and e_5, respectively. Note that $M_{\pi/6}$ is a minimal hypersurface of S^4, called the *Cartan hypersurface*. Any conjugation class meets $\gamma(t)$ for some $t \in [0, \pi/3]$, since we can always permute the eigenvalues of a diagonal matrix by conjugating it by a suitable orthogonal matrix (called a *permutation matrix*!). The interior points $\gamma(t)$ for $t \in (0, \pi/3)$ have pairwise distinct eigenvalues and hence discrete centralizers, namely, the group of diagonal matrices with ± 1 entries. The endpoints $\gamma(0)$ and $\gamma(\pi/3)$ are matrices with an eigenvalue of multiplicity two, so its centralizers are larger, namely, the block subgroups $\mathbf{S}(\mathbf{O}(2)\mathbf{O}(1))$ and $\mathbf{S}(\mathbf{O}(1)\mathbf{O}(2))$ of $\mathbf{SO}(3)$, respectively. The focal manifolds $M_+ = M_0$ and $M_- = M_{\pi/3}$ are antipodal Veronese surfaces diffeomorphic to $\mathbf{R}P^2$. In particular, the multiplicities of the isoparametric family $\{M_t\}_{t\in[0,\pi/3]}$ are $m_1 = m_2 = 1$.

There is a beautiful, unified way to generalize the above example to include all examples with $g = 3$ discovered by Cartan. The standard embeddings of the projective spaces $\mathbf{F}P^n$, where $\mathbf{F}$ is one of the four normed division algebras over $\mathbf{R}$, namely, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (quaternions) and $\mathbf{Ca}$ (Cayley numbers; here n must be 2), are constructed as follows. Let V be the space $\mathrm{Herm}_\epsilon(n, \mathbf{F})$ be the space of $n \times n$ Hermitian matrices with coefficients in $\mathbf{F}$ and constant trace ϵ; ϵ is usually taken to be equal to 0 or 1. A one-dimensional subspace of $\mathbf{F}^{n+1}$ is identified with the orthogonal projection onto it, namely, an idempotent element in V; this realizes $\mathbf{F}P^n$ as the real algebraic smooth variety $M_+ = \{\, x \in V \mid x^2 = x \,\}$. Note that $\dim \mathbf{F}P^n = dn$ and $\dim V = (n-1)(dn+2)/2$, where $d = 1$, 2, 4 or 8, according to $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{Ca}$. The squared Euclidean norm in V is $||x||^2 = \mathrm{trace}(x^2)$, so M_+ is contained in the unit sphere $S(V)$ of V. It can be shown that the tubes of constant radius $r \in [0, \pi/3]$ around M_+ in $S(V)$ comprise an isoparametric foliation of $S(V)$, where the tube with $r = \pi/3$ corresponds to the antipodal embedding $M_- = -M^+$ of $\mathbf{F}P^n$, and $M_\pm$ are the focal manifolds. These foliations are respectively homogeneous under the compact Lie groups $G = \mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ and $\mathbf{F}_4$. The representations of the group G on V are given in the first three cases by $\rho(g)x = gxg^*$, where $x \in V$, $g \in G$ and x^* denotes the transpose conjugate matrix of x, and in the fourth case by the 26-dimensional representation of $\mathbf{F}_4$.

Later Cartan discusses the case $g = 4$ and shows there are only two examples where the multiplicities of principal curvatures are all equal, namely, one in S^5 and one in S^9. Towards the end of his third paper on the subject, Cartan asks three questions, one of which asking whether every isoparametric hypersurface in a sphere is homogeneous. It is clear that any orbit of codimension one in S^n of a closed subgroup of $\mathbf{SO}(n+1)$ has constant principal curvatures and is thus isoparametric. Hsiang and Lawson [HL71] classified connected closed subgroups of $\mathbf{SO}(n+1)$ whose principal orbits have codimension one in S^n. It turns out that the actions of such groups which are maximal connected, in the sense that they are not contained in a larger connected group with the same orbits, precisely coincide with the isotropy representations of symmetric spaces of rank two. Takagi and Takahashi [TT72] refer to [HL71] and note that it implies a classification of homogeneous isoparametric hypersurfaces in spheres. They relate the geometric invariants to the invariants of the corresponding symmetric spaces and list their multiplicities. In particular, they find examples with $g = 4$ and different multiplicities, for instance the orbits of the isotropy representation of the oriented Grassmann manifold of two-planes in $\mathbf{R}^{n+3}$ is an isoparametric submanifold of S^{2n+1} with $g = 4$ and multiplicities $m_1 = 1$, $m_2 = n - 1$.

After Cartan, the subject of isoparametric hypersurfaces in spheres remained dormant until the work of Takagi and Takahashi, and the short note of Nomizu [Nom73], in which he proved that the focal manifolds of an isoparametric family are always minimal submanifolds. Around the same time, Münzner did very influential work, published in the two papers [Mue80, Mue81] much later in 1981-2. In the first paper, he proved that there are exactly two focal manifolds. In the second paper, using delicate topological arguments based on the fact that a compact isoparametric submanifold of a sphere decomposes the sphere into a union of two disk bundles over the focal manifolds, Münzner proved the striking result that the only possible values of g are 1, 2, 3, 4 and 6, namely, the same values obtained from the homogeneous examples.

In 1975, Ozeki and Takeuchi [OT75] surprised the community of researchers in the field by exhibiting examples of inhomogeneous isoparametric hyperusfaces in spheres. These examples were later systematized and generalized by Ferus, Karcher and Münzner [FKM81], who associated examples with $g = 4$ to representations of Clifford algebras, most of which are inhomogeneous.

The classification problem of isoparametric hypersurfaces in spheres starts with the determination of the possible multiplicities (m_1, m_2). Cartan had already solved the problem for $g \leq 3$. In case $g = 6$, Abresch [Abr83] proved that only $(1, 1)$ and $(2, 2)$ are possible; note that indeed there are homogeneous examples with those multiplicities. The case $g = 4$ was the most involved and, after the efforts of many mathematicians, it was finally completed by Stolz [Sto99] who, in a topological tour de force, proved that the possibilities are exactly those that appear either in the homogeneous examples or in the Clifford examples of Ferus, Karcher and Münzner.

Isoparametric hypersurfaces with $g = 6$ and $(m_1, m_2) = (1, 1)$ must indeed be homogeneous by the work of Dorfmeister and Neher [DN85]. Their proof depends on an intricate algebraic calculation, and it seems very difficult to extend their approach to the case $(m_1, m_2) = (2, 2)$. More recently, the work of Cecil, Chi and Jensen [CCJ07], Immervoll [Imm08] and Chi [Chi12] shows that isoparametric hypersurfaces with $g = 4$ must be either homogeneous or one of the known inhomogeneous examples, with the possible exception of $(m_1, m_2) = (7, 8)$.

There have been attempts to simplify Dorfmeister-Neher's result and to extend it to the case $(g, m_1, m_2) = (6, 2, 2)$ [Miy09, Miy13, Miy15, Sif16].

General structure of isoparametric hypersurfaces of spheres

Let M be a compact isoparametric hypersurface of S^{n+1}. For $p \in M$ and a unit normal vector $\xi \in \nu_p M$, consider the normal geodesic $\gamma(t) = \cos t\, p + \sin t\, \xi$ for $t \in [0, 2\pi]$. Then γ meets the parallel and focal manifolds orthogonally. Since the codimension of M in $\mathbf{R}^{n+2}$ is two, in this case the Coxeter group is a dihedral group $\mathbf{D}_g$ (with $2g$ elements) with $g \geq 3$ in case M is irreducible, or $\mathbf{Z}_2 \times \mathbf{Z}_2 = \mathbf{D}_2$ or $\mathbf{Z}_2 = \mathbf{D}_1$ otherwise. It follows that the multiplicities satisfy the periodicity condition $m_i = m_{i+2}$ (indices modulo g); in particular, M has uniform multiplicities if g is odd. It also implies that the focal distances are equidistributed along the image of γ, and hence the principal curvatures can be written (cf. Exercise 12)

$$\cot d,\ \cot\left(d + \frac{\pi}{g}\right), \ldots, \cot\left(d + (g-1)\frac{\pi}{g}\right).$$

Isoparametric submanifolds

Palais and Terng [PT87] extended Takagi and Takahashi's remark to state that the principal orbits of the isotropy representation of a symmetric space are compact isoparametric submanifolds, see chapter **??**. In the same paper, using the classification of Dadok [Dad85] they also characterized the compact homogeneous isoparametric submanifolds of Euclidean space as being exactly those orbits. There remained the inhomogeneous isoparametric submanifolds to be understood. In 1991, invoking the theory of Tits buildings, Thorbergsson proved the deep result that every compact connected full irreducible isoparametric submanifold of Euclidean space with codimension at least 3 is homogeneous, showing thus that the FKM-examples are the only inhomogeneous ones, always in codimension 2. Thorbergsson's theorem has been reproved by Olmos [Olm93] using canonical connections and normal holonomy, and by Heintze and Liu [HL99]; the latter proof in fact also applies to the infinite dimensional case, cf. add. notes.

Marked Coxeter graph

Let M be a connected compact full isoparametric submanifold of an Euclidean sphere. It follows from equation (7.6.2) that the focal hyperplanes in $p + \nu_p M$ together with the multiplicities $m_i = \dim E_i$ for $i = 1, \ldots, g$, determine the second fundamental form, as an abstract symmetric bilinear form, up to passing to a parallel submanifold. In turn, the focal hyperplanes are already determined by the Weyl group, up to scaling of the ambient metric. Thus the Weyl group together with the multiplicities essentially determine the second fundamental form; such data is usually encoded in the form of a Coxeter graph with multiplicities, as follows.

Let W be the Coxeter group of M acting on $p + \nu_p M$ for some $p \in M$. A connected component of the complement of the union of the focal hyperplanes in $p + \nu_p M$ is called a *Weyl chamber*. The *Coxeter graph* of W is constructed by fixing a Weyl chamber $\mathcal{C}$ and taking as vertices the *walls* of $\mathcal{C}$, i.e. hyperplanes bounding $\mathcal{C}$. Note that these correspond to the generators $r_1, \ldots, r_n$ of W. Associated to each wall is a curvature distribution and the corresponding multiplicity, which we write on top of the vertex; this is the marking. Since the multiplciities are preserved under the action of W, the marking already determines all multiplicities. The two vertices corresponding to generators r_i and r_j are linked by an edge if and only if the corresponding walls are not perpendicular, in which case we write the number m_{ij} on top of the edge (recall that $(r_i r_j)^{m_{ij}} = 1$ is a relation in W; for simplicity, in case $m_{ij} = 3$ one usually writes nothing and the number 3 remains implicit). It turns out that W is irreducible if and only if its Coxeter graph is connected; in this case $\mathcal{C}$ is a simplicial cone and its Coxeter graph has $n = \dim \nu_p$ vertices. The isomorphism type of the Coxeter graph is independent of the chosen Weyl chamber, as W acts simply transitively on the set of Weyl chambers, and determines W up to isomorphism. The Coxeter graph together with the marking is called the *marked Coxeter graph* of the isoparametric submanifold.

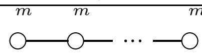
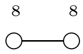
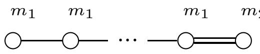
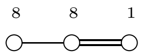
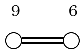
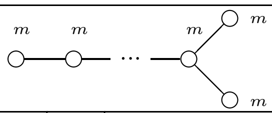
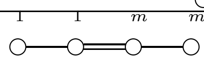
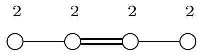
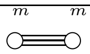
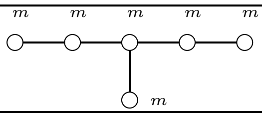
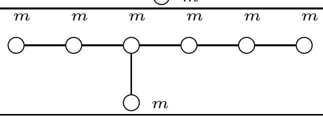
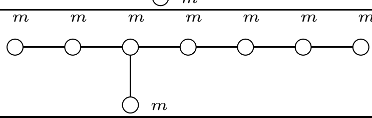
Type	*Diagram*	*Multiplicities*
$A_n,\ n \geq 2$		$m = 1, 2, 4$
A_2		–
$(BC)_n,\ n \geq 2$		m_1 m_2 1 k 2 $2,\ 2k+1$ 4 $1,\ 5,\ 4k+3$
$(BC)_3$		–
$(BC)_2$		–
$D_n,\ n \geq 4$		$m = 1, 2$
F_4		$m = 1, 2, 4, 8$
		–
G_2		$m = 1, 2$
E_6		$m = 1, 2$
E_7		$m = 1, 2$
E_8		$m = 1, 2$

Table 7.7.1: Coxeter graphs of homogeneous isoparametric submanifolds.

Recall that the connected compact full isoparametric hypersurfaces of Euclidean spaces are exactly the round hyperspheres of arbitrary radius, which have Coxeter graph of type A_1. Münzner's result quoted above says that the number g of principal curvatures of M is 1, 2, 3, 4 or 6. It follows that a rank 2 compact isoparametric submanifold of Euclidean space has Coxeter group of type A_1, $A_1 \times A_1$, A_2, B_2 or G_2. Due to Remark 7.6.6, for a compact isoparametric submanifold M of rank $n \geq 3$, any subgraph of the Coxeter graph of M which is obtained by removing some vertices of the graph of M and all edges linking to those vertices is the Coxeter graph of some slice of M. This fact shows that the admissible Coxeter graphs (groups) of isoparametric submanifolds if Euclidean space are A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4 and G_2. These are called *crystallographic Coxeter groups*, since they leave invariant a lattice. They are also called *Weyl groups*, since they appear in the theory of compact semisimple Lie algebras. In Table 7.7.1 we also list the possible multiplicities but only in the homogeneous case.

The following rigidity result shows that a homogeneous isoparametric submanifold M is completely characterized by the values of the second fundamental form B and its covariant derivative ∇B at one point $p \in M$. It is almost true that M is already determined by B, for the only exception are the adjoint orbits of the compact Lie groups $\mathbf{Spin}(2n+1)$ and $\mathbf{Sp}(n)$, whose marked

Coxeter graphs are isomorphic (with uniform multiplicity 2). For this reason, the theorem is more interesting in the infinite dimensional case, where it is also valid [GH12].

7.7.1 Theorem (Gorodski-Heintze) *Let M and M' be two connected complete full homogeneous isoparametric submanifolds of Euclidean spaces V and V', respectively. Assume there is an isometry $f : V \to V'$ and points $p \in M$, $p' \in M'$ such that $f(p) = p'$, $df_p(T_pM) = T_{p'}M'$, $df_p(B_p(u, v)) = B'_{p'}(df_p(u), df_p(v))$ and $df_p(\nabla_u B(v, w)) = \nabla_{df_p(u)} B'_{p'}(df_p(v), df_p(w))$, for all u, v, $w \in T_pM$, where B and B' denote the second fundamental forms of M and M', respectively. Then $f(M) = M'$.*

7.8 Additional notes

§1 In complex analysis of one variable, Liouville's theorem says that a bounded entire function is constant. Bernstein (1915-17) proved an analogous result in differential geometry, namely, if the graph of a function $f : \mathbf{R}^2 \to \mathbf{R}$ of class $\mathcal{C}^2$ is a minimal surface in $\mathbf{R}^3$, then the graph is a plane. He then posed the classical Bernstein problem, namely, whether the same result also holds for real functions of $n > 2$ variables. In terms of differential equations:

> **(Classical) Bernstein problem**: *Let the function $f : \mathbf{R}^n \to \mathbf{R}$ of class $\mathcal{C}^2$ be a solution of*
> $$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f/\partial x_i}{\sqrt{1 + ||\mathrm{grad} f||^2}} \right) = 0.$$
> *Must f be a linear function?*

Part of the importance of the Bernstein problem is that it has a direct bearing on the existence of minimal cones and singularities of minimal hypersurfaces in $\mathbf{R}^{n+1}$. The answer to the problem was proved to be affirmative in the cases $n = 3$ by de Giorgi (1965), $n = 4$ by Almgren (1966), and $n \leq 7$ by Simons (1968), and apparently there was some hope to extend the result to all dimensions. However, in 1969 Bombieri, de Giorgi and Giusti [EBG69] constructed a counter-example for $n = 8$, which yields a counter-example in each dimension $n > 8$ by a standard construction, closing the problem. The complete solution of the Bernstein problem turned out to involve a good deal of geometric measure theory and non-linear analysis.

§2 Let M be an isoparametric submanifold of $\bar{M} = \mathbf{R}^{n+k}$. Using the Coxeter group associated to M in an essential way, Terng proved in [Ter85] that M is the level set of a so called *isoparametric map* $F : \mathbf{R}^{n+k} \to \mathbf{R}^k$, namely, a map $F = (F_1, \ldots, F_k)$ admitting regular values and such that:

(i) the Laplacians ΔF_i are constant along the level sets of F, for $i = 1, \ldots, k$;

(ii) The inner products $\langle \mathrm{grad} F_i, \mathrm{grad} F_j \rangle$ are constant along the level sets of F, for all i, $j = 1, \ldots, k$;

(iii) The Lie brackets $[\mathrm{grad} F_i, \mathrm{grad} F_j]$ are linear combinations with constant coefficients of
$$\mathrm{grad} F_1, \ldots, \mathrm{grad} F_k$$
along the level sets of F, for all i, $j = 1, \ldots, k$.

(In case $k = 1$, conditions (i) and (ii) were classically referred to as expressing the constancy of the differential parameters ΔF_1 and $||\mathrm{grad} F_1||^2$ of F_1 along its level sets, hence the name *isoparametric*. Condition (iii) is a kind of integrability and is void in case $k = 1$.) Moreover, Terng showed that F can be taken polynomial. It follows that every connected isoparametric submanifold of Euclidean space is an open subset of a complete properly embedded isoparametric submanifold, which in addition is a real algebraic submanifold of Euclidean space. It is easy to check that, conversely, the regular level sets of an arbitrary isoparametric map are isoparametric submanifolds. The regular levels of F are exactly the parallel manifolds of M, and the singular levels are the focal manifolds of M. The resulting partition of $\bar{M}$ is called an *isoparametric foliation*, and it provides an important example of *singular Riemannian foliation* [Ale04].■2■.

§3 The theory of isoparametric submanifolds of Euclidean space was extended to separable Hilbert spaces by Terng in [Ter89]. The local differential geometry of submanifolds in Euclidean spaces generalizes without much effort to Hilbert space. One is thus tempted to use the same definition, namely, constancy of principal curvatures along parallel normal vector fields and flat normal bundle. This works if one restricts to the category of *proper Fredholm* submanifolds of Hilbert space, that is, those submanifolds of Hilbert space whose normal exponential map is a proper Fredholm map. In practice, this says that such submanifolds have finite codimension and compact (self-adjoint) Weingarten operators. Terng generalized the whole structure theory of isoparametric submanifolds to Hilbert space, including the Coxeter group, which is now an (infinite) affine Weyl group. The structure is now more involved also for the reason that the distribution of nullity does not have to split off. On one hand, there is a remarkable family of examples of isoparametric foliations of Hilbert space coming from isotropy representations of (infinite-dimensional) affine Kac-Moody symmetric spaces. On the other hand, examples of FKM-type can be also be constructed in Hilbert space (without resorting to polynomials!, though [TT95]). Thorbergsson's theorem was extended to Hilbert space by Heintze and Liu, who proved that a connected complete full irreducible isoparametric submanifold of Hilbert space of rank at least 2 is extrinsically homogeneous [HL99]; however, little is known about the group acting transitively on that submanifold. The classification problem, even in the homogeneous case, is wide open, for there is no standard theory of infinite-dimensional Lie groups and their affine representations that one can apply. A recent contribution is [GH12], which characterizes such manifolds by the values of the second fundamental form B and its covariant derivative ∇B at one point (cf. Theorem 7.7.1), and proposes a strategy to the classification, namely, first obtain restrictions on ∇B (those on B are already known) and then compare with the known examples.

7.9 Exercises

■3■

1 Let V be an inner product space. For a basis $(v_1, \ldots, v_n)$ of V, let A be the matrix of a linear transformation $T : V \to V$ in that basis. Consider also the matrices $B = (\langle Tv_i, v_j \rangle)$ and $G = (\langle v_i, v_j \rangle)$. Prove that $A^t = BG^{-1}$.

2 Let $f : M^2 \to \mathbf{R}^3$ be an isometric immersion of a surface, consider the frame of vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ along f and the corresponding coefficients g_{ij} of the induced Riemannian metric.

a. Show that the coefficients of the second fundamental form of f are given by

$$b_{ij} = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) \cdot \det(g_{ij})^{-1/2},$$

with respect to some choice of unit normal vector field ξ.

■2■Globally flat normal bundle

■3■Normal bundle; normal connection; normal component of equation.

b. Deduce that the Gaussian curvature

$$K = \det A_\xi = \frac{\det(b_{ij})}{\det(g_{ij})}$$

and the mean curvature

$$H = \operatorname{tr} A_\xi = \frac{g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11}}{\det(g_{ij})}.$$

3 *a*. Let $\gamma : (a,b) \to \mathbf{R}^3$, $\xi : (a,b) \to S^2(1)$ be smooth curves. A parametrized surface of the form $f(u,v) = \gamma(u) + v\xi(v)$ is called a *ruled surface*. Investigate sufficient conditions for f to be an immersion. Compute that

$$K = \frac{-(\gamma' \cdot \xi')^2}{||(\gamma' + v\xi') \times \xi||^2}.$$

Deduce that the plane, cilinder and cone are flat surfaces.

b. For the *helicoid*

$$f(u,v) = (v \cos u, v \sin u, au)$$

($a > 0$), show that

$$K(u,v) = \frac{-a^2}{(a^2 + v^2)^2} \qquad (7.9.1)$$

and that it is a minimal surface. Deduce its principal curvatures. It is not difficult to show that the plane and the helicoid are the only complete ruled minimal surfaces in $\mathbf{R}^3$.

4 *a*. Let $\gamma : (a,b) \to \mathbf{R}^2$ be a smooth curve. A parametrized surface of the form $f(u,v) = (\gamma_1(v)\cos u, \gamma_1(v)\sin u, \gamma_2(v))$, where γ_1, γ_2 are the components of γ, is called a *surface of revolution*. Show that

$$K = \frac{\gamma_2'(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')}{\gamma_1((\gamma_1')^2 + (\gamma_2')^2)^2)}.$$

In particular $K = -\gamma_1''/\gamma_1$ in case γ is parametrized by arc-length.

b. For the *torus of revolution*

$$f(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r \sin v)$$

($R > r > 0$), show that

$$K = \frac{\cos v}{r(R + r\cos v)}.$$

c. For the *catenoid*

$$f(u,v) = (av \cos u, av \sin u, \cosh(v/a))$$

($a > 0$), show that

$$K(u,v) = \frac{-1}{a^2 \cosh^4(v/a)} \qquad (7.9.2)$$

and that it is a minimal surface. It is not difficult to see that the only complete minimal surfaces of revolution in $\mathbf{R}^3$ are the plane and the catenoid. Interpret formulae (7.9.1) and (7.9.2) in view of exercise 2 of chapter 1.

5 Let M be a surface in $\mathbf{R}^3$ given as the pre-image of a regular value of a smooth map $f : U \to \mathbf{R}$, where U is an open subset of $\mathbf{R}^3$. Show that the second fundamental form of M is given by

$$B(u,v) = \frac{1}{||(\mathrm{grad}\, f)_p||}\mathrm{Hess}\,(f)(u,v)$$

for some choice of unit normal vector field, where $p \in M$ and u, $v \in T_pM$.

6 (The Beez-Killing theorem) *a.* Let S, $T : V \to V$ be self-adjoint linear operators on an Euclidean vector space V. Suppose that $\mathrm{rank}(S) \geq 3$ and $\Lambda^2 S = \Lambda^2 T : \Lambda^2 V \to \Lambda^2 V$. Prove that $S = \pm T$.

b. Let M be a (not necessarily complete) connected Riemannian manifold of dimension n and suppose $f : M \to \mathbf{R}^{n+1}$ is an isometric immersion such that the rank of the second fundamental form is at least 3 at every point. Prove that f is rigid.

7 Let $M \subset N \subset P$ be a chain of Riemannian submanifolds. Prove that if M is totally geodesic in N and N is totally geodesic in P, then M is totally geodesic in P.

8 Prove that each connected component of the fixed point set of an isometry of a Riemannian manifold is a properly embedded totally geodesic submanifold. Generalize the result to the fixed point set of a group of isometries.

9 Prove that the totally geodesic submanifolds of $\mathbf{R}P^n$ are the images of totally geodesic submanifolds of S^n under the projection $\pi : S^n \to \mathbf{R}P^n$. Deduce that the complete totally geodesic submanifolds of $\mathbf{R}P^n$ are isometric to $\mathbf{R}P^k$ for some $0 \leq k \leq n$; in particular, the cut-locus of a point in $\mathbf{R}P^n$ is a totally geodesic hypersurface isometric to $\mathbf{R}P^{n-1}$.

10 Consider the projection $\pi : S^{2n+1} \setminus \{0\} \to \mathbf{C}P^n$. Prove that there are exactly two kinds of complete totally geodesic submanifolds of $\mathbf{C}P^n$: (i) $\pi(V \cap S^{2n+1})$, where V is a complex subspace of $\mathbf{C}^{n+1}$; and (ii) $\pi(W \cap S^{2n+1})$, where W is a totally real subspace of $\mathbf{C}^{2n+1}$. Deduce that the complete totally geodesic submanifolds of $\mathbf{C}P^n$ are isometric to $\mathbf{C}P^k$ or to $\mathbf{R}P^k$ for some $0 \leq k \leq n$; in particular, the cut-locus of a point in $\mathbf{C}P^n$ is a totally geodesic submanifold isometric to $\mathbf{C}P^{n-1}$.

11 Let M^n be a Riemannian submanifold of $\mathbf{R}^{n+k}$. Fix a point $p \in M$ and a normal vector $\xi \in \nu_pM$. In this exercise we establish a canonical isomorphism $T_\xi(\nu M) \cong T_pM \oplus \nu_pM$.

a. Given $u \in T_pM$, consider a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$, $\gamma'(0) = u$ and take the parallel transport $\hat{\xi}$ of ξ along γ. Show that this defines a linear map $T_pM \to T_\xi(\nu M)$, and that this map is injective.

b. Given $\eta \in \nu_pM$, consider the line $s \mapsto \xi + s\eta$ in ν_pM. Show that it defines a linear map $\nu_pM \to T_\xi(\nu M)$, and that this map is injective.

c. Show that T_pM and ν_pM viewed as subspaces of $T_\xi(\nu M)$ meet only at 0. Deduce the above claim.

12 Let M^n be a Riemannian submanifold of $\bar{M} = \mathbf{R}^{n+k}$. Consider the normal exponential map $\exp^\perp : \nu M \to \mathbf{R}^{n+k}$ mapping $\xi \in \nu_pM$ to $p + \xi$.

a. Use exercise 11 to represent the differential $d(\exp^\perp)_\xi : T_pM \oplus \nu_pM \to T_pM \oplus \nu_pM$ as

$$\begin{pmatrix} \mathrm{id} - A_\xi & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

b. Assume ξ is a unit vector and prove that $q = p + t\xi$ is a focal point of multiplicity m of M along the normal line through ξ if and only if $1/t$ is an eigenvalue of A_ξ of multiplicity m. Deduce that d is a focal distance of M along ξ if and only if $1/d$ is a principal curvature of A_ξ.

c. Generalize the above to other space forms to prove that: in S^{n+k}, d is a focal distance of M along ξ if and only if $\cot d$ is a principal curvature of A_ξ; in $\mathbf{R}H^{n+k}$, d is a focal distance of M along ξ if and only if $\coth d$ is a principal curvature of A_ξ.

d. In case $\bar{M} = S^{n+k}$, note that d is a focal distance of M along ξ if and only $\pi - d$ is a focal distance of M along $-\xi$.

13 (The Morse index theorem for submanifolds of Euclidean space) Let M be a Riemannian submanifold of $\bar{M} = \mathbf{R}^n$. For $q \in \mathbf{R}^n$, consider the square distance function

$$L_q : M \to \mathbf{R}, \qquad L_q(x) = \frac{1}{2}||x - q||^2.$$

a. Prove that $\mathrm{grad}(L_q)_p = (p - q)^\top$. Deduce that $p \in M$ is a critical point of L_q if and only if $v = q - p \in \nu_p M$.

b. Let $p \in M$ be a critical point of L_q and $v = q - p \in \nu_p M$. Prove that $\mathrm{Hess}(L_q)_p = I - A_v$ (exercise 13 of chapter 4).

c. The *nullity* of L_q at a critical point p is defined to be the nullity of the symmetric bilinear form $\mathrm{Hess}(L_q)_p$; such a critical point p is called *non-degenerate* if the nullity of L_q at p is zero. Use Exercise 12 to deduce that the nullity of L_q at a critical point p equals the multiplicity of q as a focal point of M along the geodesic segment $\overline{pq}$. Deduce that p is non-degenerate as a critical point of L_q if and only if q is a non-focal point of M along the geodesic segment $\overline{pq}$.

d. The *index* $\mathrm{ind}(L_q)_p$ of L_q at a critical point p is defined to be the index of the symmetric bilinear form $\mathrm{Hess}(L_q)_p$. Show that $\mathrm{ind}(L_q)_p = \sum_{t\in(0,1)} \ker(I - t\,A_v)$, where $v = q - p$. Combine this result with part (c) to deduce that $\mathrm{ind}(L_q)_p$ equals the sum of the multiplicities of $p + tv$ as a focal point to M for $t \in (0, 1)$.

e. Check that this result is a specialization of the Morse index theorem 7.5.4 to the case of Euclidean submanifolds.

14 Let M be a submanifold of a Riemannian manifold $\bar{M}$. Prove that the kth-osculating space $\mathcal{O}^k_p(M)$ of M at a point $p \in M$ is spanned by the k-th derivatives at 0 of all smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. (Hint: Consider the reparametrizations $\gamma(\vartheta(t))$ where ϑ is a polynomial function with $\vartheta(0) = 0$.)

15 Let M be a complete isoparametric submanifold of Euclidean space $\bar{M} = \mathbf{R}^n$. Fix a parallel normal vector field ξ along M. Consider $\pi_\xi : M \to M_\xi$ and let $\hat{p} \in M_\xi$. Prove that the connected components of the level set $\pi^{-1}(\hat{p})$ are compact isoparametric submanifolds of $\nu_{\hat{p}}(M_\xi)$, with curvature normals given exactly by those curvature normals v_i of M that satisfy $\langle \xi, v_i \rangle = 1$.