Applications

6.1 Introduction

In this chapter, we collect a few basic and important theorems of Riemannian geometry that we prove by using the concepts introduced so far. We also introduce some other important techniques along the way.

We start by discussing manifolds of constant curvature. If one agrees that curvature is the main invariant of Riemannian geometry, then in some sense the spaces of constant curvature should be the simplest models of Riemannian manifolds. It is therefore very natural to try to understand those manifolds. Since curvature is a local invariant, one can only expect to get global results by further imposing other topological conditions.

Next we turn to the relation between curvature and topology. This a central and recurring theme for research in Riemannian geometry. One of its early pioneers was Heinz Hopf in the 1920's who asked to what extent the existence of a Riemannian metric with particular curvature properties restricts the topology of the underlying smooth manifold. Since then the subject has expanded so much that the scope of this book can only afford a glimpse at it.

It is worthwhile pointing out that not only the theorems in this chapter are part of a central core of results in Riemannian geometry, but also the arguments and techniques in the proofs can be applied in more general contexts to a wealth of other important problems in geometry.

6.2 Space forms

A complete Riemannian manifold with constant curvature is called a *space form*. If M is a space form, its universal Riemannian covering manifold $\tilde{M}$ is a simply-connected space form by Proposition 3.3.8. Moreover, M is isometric to $\tilde{M}/\Gamma$ with the quotient metric, where Γ is a free and proper discontinuous subgroup of isometries of $\tilde{M}$, see section 1.3. So the classification of space forms can be accomplished in two steps, as follows:

a. Classification of the simply-connected space forms.

b. For each simply-connected space form, classification of the subgroups of isometries acting freely and properly discontinuously.

In this section, we will prove the Killing-Hopf theorem that solves part (a) in this program. Despite a lot being known about part (b), it is yet an unsolved problem, and we include a brief discussion about it after the proof of the theorem.

We first prove a local result.

6.2.1 Theorem *Fix $k \in \mathbf{R}$. Then any two Riemannian manifolds of constant curvature k of the same dimension are locally isometric.*

Proof. Let M, $\tilde{M}$ be two Riemannian manifolds of constant curvature k. Fix points $p \in M$, $\tilde{p} \in \tilde{M}$ and choose a linear isometry $f : T_pM \to T_{\tilde{p}}\tilde{M}$. Choose open balls $U \subset T_pM$, $\tilde{U} \subset T_{\tilde{p}}\tilde{M}$ with $\tilde{U} = f(U)$ that determine normal neighborhoods $V = \exp_p(U)$, $\tilde{V} = \exp_{\tilde{p}}(\tilde{U})$. Now we have a diffeomorphism $F : V \to \tilde{V}$ given by

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & \tilde{U} \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{\tilde{p}}} \\ V & \xrightarrow[F]{} & \tilde{V} \end{array}$$

namely, $F \circ \exp_p = \exp_{\tilde{p}} \circ f$. Note that $F(p) = \tilde{p}$ and $dF_p = f$. We shall prove that F is an isometry.

We need to prove that $dF_q : T_qM \to T_{\tilde{q}}\tilde{M}$ is a linear isometry, where $q \in V$ is arbitrary and $\tilde{q} = F(q)$. Write $q = \gamma_v(t_0)$ where γ_v is the radial geodesic from p with initial unit velocity $v \in T_pM$ and $t_0 \in [0, \epsilon)$. We orthogonally decompose $T_qM = \mathbf{R}\gamma_v'(t_0) \oplus W$, where W is the orthogonal complement, and similarly $T_{\tilde{q}}\tilde{M} = \mathbf{R}\gamma_{\tilde{v}}'(t_0) \oplus \tilde{W}$, where $\tilde{v} = f(v)$.

Note $F \circ \gamma_v$ is the geodesic $\gamma_{\tilde{v}}$ in $\tilde{M}$, so by the chain rule

$$||dF_q(\gamma_v'(t_0))|| = ||\gamma_{\tilde{v}}'(t_0)|| = ||\tilde{v}|| = ||v|| = ||\gamma_v'(t_0)||.$$

Furthermore, by the Gauss lemma 5.5.1 (or 3.2.1), $d(\exp_p)_{t_0v} : T_pM \to T_qM$ sends the orthogonal decomposition $T_pM = \mathbf{R}v \oplus (\mathbf{R}v)^\perp$ to the orthogonal decomposition $T_qM = \mathbf{R}\gamma_v'(t_0) \oplus W$, and similarly for $d(\exp_{\tilde{p}})_{t_0\tilde{v}}$. It follows that dF_q sends the orthogonal decomposition $T_qM = \mathbf{R}\gamma_v'(t_0) \oplus W$ to $T_{\tilde{q}}\tilde{M} = \mathbf{R}\gamma_{\tilde{v}}'(t_0) \oplus \tilde{W}$. It remains only to check that dF_q restricts to an isometry $W \to \tilde{W}$.

It is here and only here that we use the assumption on the sectional curvatures. Let $u \in T_pM$ be orthogonal to v and let $\tilde{u} = f(u) \in T_{\tilde{p}}\tilde{M}$. Extend u, $\tilde{u}$ to parallel vector fields U, $\tilde{U}$ along γ_v, $\gamma_{\tilde{v}}$, respectively. On one hand, the Jacobi fields Y, $\tilde{Y}$ along γ_v, $\gamma_{\tilde{v}}$, resp., with initial conditions $Y(0) = \tilde{Y}(0) = 0$, $Y'(0) = u$, $\tilde{Y}'(0) = \tilde{u}$ are given by $Y(t) = d(\exp_p)_{tv}(tu)$, $\tilde{Y}(t) = d(\exp_{\tilde{p}})_{t\tilde{v}}(t\tilde{u})$, due to Scholium 5.4.5. On the other hand, the Jacobi equation along a geodesic in a space of constant curvature k is given by $Y'' + kY = 0$. It follows that

$$Y(t) = \begin{cases} \frac{\sin(\sqrt{k}t)}{\sqrt{k}}U(t), & \text{if } k > 0, \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}U(t), & \text{if } k < 0, \end{cases} \quad \text{and} \quad \tilde{Y}(t) = \begin{cases} \frac{\sin(\sqrt{k}t)}{\sqrt{k}}U(t), & \text{if } k > 0, \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}U(t), & \text{if } k < 0, \end{cases}$$

and

$$Y(t) = tU(t) \quad \text{and} \quad \tilde{Y}(t) = t\tilde{U}(t)$$

if $k = 0$. In any case

$$||\tilde{Y}(t)|| = ||Y(t)||.$$

Since $Y(t_0) \in W$ is an arbitrary vector and

$$\begin{aligned} dF_q(Y(t_0)) &= dF_q\left(d(\exp_p)_{t_0v}(t_0u)\right) \\ &= d(\exp_{\tilde{p}})_{t\tilde{v}}(t_0f(u)) \\ &= \tilde{Y}(t_0), \end{aligned}$$

it follows that $dF_q : W \to \tilde{W}$ is an isometry, and this finishes the proof. □

If (M, g) is a space form of curvature k, then, for a positive real number λ, $(M, \lambda g)$ is a space form of curvature $\lambda^{-1}k$, see Exercise 2 in chapter 4. Therefore, the metric g can be normalized so that k becomes equal to one of 0, 1, or -1.

6.2.2 Theorem (Killing-Hopf) *Let M be a simply-connected space form of curvature k and dimension bigger than one. Then M is isometric to:*

a. the Euclidean space $\mathbf{R}^n$, if $k = 0$;
b. the real hyperbolic space $\mathbf{R}H^n$, if $k = -1$;
c. the unit sphere S^n, if $k = 1$.

Proof. Let $\tilde{M}$ be $\mathbf{R}^n$, $\mathbf{R}H^n$ or S^n according to whether $k = 0$, -1 or 1. Fix $\tilde{p} \in \tilde{M}$, $p \in M$ and choose a linear isometry $f : T_{\tilde{p}}\tilde{M} \to T_pM$. As in the proof of Theorem 6.2.1, this data can be used to define an isometry $F : \tilde{V} \to V$ with $F(\tilde{p}) = p$ and $dF_{\tilde{p}} = f$, where V, $\tilde{V}$ are certain normal neighborhoods of p, $\tilde{p}$. We shall see that F can be extended to an isometry $\tilde{M} \to M$.

Consider first the case $k = 0$ or -1. Since the cut locus of a point in $\mathbf{R}^n$ or $\mathbf{R}H^n$ is empty, we can take $\tilde{V} = \tilde{M}$ as a normal neighborhood, and using the completeness of M, extend F to a map $\tilde{M} \to M$ by the same formula, namely, $F \circ \exp_{\tilde{p}} = \exp_p \circ f$. Note, however, that in principle F does not have to be a diffeomorphism, because $f(T_{\tilde{p}}\tilde{M}) = T_pM$ does not in principle exponentiate to a normal neighborhood of p. Nevertheless, the proof of Theorem 6.2.1 (using the global Gauss lemma 5.5.1) carries through to show that F is a local isometry. Since $\tilde{M}$ is complete, Proposition 3.3.8(b) can be applied to yield that F is a Riemannian covering map and hence, since M is assumed to be simply-connected, F must be an isometry.

Consider now $k = 1$. Here the above argument yields a local isometry $F : \tilde{V}_{\tilde{p}} \to M$, where $\tilde{V}_{\tilde{p}} = S^n \setminus \{-\tilde{p}\}$ is the maximal normal neighborhood of $\tilde{p}$. To finish, we choose another point $\tilde{q} \in S^n \setminus \{\tilde{p}, -\tilde{p}\}$ and construct a similar local isometry $G : \tilde{V}_{\tilde{q}} \to S^n$, with initial data $G(\tilde{q}) = F(\tilde{q})$ and $dG_{\tilde{q}} = dF_{\tilde{q}}$, where $V_{\tilde{q}} = S^n \setminus \{-\tilde{q}\}$. By exercise 15 of chapter 3, F and G can be pasted together to define a local isometry $S^n \to M$. The rest of the proof is as above, using the completeness of S^n and the simple-connectedness of M. □

Depending on the context in which one is interested, it is possible to find in the literature other proofs of Theorem 6.2.2 different from the above one. The argument that we chose to use, based on Jacobi fields, works in a more general context, and will be used to prove a generalization of this theorem in chapter ??? of part 2.

Next, we discuss the case of non-simply-connected space forms. In the flat case, the main result is the following theorem.

6.2.3 Theorem (Bieberbach) *A compact flat manifold M is finitely covered by a torus.*

Namely, Bieberbach showed that the fundamental group $\pi_1(M)$ contains a free Abelian normal subgroup Γ of rank $n = \dim M$ and finite index, so there is a finite covering

$$\pi_1(M)/\Gamma \to \mathbf{R}^n/\Gamma \to \mathbf{R}^n/\pi_1(M) = M.$$

(For an example, review the contents of exercise 10 of chapter 1.) The complete classification of compact flat Riemannian manifolds is known only in the cases $n = 2$, 3; see [Wol84, Cha86] for proofs of Bieberbach's theorem and these classifications.

Next we consider non-simply-connected space forms of positive curvature. In even dimensions, the only examples are the real projective spaces, as the following result shows.

6.2.4 Theorem *An even-dimensional space form of positive curvature is isometric either to S^{2n} or to $\mathbf{R}P^{2n}$.*

Proof. We know that $M = S^{2n}/\Gamma$, where Γ is a subgroup of $\mathbf{O}(2n + 1)$ acting freely and properly discontinuously on S^{2n}. Since this action is free, if an element of Γ admits a $+1$-eigenvalue then it must be the identity id. Recall that the eigenvalues of an orthogonal transformation are unimodular complex numbers, and the non-real ones must occur in complex conjugate pairs.

Next, let $\gamma \in \Gamma$. Then $\gamma^2 \in \mathbf{SO}(2n+1)$, and since $2n+1$ is odd, γ^2 admits an eigenvalue $+1$, thus $\gamma^2 = \mathrm{id}$. This implies that all the eigenvalues of γ are ± 1. If $\gamma \neq \mathrm{id}$, it follows that all the eigenvalues of γ are -1, namely, $\gamma = -\mathrm{id}$. Hence $\Gamma = \{\mathrm{id}\}$ or $\Gamma = \{\pm\mathrm{id}\}$. □

The odd-dimensional space forms of positive curvature have been completely classified by J. Wolf [Wol84]. Here we just present a very rich family of examples.

6.2.5 Example (Lens spaces) Let p, q be relatively prime integers. The *lens space* $L_{p;q}$ is the quotient Riemannian manifold S^3/Γ, where we view

$$S^3 = \{\, (z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \,\},$$

and Γ is the cyclic group of order p generated by the element

$$t_{p;q}(z_1, z_2) = (\omega z_1, \omega^q z_2),$$

where ω is a pth root of unity. Note that $L_{2;1} = \mathbf{R}P^3$. More generally, let $q_2, \ldots, q_n$ be integers relatively prime to an integer p. The *lens space* $L_{p;q_2,\ldots,q_n}$ is the quotient Riemannian manifold S^{2n-1}/Γ, where we view

$$S^{2n-1} = \{\, (z_1, \ldots, z_n) \in \mathbf{C}^2 \mid |z_1|^2 + \cdots + |z_n|^2 = 1 \,\},$$

and Γ is the cyclic group of order p generated by the element

$$t_{p;q_2,\ldots,q_n}(z_1, z_2, \ldots, z_n) = (\omega z_1, \omega^{q_2} z_2, \ldots, \omega^{q_n} z_n).$$

Of course, a lens space is a non-simply-connected space form of positive curvature. The 3-dimensional lens spaces were introduced by Tietze in 1908. In general, lens spaces are important in topology because they provide examples of non-homeomorphic compact manifolds which are homotopy-equivalent (see [Mun84, §40, §69]). Historywise they can thus be seen as representing the birth of geometric topology of manifolds as distinct from algebraic topology. ⋆

A space form of negative curvature is called a *hyperbolic manifold*. Of course, a hyperbolic manifold is isometric to the quotient of $\mathbf{R}H^n$ by a group of isometries Γ acting freely and proper discontinously. A compact orientable surface of genus $g \geq 2$ admits many hyperbolic metrics, which are constructed as follows. It is a theorem of Radó [Rad24] that any compact surface is homeomorphic to the identification space of a polygon whose sides are identified in pairs. In particular, a compact orientable surface S_g of genus g is realized as a regular $4g$-sided polygon P with a certain identification of the sides. The vertices of P are all identified to one point, so in order to get a smooth surface it is necessary that the sum of the inner angles of P be 2π. Note that P cannot be taken to be an Euclidean polygon, for in that case the sum of the inner angles is known to be $(4g-2)\pi > 2\pi$ for $g \geq 2$. Instead, we construct P as a regular polygon in the disk model $\mathbf{D}^2$ of $\mathbf{R}H^2$ having the center at $(0,0)$ and with the sides being geodesic segments. In this case, by the Gauss-Bonnet theorem the sum of the inner angles is $(4g-2)\pi - A$, where A denotes the area of P. It is clear that there exist such polygons in $\mathbf{D}^2$ with arbitrary diameter, and that A varies continuously with the diameter, between zero (when the diameter is near zero) and $(4g-2)\pi$ (when the angles are near zero). Since $(4g-2)\pi > 2\pi$, it follows from the intermediate value theorem that it is possible to construct P such that the sum of the inner angles is 2π. Next one sees that the identifications between pairs of sides can be realized by isometries of $\mathbf{D}^2$ such that these isometries generate a discrete subgroup Γ of the isometry group of $\mathbf{D}^2$ acting freely and properly discontinuously. This shows that $S_g = \mathbf{D}^2/\Gamma$ admits a hyperbolic metric. Further, it is known that the hyperbolic metric on S_g for $g \geq 2$ is not unique. It is a classical result that there exist natural bijections between the following sets of structures on a compact oriented surface S_g: conformal classes of Riemannian metrics; complex structures compatible with the orientation; hyperbolic metrics (see e.g. [Jos06]). The *moduli space* $\mathcal{M}_g$ of S_g is the space of equivalence classes of hyperbolic metrics on S_g, where two hyperbolic metrics belong to the same class if and only if they differ by a diffeomorphism of S_g. It turns out that $\mathcal{M}_g$ is not a manifold: singularities develop exactly at the hyperbolic metrics admitting nontrivial isometry groups. For this reason, Teichmüller introduced a weaker equivalence relation on the space of hyperbolic metrics on S_g by requiring two of them to be equivalent if they differ by a diffeomorphism which is homotopic to the identity; the *Teichmüller space* $\mathcal{T}_g$ of S_g is the resulting space of equivalence classes. It is known that $\mathcal{T}_g$ admits the structure of a smooth manifold of dimension $6g-6$ if $g \geq 2$ [EE69].

In the higher dimensional case, it is much more difficult to construct hyperbolic metrics, and most of the progress in this direction has been made in the 3-dimensional case, see [Thu97].

6.3 Synge's theorem

We will use the following lemma in the proofs of Synge's and Preissmann's theorems. It is easy to see that the compactness assumption in it is essential.

6.3.1 Lemma (Cartan) *Let M be a compact Riemannian manifold. Assume that M is not simply-connected. Then every nontrivial free homotopy class $\mathcal{C}$ of loops contains a closed geodesic of minimal length in $\mathcal{C}$.*

Proof. We first claim that since M is compact, it is possible to find $\epsilon > 0$ such that any two points of M within distance less than ϵ can be joined by a unique minimizing geodesic, and this geodesic depends smoothly on its endpoints. Indeed, cover M by finitely many balls $B(p_i, \epsilon_i/2)$ where $p_i \in M$, $\epsilon_i > 0$, and $B(p_i, \epsilon_i)$ is a δ_i-totally normal ball for some $\delta_i > 0$ as in Proposition 2.4.7, for $i = 1, \ldots, k$. Take $\epsilon = \min_i\{\frac{1}{2}\epsilon_i, \delta_i\}$. If $d(x, y) < \epsilon$ for points x, $y \in M$, then $x \in B(p_{i_0}, \epsilon_{i_0}/2)$ for some i_0, and then

$$d(y, p_{i_0}) \leq d(y, x) + d(x, p_{i_0}) < \epsilon + \frac{\epsilon_{i_0}}{2} \leq \epsilon_{i_0}.$$

Hence x, $y \in B(p_{i_0}, \epsilon_{i_0})$ with $d(x, y) < \delta_{i_0}$, so the claim follows from the quoted proposition.

Let ℓ be the infimum of the lengths of the piecewise smooth curves in $\mathcal{C}$, and take a minimizing sequence (η_j) in $\mathcal{C}$ such that each η_j is parametrized on $[0, 1]$ with constant speed. Since (η_j) is a minimizing sequence, $L = \sup_j L(\eta_j)$ is finite. Choose a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ with $t_i - t_{i-1} < \epsilon/2L$ for $i = 1, \ldots, n$. Then

$$d(\eta_j(t_{i-1}), \eta_j(t)) \leq \int_{t_{i-1}}^{t} ||\eta_j'(t)||\, dt \leq L(t_i - t_{i-1}) < \frac{\epsilon}{2}$$

for $t_{i-1} \leq t \leq t_i$. This estimate allows us to replace each curve η_j by the broken geodesic γ_j joining the points $\eta_j(0), \eta_j(t_1), \ldots, \eta_j(1)$. For every j, γ_j is homotopic to η_j; this can be seen as follows. Owing to

$$d(\gamma_j(t), \eta_j(t)) \leq d(\gamma_j(t), \gamma_j(t_{i-1})) + d(\eta_j(t_{i-1}), \eta_j(t)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $t_{i-1} \leq t \leq t_i$, we can construct a smooth homotopy from $\eta_j|_{[t_{i-1}, t_i]}$ into $\gamma_j|_{[t_{i-1}, t_i]}$ by using the shortest geodesic from $\eta_j(t)$ to $\gamma_j(t)$.

It is clear that $L(\gamma_j) \le L(\eta_j)$, so (γ_j) is also a minimizing sequence in $\mathcal{C}$. Using again the compactness of M, we can select a subsequence of (γ_j), denoted by the same symbol, such that $(\gamma_j(t_i))$ converges to a point p_i as $j \to \infty$ for all i. It follows that (γ_j) converges in the C^1-topology to the broken geodesic γ joining the p_i. It is clear that γ belongs to $\mathcal{C}$ and has length ℓ. Since γ is of minimal length in $\mathcal{C}$, it is locally minimizing. By Theorem 3.2.6, γ is a geodesic. □

In the case of a simply connected compact Riemannian manifold, it is still true that there exists at least one closed geodesic (Lyusternik-Fet [LF51]). More specifically, in the case of S^2, it is known that every Riemannian metric must admit at least 3 geometrically distinct closed geodesics (Lyusternik-Schnirelmann [LŠ47] [1]).

6.3.2 Theorem (Synge) *An even-dimensional orientable compact Riemannian manifold M of positive sectional curvature must be simply connected.*

We remark that each one of the hypotheses in the statement of Synge's theorem is essential. In fact, the following manifolds are not simply-connected: $\mathbf{R}P^2$ is even-dimensional, compact and positively curved, and nonorientable; $\mathbf{R}P^3$ is compact, orientable and positively curved, and odd-dimensional; and a flat 2-torus is even-dimensional, compact and orientable and flat.

Proof of Theorem 6.3.2. Suppose, on the contrary, that M is not simply-connected and let $\mathcal{C}$ denote a nontrivial free homotopy class of loops. By Lemma 6.3.1, there exists a closed geodesic $\gamma : [0,\ell] \to M$, parametrized with unit speed, such that $L(\gamma) = \ell = \inf_{\eta\in\mathcal{C}} L(\eta)$. Let $p = \gamma(0) = \gamma(\ell)$, and denote by $P : T_pM \to T_pM$ the parallel translation map along γ from 0 to ℓ. Fix an orientation of M. Since the parallel translation maps along γ from 0 to t, for $0 \le t \le \ell$, join P to the identity map of T_pM, we have that P is orientation-preserving. Since γ is a geodesic, $\gamma'(0)$ is a fixed vector of P. Now P, being an isometry, leaves the orthogonal complement $\langle\gamma'(0)\rangle^\perp$ invariant. Since the dimension of this subspace is odd, it contains a nonzero vector y that is fixed under P. Let Y be the parallel vector field along γ that extends y, and construct a variation $\{\gamma_t\}$ of γ through closed curves with associated variational vector field given by Y. Since M is positively curved, $\langle R(Y,\gamma')Y,\gamma'\rangle < 0$. Using the variation formulas (5.3.3) and (5.3.9), we get that

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}\Big|_{t=0} E(\gamma_t) < 0.$$

Then, for t sufficiently small, we have that $E(\gamma_t) < E(\gamma)$ and

$$L(\gamma_t)^2 \le 2\ell\; E(\gamma_t) < 2\ell\, E(\gamma) = L(\gamma)^2,$$

and this contradicts the fact that γ is of minimal length in $\mathcal{C}$. Hence $\mathcal{C}$ cannot exist and M is simply-connected. □

6.3.3 Corollary *An even-dimensional compact Riemannian manifold M of positive sectional curvature has fundamental group of order at most two.*

Proof. We may assume M is non-orientable. Let $\tilde{M}$ be the orientable double cover of M. Then $\tilde{M}$ is connected and satisfies the hypotheses of Synge's theorem 6.3.2, so it is simply connected. The result follows. □

It follows from Corollary 6.3.3 that there exists no Riemannian metric of positive sectional curvature in $\mathbf{R}P^m \times \mathbf{R}P^n$ if $m+n$ is even. Indeed, otherwise this manifold would satisfy the hypotheses of the corollary but its fundamental group is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. It is interesting to compare this example with the fact that the nonexistence of a positively curved Riemannian metric in $S^2 \times S^2$ is still an unsettled question (see Add. note 4).

[1] Check Klingenberg and simpleness of curves.

6.4 Bonnet-Myers' theorem

The following result is an elementary example of a comparison theorem in Riemannian geometry. Note that the right-hand side in (6.4.2) is exactly the Ricci curvature of the sphere $S^n(R)$.

6.4.1 Theorem (Bonnet-Myers) *Let M be a complete Riemannian manifold of dimension n. Assume there exists a constant $R > 0$ such that*

$$\text{Ric}(v, v) \geq \frac{n-1}{R^2}\, g(v, v) \tag{6.4.2}$$

for every $v \in TM$. Then

$$\text{diam}(M) \leq \text{diam}(S^n(R)) = \pi R.$$

In particular, M is compact and has finite fundamental group $\pi_1(M)$.

Proof. Recall that $\text{diam}(M) = \sup\{\, d(x, y) \mid x,\ y \in M\,\}$. We will show that the distance of two given points p, $q \in M$ is bounded above by πR. Since M is complete, there exists a minimal geodesic $\gamma : [0, L] \to M$ with unit speed and such that $\gamma(0) = p$ and $\gamma(L) = q$. Because γ is minimal, $I(Y, Y) \geq 0$ for all vector fields Y along γ vanishing at the endpoints. We will use this remark below for some suitable vector fields.

Select an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM with $e_1 = \gamma'(0)$, and extend it to parallel orthonormal frame $\{E_1, \ldots, E_n\}$ along γ; of course, $E_1 = \gamma'$. Set

$$Y_i(s) = \sin\frac{\pi s}{L} E_i(s)$$

for $i = 2, \ldots, n$. Then

$$\begin{aligned} I(Y_i, Y_i) &= \int_0^L -\langle Y_i'', Y_i\rangle + \langle R(\gamma', Y_i)\gamma', Y_i\rangle\, ds \\ &= \int_0^L \sin^2\frac{\pi s}{L}\left(\frac{\pi^2}{L^2} + \langle R(\gamma', E_i)\gamma', E_i\rangle\right) ds. \end{aligned}$$

Noting that each Y_i vanishes at the endpoints of γ, we have

$$\begin{aligned} 0 \leq \sum_{i=2}^n I(Y_i, Y_i) &= \int_0^L \sin^2\frac{\pi s}{L}\left((n-1)\frac{\pi^2}{L^2} - \text{Ric}(\gamma', \gamma')\right) ds \\ &\leq (n-1)\left(\frac{\pi^2}{L^2} - \frac{1}{R^2}\right)\int_0^L \sin^2\frac{\pi s}{L}\, ds, \end{aligned}$$

using the assumption on the Ricci curvature. This proves that $d(p, q) = L \leq \pi R$. We conclude that $\text{diam}(M) \leq \pi R$.

The other assertions in the statement can now be easily verified. The manifold M is complete and bounded, thus, in view of Corollary 3.3.7, compact. Let $\tilde{M}$ denote the Riemannian universal covering manifold of M. Since $\tilde{M}$ is complete and satisfies the same estimate on the Ricci curvature as M, the previous results imply that $\tilde{M}$ is compact, forcing $\pi_1(M)$ to be finite. This completes the proof of the theorem. □

6.4.3 Corollary *No compact nontrivial product manifold $S^1 \times M$ admits a metric of positive Ricci curvature.*

6.4.4 Remark The assumption about the Ricci curvature in the statement of the Bonnet-Myers theorem cannot be relaxed in the sense of requiring that the Ricci curvature only be positive, as the following example shows. The two-sheeted hyperboloid

$$\{ (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 - z^2 = -1 \}$$

with the metric induced from $\mathbf{R}^3$ is complete, non-compact, and has Gaussian curvature at a point (x, y, z) given by $(x^2 + y^2 + z^2)^{-2}$, which, despite being positive, goes to zero as the point tends to infinity. ⋆

6.5 Nonpositively curved manifols

One of the main features of nonpositively curved manifols is the abundance of convex functions. Recall that a continuous function $f : I \to \mathbf{R}$ defined on an interval I is called convex if $f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$ for every $t \in (0, 1)$ and x, $y \in I$. If f is of smooth, this condition is equivalent to requiring that its second derivative $f'' \geq 0$. In the case of a continous function f on a complete Riemannian manifold M, we say that f is *convex* if its restriction $f \circ \gamma$ is convex for every geodesic γ of M. Strict convexity is defined analogously by replacing the inequalities above by the strict inequalities. Our point of view in this section is that most of the important results about the geometry of manifolds with nonpositive curvature can be derived by using appropriate convex functions on the manifold.

We will use the following remark in the proof of Lemma 6.5.1. If a convex function admits two global minima, then a geodesic connecting these two points also consists of global minima of the function. In fact, the function restricted to the geodesic is convex, and this implies that it cannot have bigger values on the interior of the segment than at the endpoints forcing it to be constant along the geodesic segment. A similar argument shows that any local minimum of a convex function on a complete Riemannian manifold must in fact be a global one.

6.5.1 Lemma *Let γ be a geodesic in a Riemannian manifold M. If the sectional curvature along γ is nonpositive, then there are no conjugate points along γ.*

Proof. Let Y be a Jacobi field along γ. We claim that the fuction $f = ||Y||^2$ is convex. In order to prove this, we recall the Jacobi equation $-Y'' + R(\gamma', Y)\gamma' = 0$ and differentiate f twice to get

$$\begin{aligned} f'' &= 2(\langle Y'', Y\rangle + ||Y'||^2) \\ &= 2(\langle R(\gamma', Y)\gamma', Y\rangle + ||Y'||^2) \\ &\geq 0, \end{aligned}$$

in view of the assumption on the curvature; this proves the claim. Now if $f(t_1) = f(t_2) = 0$ for some $t_1 < t_2$, then $f|_{[t_1,t_2]} \equiv 0$, whence Y is trivial. Hence there are no conjugate points along γ. □

6.5.2 Theorem (Hadamard-Cartan) *Let M be a complete Riemannian manifold with nonpositive sectional curvature. Then, for every point $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a smooth covering. In particular, M is diffeomorphic to $\mathbf{R}^n$ if it is simply-connected.*

Proof. Fix a point $p \in M$. In view of Lemma 6.5.1, we know that $\exp_p : T_pM \to M$ is a local diffeomorphism. This being so, we may endow T_pM with the pull-back metric $\tilde{g} = \exp_p^* g$. Since a local isometry maps geodesics to geodesics, the geodesics of $(T_pM, \tilde{g})$ through the origin 0_p are the straight lines, thus, defined on all of $\mathbf{R}$ due to the completeness of M. In view of Theorem 3.3.5(c), this implies that $(T_pM, \tilde{g})$ is complete. Now $\exp_p$ is a covering because of Proposition 3.3.8(b), and the last assertion in the statement is obvious. □

A complete simply-connected manifold of nonpositive sectional curvature is called a *Hadamard manifold*.

6.5.3 Corollary *Let M be a Hadamard manifold. Then, given p, $q \in M$, there is a unique geodesic joining p to q.*

Proof. Let γ be a geodesic joining p to q. Consider the diffeomorphism $\exp_p : T_pM \to M$. Then $\exp_p^{-1} \circ \gamma$ is the straight line in T_pM joining the origin and $\exp_p^{-1}(q)$, as in the proof of Theorem 6.5.2, and this proves the uniqueness of γ. □

In particular, the preceding corollary implies that the cut-locus of an arbitrary point in a Hadamard manifold is empty.

The Hadamard-Cartan theorem says that the universal covering manifold of a complete Riemannian manifold M of nonpositive sectional curvature is $\mathbf{R}^n$. Since $\mathbf{R}^n$ is contractible, the higher homotopy groups $\pi_i(M)$, where $i \geq 2$, are all trivial. Consequently, the topological information about M is contained in its fundamental group $\pi_1(M)$. In the sequel, we prove some classical results about the fundamental group of nonpositively curved manifolds. We start with a lemma.

6.5.4 Lemma *Let M be a Hadamard manifold. Then, for any point $p \in M$, the function $f_p : M \to \mathbf{R}$ given by $f_p(x) = \frac{1}{2}d(p,x)^2$ is smooth and strictly convex.*

Proof. Fix a point $p \in M$. Denote by $\gamma^x : [0,1] \to M$ the unique geodesic parametrized with constant speed joining p to x. Plainly, γ^x is minimizing, so

$$f_p(x) = \frac{1}{2}L(\gamma^x)^2 = E(\gamma^x) = \frac{1}{2}||\gamma^{x\prime}(0)||^2 = \frac{1}{2}||\exp_p^{-1}(x)||^2,$$

showing that f_p is smooth.

Next, let η be a geodesic; we intend to verify that $f \circ \eta$ is strictly convex. For that purpose, we set $\gamma_t = \gamma^{\eta(t)}$ and invoke the second variation formula (5.3.9) to write:

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0}(f_p \circ \eta)(t) &= \frac{d^2}{dt^2}\Big|_{t=0} E(\gamma_t) \\ &= \langle \nabla_{\frac{\partial}{\partial t}} \frac{\bar{\partial}}{\partial t}\Big|_{t=0}, \gamma' \rangle \Big|_0^1 + \int_0^1 ||Y'||^2 + \langle R(\gamma', Y)\gamma', Y\rangle \, ds. \end{aligned} \tag{6.5.5}$$

Since the variational vector field $Y = \frac{\bar{\partial}}{\partial t}|_{t=0}$ vanishes at $s=0$ and $\nabla_{\frac{\partial}{\partial t}} \frac{\bar{\partial}}{\partial t}\Big|_{\substack{s=1\\t=0}} = \eta''(0) = 0$, the first term in the sum is zero; the assumption on the curvature and the fact that Y is nonzero imply that the second term there is positive. We conclude that f is strictly convex. □

6.5.6 Remark We can get more refined information about the second derivatives of f_p. It immediately follows from the Cauchy-Schwarz inequality that a smooth function $f : [0,1] \to \mathbf{R}$ with $f(0) = 0$ must satisfy the inequality $\int_0^1 (f')^2 \, ds \geq f(1)^2$. Retaining the notation in the proof of Lemma 6.5.4, we write $Y(s) = \sum_i a_i(s)E_i(s)$ for smooth functions $a_i : [0,1] \to \mathbf{R}$ and an orthonormal frame $\{E_i\}$ of parallel vectors along γ_0. Then

$$\begin{aligned}\int_0^1 ||Y'||^2\,ds &= \sum_i \int_0^1 (a_i)'^2\,ds \\ &\geq \sum_i a_i(1)^2 \\ &= ||Y(1)||^2 \\ &= ||\eta'(0)||^2.\end{aligned}$$

Together with (6.5.5), this shows that (see exercise 13 in chapter 4)

$$\mathrm{Hess}(f_p) \geq g$$

at every point of M, as bilinear symmetric forms. ⋆

Lemma 6.5.4 allows one to generalize the notion of center of mass of a finite set of points in Euclidean space to the context of Hadamard manifolds. For that purpose, two remarks are in order. First, we note that a non-negative strictly convex proper function has a unique minimum. In fact, because of properness, there must a minimum. If there were two minima, the function would be strictly convex when restricted to a geodesic joining the two minima, and this would imply that the function has smaller values on the interior of this segment than at the endpoints, contradicting the fact that the endpoints are minima. The second remark is that the maximum of any number of strictly convex functions is still strictly convex, as one sees easily. Now, given a finite set of points $p_1, \ldots, p_k$ in a Hadamard manifold, the *center of mass* of the set $\{p_1, \ldots, p_k\}$ is defined to be the uniquely defined minimum of the non-negative strictly convex proper function

$$x \mapsto \max\{f_{p_1}(x), \ldots, f_{p_k}(x)\}.$$

6.5.7 Theorem (Cartan) *Let M be a Hadamard manifold. Then any isometry of finite order of M has a fixed point.*

Proof. Let φ be an isometry of M of order $k \geq 1$. For an arbitrary point $p \in M$, set q to be the center of mass of the finite set $\{p, \varphi(p), \ldots, \varphi^{k-1}(p)\}$. This means that q is the unique minimum of the function

$$f(x) = \max\{f_p(x), f_{\varphi(p)}(x), \ldots, f_{\varphi^{k-1}(p)}(x)\}.$$

Since $\varphi^k(p) = p$ and φ is distance-preserving,

$$\begin{aligned} f(\varphi(q)) &= \frac{1}{2}\max\left\{d(p,\varphi(q))^2, d(\varphi(p),\varphi(q))^2, \ldots, d(\varphi^{k-1}(p),\varphi(q))^2\right\} \\ &= \frac{1}{2}\max\left\{d(\varphi^{k-1}(p),q)^2, d(p,q)^2, \ldots, d(\varphi^{k-2}(p),q)^2\right\} \\ &= f(q), \end{aligned}$$

which shows that also $\varphi(q)$ is a minimum of f. Hence, $\varphi(q) = q$. □

6.5.8 Corollary *Let M be a complete Riemannian manifold of nonpositive sectional curvature. Then the fundamental group of M is torsion-free.*

Proof. The Riemannian universal covering $\tilde{M}$ of M is a Hadamard manifold, and the elements of $\pi_1(M)$ act on $\tilde{M}$ as deck transformations, thus, without fixed points; Theorem 6.5.7 implies that they cannot have finite order. $\square$

Before proving the next theorem, we recall some facts about the relation between the fundamental group $\pi_1(M,p)$ and the set of free homotopy classes of loops, which we denote by $[S^1,M]$, for a connected manifold M and $p \in M$.

6.5.9 Lemma *The 'forgetful' map $\mathcal{F} : \pi_1(M,p) \to [S^1,M]$, which is obtained by ignoring basepoints, sets up a one-to-one correspondence between $[S^1,M]$ and the set of conjugacy classes in $\pi_1(M,p)$.*

Proof. First we remark that $\mathcal{F}$ is onto. In fact, let $\zeta_1 : [0,1] \to M$ be a loop in M, with $\zeta_1(0) = \zeta_1(1) = q$, representing a class in $[S^1,M]$. Since M is arcwise connected, there is a continuous path c joining p to q. Then $\zeta_t := c|_{[t,1]} \cdot \zeta \cdot (c|_{[t,1]})^{-1}$ is a continuous homotopy between ζ_0 and ζ_1, and ζ_0 lies in the image of $\mathcal{F}$.

Next, if γ, η are loops based at p then $\mathcal{F}[\eta\cdot\gamma\cdot\eta^{-1}] = \mathcal{F}[\eta]\cdot\mathcal{F}[\gamma]\cdot\mathcal{F}[\eta^{-1}] = \mathcal{F}[\eta^{-1}]\cdot\mathcal{F}[\eta]\cdot\mathcal{F}[\gamma] = \mathcal{F}[\gamma]$, where for the second equality we cyclically permute the order of concatenation by changing the basepoint. This proves that $\mathcal{F}$ is constant on conjugacy classes.

Conversely, let γ_0, $\gamma_1 : [0,1] \to M$ be loops based at p with $\mathcal{F}[\gamma_0] = \mathcal{F}[\gamma_1]$. This means there is a homotopy γ_t between those curves without necessarily preserving basepoints. The curve $c(t) = \gamma_t(0) = \gamma_t(1)$ traces out the path taken by the basepoints and thus is a loop. Now the concatenation $\tilde{\gamma}_t := c|_{[0,t]} \cdot \gamma_t \cdot (c|_{[0,t]})^{-1}$ is a homotopy from γ_0 to $c\cdot\gamma_1\cdot c^{-1}$ preserving basepoints. $\square$

6.5.10 Lemma *Let γ, η be loops in M based at p, q, respectively. Then the classes $[\gamma] = [\eta]$ in $[S^1,M]$ if and only if $[\gamma] \in \pi_1(M,p)$ and $[\eta] \in \pi_1(M,q)$ act by the same deck transformation on the universal cover $\tilde{M}$.*

Proof. Let ζ be a curve joining p to q. Then $\zeta\cdot\eta\cdot\zeta^{-1}$ is in the same free homotopy class as η. Using Lemma 6.5.9, by concatenating ζ with a loop at p, we may assume that $[\zeta\cdot\eta\cdot\zeta^{-1}] = [\gamma]$ in $\pi_1(M,p)$. The desired result follows from the standard relation between the fundamental group and deck transformations. $\square$

6.5.11 Theorem (Preissmann) *Let M be a compact Riemannian manifold of negative sectional curvature. Then every nontrivial Abelian subgroup of its fundamental group is infinite cyclic.*

Proof. We can assume that M is not simply-connected. Let $\tilde{M}$ be the Riemannian universal covering of M, and let $\varphi \in \pi_1(M)$ an element different from the identity which we view as an isometry of $\tilde{M}$. Recall that φ acts on $\tilde{M}$ without fixed points. The fundamental remark is that the *displacement function* $f : \tilde{M} \to \mathbf{R}$ given by $f(x) = d(x,\varphi(x))$ is smooth and convex. For the purpose of proving this claim, consider the function $\Phi : TM \to M\times M$, given by $\Phi(v) = (x, \exp_x(v))$ for $v \in T_xM$, that was introduced in Lemma 2.4.6. Since $\tilde{M}$ is a Hadamard manifold, we easily see that Φ is well defined and a global diffeomorphism. Now $d : \tilde{M}\times\tilde{M}\setminus\Delta_{\tilde{M}} \to \mathbf{R}$ is given by $d(x,y) = g_x(\Phi^{-1}(x,y),\Phi^{-1}(x,y))^{1/2}$, so it is also smooth; here $\Delta_{\tilde{M}}$ denotes the diagonal of $\tilde{M}$. This proves that f is smooth. In order to prove the convexity of f, we resort to the second variation formula of the length given in exercise 1 of chapter 5. Let η be a geodesic; similarly to in (6.5.5), we can write

$$\frac{d^2}{dt^2}\Big|_{t=0}(f\circ\eta)(t) = \int_0^1 ||Y_\perp'||^2 + \langle R(\gamma',Y_\perp)\gamma',Y_\perp\rangle\,ds \geq 0, \tag{6.5.12}$$

where γ_t is the geodesic joining $\eta(t)$ to $\varphi(\eta(t))$, Y is the variational vector field along γ_0 and $Y_\perp$ denotes its normal component, and we have used that $\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar{\partial}}{\partial t}\big|_{t=0}$ is equal to $\eta''(0) = 0$ and $(\varphi\circ\eta)''(0) = 0$ for $s = 0$ and 1, respectively. Although f is not strictly convex, we can derive more refined information from formula (6.5.12). Since $\tilde{M}$ has negative curvature, the equality holds in (6.5.12) if and only if Y is a constant multiple of γ', so at any given point $x \in \tilde{M}$, f is stricly convex in any direction different from the direction of the geodesic joining x to $\varphi(x)$.

Next, we introduce a definition. An *axis* of φ is a geodesic of $\tilde{M}$ that is invariant under φ. Note that φ cannot reverse the orientation of an axis γ for otherwise the midpoint of the geodesic segment between $\gamma(t)$ and $\varphi(\gamma(t))$ would be a fixed point of φ for any $t \in \mathbf{R}$. Hence the restriction of φ to γ must be translation along it:

$$\varphi(\gamma(t)) = \gamma(t + t_0)$$

for some $t_0 \in \mathbf{R}$ and all $t \in \mathbf{R}$. The number t_0 will be called the *period* of φ along the axis γ. For later reference, we also note that

$$f(\varphi(x)) = d(\varphi(x), \varphi^2(x)) = d(x, \varphi(x)) = f(x)$$

for every $x \in \tilde{M}$.

Now we give three important properties of axes. The first one is that f is constant along an axis γ of φ. Indeed,

$$f(\gamma(t + t_0)) = f(\varphi(\gamma(t))) = f(\gamma(t))$$

for all $t \in \mathbf{R}$, where t_0 is the period of γ. It follows that $f \circ \gamma$ is convex and periodic, and it is easy to see that such a function must be constant. The second one is that an axis of φ is a set of minima of f. This follows immediately from the formula of the first variation of length. The last one is that if f is constant on a geodesic segment $\overline{xy}$ for points $x \neq y$, then the supporting geodesic γ of that segment is an axis of φ. Indeed, f is not stricly convex along $\overline{xy}$, so γ must coincide with the geodesic joining x and $\varphi(x)$. It follows that $\varphi(x)$ lies in the image of γ. Similarly, $\varphi(y)$ lies in the image of γ. Since a geodesic in $\tilde{M}$ is uniquely defined by two points on it, γ must be an axis of φ.

The next step is to prove that φ admits one and only one axis, up to reparametrization and reorientation. Note that the value f at a point $x \in \tilde{M}$ is the length of the unique geodesic in $\tilde{M}$ joining x to $\varphi(x)$. Such geodesics project to geodesics in M all lying in the same free homotopy class of loops in M, independent of the point $\tilde{x}$, according to Lemma 6.5.10. Since M is compact, f admits a global minimum $p \in \tilde{M}$ by Lemma 6.3.1. Since $f(\varphi(p)) = f(p)$, we have that $\varphi(p)$ is also a global minimum. By convexity, f is constant along the geodesic segment joining p and $\varphi(p)$; let γ be the unit speed geodesic that supports this segment. By the above, γ is an axis of φ. Now the points in the image of γ comprise a set of minima at each point of which f is strictly convex in any direction different from γ. It follows that there cannot be another axis.

Finally, suppose that H is an Abelian subgroup of $\pi_1(M)$, and that φ belongs to H and has γ as an axis as above. Since the elements of H commute with φ, they map γ to a geodesic which is invariant under φ; by the above uniqueness result, γ is an axis for all the elements of H. Consider now the period map $H \to \mathbf{R}$. This map is clearly an injective homomorphism, thus its image is a subgroup of $\mathbf{R}$ isomorphic to H. It is not difficult to see that every subgroup of $\mathbf{R}$ is either infinite cyclic or dense. Since the orbits of H on $\tilde{M}$ are discrete, H must be infinite cyclic. □

6.5.13 Corollary *No compact nontrivial product manifold $M \times N$ admits a metric with negative sectional curvature.*

Proof. Suppose, on the contrary, that $M \times N$ supports a metric of negative sectional curvature. By the Hadamard-Cartan theorem 6.5.2, its universal covering, which is the product of the universal coverings $\tilde{M}$ of M and $\tilde{N}$ of N, is contractible. Since a compact manifold can never be contractible (unless it is a point), neither $\tilde{M}$ nor $\tilde{N}$ is compact. In particular, neither M nor N is simply-connected. Now $\pi_1(M)$ and $\pi_1(N)$ are both non-trivial, and as subgroups of $\pi_1(M \times N)$, each of its elements have infinite order by Corollary 6.5.8. We deduce that $\pi_1(M)$ and $\pi_1(N)$ contain infinite cyclic groups H and K, respectively. But then $H \times K$ is a non-trivial Abelian subgroup of $\pi_1(M \times N)$ which is not cyclic, contradicting Preissmann's theorem. This proves the corollary. □

6.5.14 Remark An isometry φ of a Hadamard manifold $\tilde{M}$ can be of three types. Let f be the displacement function associated to φ as in Preissmann's theorem 6.5.11. Then φ is said to be:

a. elliptic if f attains the value zero (i.e. φ admits a fixed point);

b. hyperbolic if f attains a positive minimum;

c. parabolic if f attains no minimum.

The argument in Preissmann's theorem proves that a hyperbolic isometry of a Hadamard manifold admits an axis (which is unique in the case in which the curvature of $\tilde{M}$ is negative).

6.6 Rauch's theorem

In this section we present a version of Rauch's theorem, which is another example of comparison theorem in Riemannian geometry, and derive as an application the existence of convex neighborhoods in Riemannian manifolds.

6.6.1 Theorem (Rauch) *Let $\gamma : [0,\ell] \to M$ be a unit speed geodesic in a Riemannian manifold M and assume that the sectional curvatures of M along γ are bounded above by a real constant κ. If Y is a Jacobi field along γ which is always orthogonal to γ', then the function $||Y||$ along γ satisfies the differential inequality*

$$||Y||'' + \kappa ||Y|| \geq 0 \tag{6.6.2}$$

on the complement of the zero set of Y on $(0,\ell)$.

Moreover, if ψ denotes the solution on $[0,\ell]$ of the differential equation

$$\psi'' + \kappa\psi = 0, \quad \psi(0) = ||Y||(0), \quad \psi'(0) = ||Y||'(0),$$

and ψ does not vanish on $(0,\ell)$, then Y does not vanish on $(0,\ell)$ and

$$\left(\frac{||Y||}{\psi}\right)' \geq 0 \quad \text{and} \quad ||Y|| \geq \psi \tag{6.6.3}$$

on $(0,\ell)$.

Finally, the first inequality in (6.6.3) is an equality at s_0 for some $s_0 \in (0,\ell)$ if and only if the sectional curvatures $K(\gamma', Y) = \kappa$ along $[0, s_0]$ and there exists a parallel unit vector field E along γ for which

$$Y(t) = \psi(t)E(t)$$

along $[0, s_0]$.

Proof. We differentiate $||Y||$ twice along γ to obtain

$$||Y||' = \frac{\langle Y', Y\rangle}{||Y||}$$

and

$$\begin{aligned}
||Y||'' &= \frac{\left(||Y'||^2 + \langle Y'', Y\rangle\right)||Y|| - \langle Y', Y\rangle^2/||Y||}{||Y||^2} \\
&= \frac{||Y'||^2||Y||^2 - \langle Y', Y\rangle^2}{||Y||^3} + \frac{1}{||Y||}\langle R(\gamma', Y)\gamma', Y\rangle \\
&\geq -\kappa||Y||,
\end{aligned}$$

where we have used the Jacobi equation, the Cauchy-Schwarz inequality and the assumption that the sectional curvature of the plane spanned by γ', Y is bounded above by κ, proving the differential inequality.

Moreover, if ψ is as in the statement, then

$$\left(\frac{||Y||}{\psi}\right)' = \frac{||Y||'\psi - ||Y||\psi'}{\psi^2},$$

where the numerator satisfies

$$(||Y||'\psi - ||Y||\psi')(0) = 0$$

by the assumptions, and

$$\left(||Y||'\psi - ||Y||\psi'\right)' = ||Y||''\psi - ||Y||\psi'' \geq 0$$

on $(0, s_0)$ by the differential inequality, where $s_0 > 0$ is the first parameter value where $Y(s_0) = 0$. It follows that the numerator is also non-negative, proving that $(||Y||/\psi)' \geq 0$ on $[0, s_0]$. Since $\lim_{s\to 0^+} \frac{||Y(s)||}{\psi(s)} = 1$, this implies that $||Y|| \geq \psi$ on $[0, s_0]$. Finally, the assumption that ψ does not vanish on $(0, \ell)$ shows that $s_0 \geq \ell$.

If we have equality in the first equation in (6.6.3) for some $s_0 \in (0, \ell)$, then we have equality on all of $(0, s_0]$, which implies $||Y|| = \psi$ on all of $[0, s_0]$. Write $Y = \psi E$ where $||E|| = 1$ along γ. Then $Y' = \psi' E + \psi E'$. We have equality in (6.6.2), which implies $K(\gamma', Y) = \kappa$ and also equality in the Cauchy-Schwarz inequality above, meaning that Y and Y' are linearly dependent at every point of $(0, s_0]$; hence E is parallel along $\gamma|_{[0,s_0]}$. □

The following corollary of Rauch's theorem is attributed to M. Morse (1930) and I. J. Schönberg (1932), and is a strengthening of Lemma 6.5.1.

6.6.4 Corollary *Let $\gamma : [0, \ell] \to M$ be a unit speed geodesic in a Riemannian manifold M and assume that the sectional curvatures of M along γ are bounded above by a positive real constant κ. Then the first conjugate point of $\gamma(0)$ along γ can only occur at $s \geq \pi/\sqrt{\kappa}$.*

Proof. Let Y be a Jacobi field along γ with $Y(0) = 0$; by rescaling, we may assume $||Y||'(0) = 1$. In Rauch's theorem, we have $\psi(s) = \frac{\sin(\sqrt{\kappa}s)}{\sqrt{\kappa}}$, and $||Y(s)|| \geq \frac{\sin(\sqrt{\kappa}s)}{\sqrt{\kappa}} > 0$ for $s < \pi/\sqrt{\kappa}$. □

6.6.5 Lemma *Let p be a point in a Riemannian manifold and choose a sufficiently small $r > 0$ such that $B(p, r)$ is a normal neighborhood of p and $r < \frac{\pi}{2\sqrt{\kappa}}$, where κ is the supremum of sectional curvatures of M at points in a given compact neighborhood of p, and we interpret $\frac{\pi}{2\sqrt{\kappa}}$ as $+\infty$ in case $\kappa \leq 0$. If $\eta : [0, 1] \to B(p, r)$ is a geodesic segment, then the function $f(t) = d(p, \eta(t))$ has at most one critical point for $t \in (0, 1)$, and such a critical point must be a point of minimum.*

Proof. It suffices to prove that any critical point $t_0 \in (0,1)$ of f is a point of minimum. Construct a smooth variation through geodesics $\{\gamma_t\}$ where $\gamma_t : [0,\ell] \to B(p,r)$ is the unique constant speed geodesic joining p to $\eta(t)$ and γ_{t_0} has unit speed. Note that $\ell < r < \frac{\pi}{2\sqrt{\kappa}}$. The variational vector field Y is a Jacobi field orthogonal to γ' at the endpoints, and thus everywhere. By the second variation formula of length (exercise 1 of chapter 5),

$$\frac{d^2}{dt^2}\Big|_{t=t_0} L(\gamma_t) = \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\Big|_{t=0}, \gamma' \rangle\Big|_0^\ell + \int_0^\ell ||Y'||^2 + \langle R(\gamma', Y)\gamma', Y\rangle \, ds.$$

Since $\gamma_t(0) = p$ for all t, and $t \mapsto \gamma_t(\ell) = \eta(t)$ is a geodesic, the first term on the right-hand side vanishes. In case $\kappa \leq 0$, this already shows that $\frac{d^2}{dt^2}|_{t=t_0} L(\gamma_t) > 0$ and hence t_0 is a point of minimum. Otherwise $\kappa > 0$ and, using $||Y'||^2 = \langle Y, Y'\rangle' - \langle Y, Y''\rangle$ and the Jacobi equation, we can write

$$\frac{d^2}{dt^2}\Big|_{t=t_0} L(\gamma_t) = \langle Y'(\ell), Y(\ell)\rangle.$$

By Rauch's theorem 6.6.1,

$$\frac{||Y||'}{||Y||} \geq \frac{\psi'}{\psi}$$

on $(0,\ell)$, where $\psi(s) = \sin(\sqrt{\kappa}s)\frac{||Y||'(0)}{\sqrt{\kappa}}$. It follows that

$$\langle Y'(\ell), Y(\ell)\rangle = ||Y||'(\ell)||Y(\ell)|| \geq \frac{\psi'(\ell)}{\psi(\ell)}||Y(\ell)||^2 = \sqrt{\kappa}\cot(\sqrt{\kappa}\ell)||Y(\ell)||^2 > 0,$$

which proves that t_0 is a point of minimum. ☐

In Proposition 2.4.7, we proved the existence of a totally normal neighborhood U of any point in a Riemannian manifold, namely, any two points in U can be connected by a unique minimizing geodesic. We next show that U can be chosen so that the minimizing geodesic lies entirely in U.

A subset C of a Riemannian manifold M is called *strongly convex* if every two points p, q lying in the topological closure $\bar{C}$ can be connected by a unique minimizing geodesic $\eta : [0,1] \to M$ such that $\eta(0,1) \subset C$. J. H. C. Whitehead proved in 1932 that any point in any Riemannian manifold is the center of a sufficiently small open metric ball which is strongly convex. Recall that the injectivity radius inj as a function on a Riemannian manifold, namely, the distance of a point to its cut locus, is a continuous function.

6.6.6 Theorem (Whitehead) *Let p be a point in a Riemannian manifold. Fix a compact neighborhood K of p in M, let ι denote the infimum of* inj *over K, and let κ denote the supremum of sectional curvatures at points in K. If $r < \frac{1}{2}\min\{\frac{\pi}{\sqrt{\kappa}}, \iota\}$ and $B(p,r) \subset K$, then $B(p,r)$ is strongly convex; here we interpret $\frac{\pi}{\sqrt{\kappa}}$ as $+\infty$ in case $\kappa \leq 0$.*

Proof. Let q, $q' \in \bar{B}(p,r)$. Then $d(q,q') \leq 2r < \iota$, so there is a unique minimizing geodesic $\gamma; [0,1] \to M$ connecting q to q' and γ depends continuously on q, q'. Choose $\epsilon > 0$ such that $r + \epsilon < \frac{1}{2}\min\{\frac{\pi}{\kappa}, \iota\}$, and put

$$V_{r+\epsilon} = \{\, (q,q') \in \bar{B}(p,r) \times \bar{B}(p,r) \mid \gamma(0,1) \subset B(p, r+\epsilon) \,\}.$$

This set is clearly non-empty, and also open in $\bar{B}(p,r) \times \bar{B}(p,r)$ since γ depends continuously on its endpoints. Owing to Lemma 6.6.5,

$$\gamma(0,1) \subset B(p,r) \quad \text{for all} \quad (q,q') \in V_{r+\epsilon}. \tag{6.6.7}$$

Again by continuous dependence of γ on its endpoints, we have that $(q, q') \in \bar{V}_{r+\epsilon}$ implies $\gamma[0,1] \subset \bar{B}(p, r) \subset B(p, r+\epsilon)$, therefore $\bar{V}_{p+\epsilon} \subset V_{r+\epsilon}$, which means that $V_{r+\epsilon}$ is closed. By connectedness, $V_{r+\epsilon} = \bar{B}(p, r) \times \bar{B}(p, r)$, and we finish the proof by referring to (6.6.7). □

The *convexity radius at p* is the supremum (which may be $+\infty$) of all $r \in \mathbb{R}$ such that, for all $\eta < r$, the geodesic ball $B(p, \eta)$ is strongly convex. The *convexity radius* of M is the infimum of convexity radii at all points of M. For instance, the convexity radius of the sphere is $\pi/2$.

6.7 Additional notes

§1 The Gauss-Lobatchevsky-Bolyai discovery of hyperbolic geometry in the early nineteenth century finally pointed out the impossibility of proving Euclid's fifth postulate from the other postulates of Euclidean geometry. In 1868, Beltrami proved the consistency of hyperbolic geometry by realizing it as the intrinsic geometry of a well known surface in Euclidean 3-space — the so-called pseudo-sphere — which has constant negative curvature. In his *Habilitationsvortrag* of 1854 in which Riemann laid the foundations of Riemannian geometry were also exhibited examples of metrics of arbitrary constant curvature. Based on Riemann's ideas, Beltrami published another article in 1869 in which he discussed spaces of constant curvature in arbitrary dimensions. In this way, the non-Euclidean geometries were for the first time incorporated into the realm of Riemannian geometry. In 1890, Klein drew attention to Clifford's 1873 discovery of a 2-torus — nowadays known as the *Clifford torus* — sitting in S^3 with constant zero curvature and formulated the problem of classifying Riemannian manifolds of arbitrary constant curvature in arbitrary dimensions. The problem, referred to as the *Clifford-Klein space forms problem*, was extensively studied by Killing in an article in 1891 and a book in 1893, and then again by Heinz Hopf in 1925 culminating in Theorem 6.2.2.

§2 The argument in the proof of the Hadamard-Cartan theorem 6.5.2 shows that if there is a point in a simply-connected Riemannian manifold possessing no conjugate points, then the manifold is diffeomorphic to Euclidean space. Eberhard Hopf [Hop48] proved that a compact Riemannian manifold M without conjugate points satisfies the inequality

$$\int_M \mathrm{scal} \leq 0$$

where the integral is taken with respect to the canonical Riemannian measure (exercise 12 of chapter 4), and the equality holds if and only if M is flat. In the 2-dimensional case, the left-hand side equals 2π times the Euler characteristic of M by the Gauss-Bonnet theorem. It follows from E. Hopf's result that a metric without conjugate points on T^2 must be flat. It was a long standing conjecture that the same result should be also valid for the higher dimensional tori. In 1994, Burago and Ivanov [BI94] finally settled the conjecture in the positive sense.

§3 Techniques from geometric analysis have been proved to be very powerful in dealing with problems involving curvature in Riemannian manifolds. We would like to mention two spectacular instances of this fact. In 1960, Yamabe [Yam60] tried to deform conformally a given Riemannian metric g on a manifold M into a metric $f \cdot g$ of constant scalar curvature, where f is an unknown positive smooth function on M. If $n = \dim M = 2$, this is a classical result and amounts to showing that M admits isothermal coordinates [Jos06], so he was dealing with the case $n \geq 3$. There was a problem with Yamabe's arguments, though, and the question became the *Yamabe problem*. In order to find f, one needs to solve the nonlinear partial differential equation

$$\Delta f + \frac{n-2}{4(n-1)} \mathrm{scal}(M, g) = f^{\frac{n+2}{n-2}}.$$

This is an extremely difficult question in analysis because the exponent of f is exactly the "critical exponent" in regard to which the standard Sobolev embedding theorems do not apply. The problem was eventually solved through the work of of Aubin [Aub76] and Schoen [Sch84]. Thanks to contributions by other mathematicians, the Yamabe problem is today almost completely understood and it is known that the set of metrics of constant scalar curvature in a given conformal class of metrics is an infinite-dimensional space if $n > 2$. See [Aub98] for these results in book form.

Deformation techniques like that concerning the Yamabe problem are used to prove the existence of several objects in geometry. An interesting approach is to consider deformations on the level of the space of Riemannian metrics on a given smooth manifold M. For instance, Hamilton [Ham82] introduced the following normalized *Ricci flow* equation in the space of Riemannian metrics on a compact n-dimensional manifold M:

$$\frac{d}{dt}g(t) = -2\mathrm{Ric}(g(t)) + 2\frac{\tau}{n}g(t),$$

where $\mathrm{Ric}(g(t))$ denotes the Ricci curvature of the metric $g(t)$, and τ denotes the integral of the scalar curvature of $g(t)$. The fixed points of this equation are the metrics of constant Ricci curvature. One considers t as time and studies the equation as an initial value problem for a fixed Riemannian metric $g_0 = g(0)$ on M. Hamilton proved that if $n = 3$ and the Ricci curvature of g_0 is positive, then the Ricci flow converges smoothly to a metric of constant Ricci curvature. In particular, the manifold is diffeomorphic to a spherical space form. At that time, this was a very interesting application of Riemannian geometry to provide a partial answer to a long-standing open problem in topology, the so called *Poincaré conjecture*: Is every simply-connected compact 3-dimensional manifold homeomorphic to S^3? The difficulty in using Hamilton's method to prove the full Poincaré conjecture was that if one removes the assumption that $\mathrm{Ric}(g_0) > 0$, then the Ricci flow develops finite-time singularities that impede the convergence to a nice metric, and those singularities were not completely understood. As it turns out, Perelman was able to overcome those analytic difficulties. He extended Hamilton's results and in particular proved the full Poincaré conjecture (see e.g. [MT06]).

§4 A famous, open conjecture of Heinz Hopf asserts that $S^2 \times S^2$ does not admit a metric of positive sectional curvature. Indeed, known examples of simply-connected compact manifolds with positive sectional curvature are relatively rare (owing to the Bonnet-Myers theorem 6.4.1, the non-simply-connected examples are quotients of the simply-connected ones by finite subgroups of isometries). The standard examples are the compact rank one symmetric spaces (see Add. notes ? of chapter ?). Apart from these, the homogeneous examples have been classified by Wallach [Wal72] in the odd-dimensional case and by Bérard-Bergery [BB76] in the even dimensional case. These examples occur only in dimensions 6, 7, 12, 13 and 24, and are due to Berger, Wallach and Allof-Wallach. The only other examples known are given by *biquotients* $G//H$. Here G is a Lie group equipped with a bi-invariant metric and H is subgroup of $G\times G$ acting on G by $(h_1, h_2)\cdot g = h_1gh_2^{-1}$. This action is always proper and isometric, and if it is also free, then the quotient space is a manifold denoted by $G//H$. In this case, there is a unique metric on $G//H$ making the projection $G \to G//H$ into a Riemannian submersion and it follows from Proposition 4.5.8 that $G//H$ has always non-negative curvature. More generally, one can also construct bi-quotients by considering left-invariant metrics on G more general than the bi-invariant ones. It turns out that the only known examples of positively curved biquotients occur in dimensions 6, 7 and 13, and these are due to Eschenburg and Bazaikin. There is no general classification of positively curved biquotients. See [Zil07] for a recent survey on these results and related ones.

6.8 Exercises

1 Some definitions: a Riemannian manifold M is called *locally homogeneous* is any two points admit isometric neighborhoods. The *local isotropy group* of M at a point p is the group of germs of isometries defined on connected neighborhoods of p; note that this group is well defined in view of exercise 15 of chapter 3. Finally, M is called *locally 3-point homogeneous* if for any two points p_0, p_0' there exist neighborhoods U, U' of p_0, p_0', resp., such that given two triples (p,q,r), (p',q',r') of points in U, U', resp., with $d(p,q)=d(p',q')$, $d(q,r)=d(q',r')$, $d(r,p)=d(r',p')$, there exists an isometry of a neighborhood V of $\{p,q,r\}$ onto a neighborhood V' of $\{p',q',r'\}$ that maps the first triple to the second one.

Let M be a complete Riemannian manifold of dimension n. Prove that the following assertions are equivalent:

a. M has constant sectional curvature.
b. M is locally homogeneous, and its local isotropy group at any point is isomorphic to $O(n)$.
c. M is locally 3-point homogeneous.

2 Prove that an odd-dimensional compact Riemannian manifold of positive sectional curvature is orientable.

3 Let M be a complete Riemannian manifold of nonpositive curvature. Prove that each homotopy class of curves with given endpoints in M contains a unique geodesic.

4 Consider the disk model $\mathbf{D}^n$ of $\mathbf{R}H^n$ and let φ be an isometry of $\mathbf{R}H^n$.

a. Prove that φ uniquely extends to a homeomorphism of the closed ball $\overline{\mathbf{D}^n}$. (Hint: Use exercise 4 of chapter 3.)
b. Prove that φ is hyperbolic if and only if its extension to $\overline{\mathbf{D}^n}$ admits exactly two fixed points and those lie in the boundary S^{n-1}.
c. Prove that φ is parabolic if and only if its extension to $\overline{\mathbf{D}^n}$ admits exactly one fixed point and that lies in the boundary S^{n-1}.

5 Let G be an Abelian subgroup of the fundamental group of a nonflat space form M. Prove that G is cyclic.

6 An isometry φ of a Riemannian manifold M is called a *Clifford translation* if the associated displacement function $x \mapsto d(x,\varphi(x))$ is constant. Prove that:

a. The Clifford translations for $\mathbf{R}^n$ are just the ordinary translations.
b. The only Clifford translation of $\mathbf{R}H^n$ is the identity transformation.
c. A linear transformation $A \in \mathbf{O}(n+1)$ is a Clifford translation of S^{n+1} if and only if there is a unimodular complex number λ such that half the eigenvalues of A are λ and the other half are $\bar{\lambda}$.

7 Let M be a Hadamard manifold. Prove that an isometry φ of M is a Clifford translation (cf. exercise 6) if and only if the vector field X on M given by $\exp_p(X_p)=\varphi(p)$ is parallel.

8 Extend Preissmann's theorem 6.5.11 to show that every solvable subgroup of the fundamental group of a compact Riemannian manifold of negative curvature must be infinite cyclic.

9 In this exercise, we prove that a compact homogeneous Riemannian manifold M whose Ricci tensor is negative semidefinite everywhere is isometric to a flat torus.

a. Use exercise 9 of chapter 5 to show that the identity component of the isometry group of M is Abelian.
b. Check that M can be identified with an n-torus equipped with a left-invariant Riemannian metric.
c. Show that an n-torus equipped with a left-invariant Riemannian metric admits a global parallel orthonormal frame and hence is flat.

10 A Riemannian manifold M is called *locally symmetric* if every point $p \in M$ admits a normal neighborhood V and an isometry $\varphi : V \to V$ such that $\varphi(p) = p$ and $d\varphi_p = -\mathrm{id}$.
a. Show that space forms and Lie groups with bi-invariant metrics are locally symmetric. (Hint: for the second example, use group inversion.)
b. Prove that the curvature tensor of a locally symmetric manifold is parallel. (Hint: Use the version of equation (4.2.6) for ∇R.)

11 Let M be a Riemannian manifold with curvature tensor R.
a. Prove that R is parallel if and only if for every smooth curve γ in M and parallel vector fields X, Y, Z, W along γ we have that $\langle R(X,Y)Z, W\rangle$ is constant.
b. Prove that if R is parallel then the Jacobi equation along a geodesic has constant coefficients in a suitable basis.

12 In this exercise, we prove the converse of the result of exercise 10(a).
a. Let M and $\tilde{M}$ be a Riemannian manifolds with parallel curvature tensors. Suppose there are points $p \in M$, $\tilde{p} \in \tilde{M}$ and a linear isometry $f : T_pM \to T_{\tilde{p}}\tilde{M}$ such that takes any 2-plane in T_pM to a 2-plane in $T_{\tilde{p}}\tilde{M}$ with the same sectional curvature. Prove that there exists normal neighborhoods V, $\tilde{V}$ of p, $\tilde{p}$, resp., and an isometry $F : V \to \tilde{V}$ such that $F(p) = \tilde{p}$ and $dF_p = f$. (Hint: combine the idea in the proof of Theorem 6.2.1 with exercise 11(b)).
b. Prove that a Riemannian manifold with parallel curvature tensor is locally symmetric. (Hint: Apply part (a) to $M = \tilde{M}$ and $f = -\mathrm{id}$.)