Variational calculus

5.1 Introduction

We continue to study the problem of minimization of geodesics in Riemannian manifolds that was started in chapter 3. We already know that geodesics are the locally minimizing curves. Also, long segments of geodesics need not be minimizing, and the study of this phenomenon in complete Riemannian manifolds motivates the definition of cut locus.

Herein we take a different standpoint in that we consider finite segments of curves. Namely, consider a complete Riemannian manifold M. Given two points p, $q \in M$, the Hopf-Rinow theorem ensures the existence of at least one minimizing geodesic γ joining p and q. It follows that γ is a global minimum for the length functional L defined in the space of piecewise smooth curves joining p and q. Of course, the calculus approach to finding global minima of a function is to differentiate it, compute critical points and decide which of them are local minima by using the second derivative. In our case, the apparatus of classical calculus of variations can be applied to carry out this program.

To begin with, we show that the critical points of the length functional in the space of piecewise smooth curves joining p and q are exactly the geodesic segments, up to reparametrization. The main result of this chapter is the Jacobi-Darboux theorem that gives a necessary and sufficient condition for a geodesic segment between p and q to be a local minimum for L. In order to prove this theorem, we introduce Jacobi fields and conjugate points. Finally, we study the relation between the concepts of cut locus and conjugate locus. These results will be generalized in chapter 7, where we will prove the Morse index theorem.

Throughout this chapter, (M, g) denotes a Riemannian manifold.

5.2 The energy functional

Instead of working with the length functional L, we will be working with the energy functional E, which will be defined in a moment. The reason for that is that the critical point theory of E is very much related to the one of L and, from a variational calculus point of view, E is easier to work with than L.

The *energy* of a piecewise smooth curve $\gamma : [a, b] \to M$ is defined to be

$$E(\gamma) = \frac{1}{2}\int_a^b ||\gamma'(t)||^2 \, dt.$$

The factor 1/2 in this expression is a normalization constant and it is not very important.

It is interesting to note that, in contrast to L, E is not invariant under reparametrizations of the curve. On the one hand, this points out the fact that E is not a geometrical invariant like L. On the other hand, this can be seen as an advantage since, as we will soon see, critical points of E come already equipped with a very specific parametrization.

5.2.1 Lemma *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve, and let $\gamma(a) = p$ and $\gamma(b) = q$.*

a. If γ is minimizing, that is $L(\gamma) = d(p,q)$, then γ is a geodesic, up to reparametrization.

b. If γ minimizes the energy in the space of piecewise smooth curves defined on $[a,b]$ and joining p and q, then γ is a minimizing geodesic.

Proof. (a) If γ is minimizing, then it is locally minimizing (Lemma 3.2.5) and hence a geodesic (Theorem 3.2.6).

(b) In the space of continuous functions $[a,b] \to \mathbf{R}$, consider the scalar product $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$. The Cauchy-Schwarz inequality says that $\langle f,g\rangle^2 \leq ||f||^2||g||^2$ with the equality holding if and only if $\{f,g\}$ is linearly dependent. Applying this to $f = ||\gamma'||$ and $g = 1$ yields that

$$\left(\int_a^b ||\gamma'(t)||\,dt\right)^2 \leq (b-a)\int_a^b ||\gamma'(t)||^2\,dt,$$

and hence

$$L(\gamma)^2 \leq 2E(\gamma)(b-a) \tag{5.2.2}$$

with the equality holding if and only if γ is parametrized with constant speed. Let η be any piecewise smooth curve defined on $[a,b]$ and joining p and q, and assume that it is parametrized with constant speed. By assumption $E(\gamma) \leq E(\eta)$, so using (5.2.2)

$$L(\gamma)^2 \leq 2E(\gamma)(b-a) \leq 2E(\eta)(b-a) = L(\eta)^2.$$

Since the length of a curve does not depend on its parametrization, this shows that γ is a minimizing curve. Due to the result of (a), γ is a geodesic, up to reparametrization. Finally, we observe that γ must be parametrized with constant speed for otherwise it would not minimize the energy by the same (5.2.2) and the condition of equality thereto pertaining. ☐

5.3 Variations of curves

A *variation* of a piecewise smooth curve $\gamma : [a,b] \to M$ is a continuous map $H : [a,b] \times (-\epsilon,\epsilon) \to M$, where $\epsilon > 0$, such that $H(s,0) = \gamma(s)$ for all $s \in [a,b]$, and there exists a subdivision

$$a = s_0 < s_1 < \cdots < s_n = b$$

such that $H|_{[s_{i-1},s_i]\times(-\epsilon,\epsilon)}$ is smooth for all $i = 1,\ldots,n$. For each $t \in (-\epsilon,\epsilon)$, the curve

$$t \mapsto H(s,t)$$

will be denoted by γ_t. We say that H is a *variation with fixed endpoints* if H is a variation satisfying

$$H(a,t) = \gamma_t(a) = \gamma(a) \quad \text{and} \quad H(b,t) = \gamma_t(b) = \gamma(b)$$

for every $t \in (-\epsilon,\epsilon)$. A variation H is called *smooth* if $H : [a,b] \times (-\epsilon,\epsilon) \to M$ is smooth. Finally, we say that H is a *variation through geodesics* if H is a variation such that γ_t is a geodesic for every $t \in (-\epsilon,\epsilon)$.

For a variation H of a piecewise smooth curve $\gamma : [a,b] \to M$, we will denote by $\overline{\nabla}$ the connection induced along H according to Proposition 2.6.1, and we will consider the following vector fields along H:

$$\frac{\bar\partial}{\partial t} = dH\left(\frac{\partial}{\partial t}\right) \quad \text{and} \quad \frac{\bar\partial}{\partial s} = dH\left(\frac{\partial}{\partial s}\right).$$

Note that

$$\frac{\bar\partial}{\partial s} = \gamma_t'$$

may be discontinuous at $s = s_i$. On the other hand, $\frac{\bar\partial}{\partial t}$ and $\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t}$ are continuous vector fields; this is true because $[a,b] \times (-\epsilon,\epsilon) = \cup_{i=1}^n [s_{i-1}, s_i] \times (-\epsilon,\epsilon)$ is a decomposition into a finite union of closed subsets, and the restrictions of those vector fields to $[s_{i-1}, s_i] \times (-\epsilon,\epsilon)$ are continuous for $i = 1, \ldots, n$. Hence we have that

$$Y = \frac{\bar\partial}{\partial t}\Big|_{t=0}$$

is a piecewise smooth vector field along γ called the *variational vector field associated to H*. Conversely, we have the following result.

5.3.1 Lemma *Given a piecewise smooth vector field Y along a piecewise smooth curve $\gamma : [a,b] \to M$, there exists a piecewise smooth variation H of γ whose associated variational vector field is Y.*

Proof. Set $H(s,t) = \exp_{\gamma(s)}(tY(s))$. Since the interval $[a,b]$ is compact, we can find $\epsilon > 0$ such that H is well defined on $[a,b] \times (-\epsilon,\epsilon)$, and

$$\frac{\bar\partial}{\partial t}\Big|_{t=0} = d(\exp_{\gamma(s)})_{0_{\gamma(s)}}(Y(s)) = Y(s).$$

□

5.3.2 Proposition (First variation of energy) *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve, and let H be a variation of γ with associated variational vector field Y. Then*

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = \sum_{i=1}^n \langle Y, \gamma' \rangle \Big|_{s_{i-1}^+}^{s_i^-} - \int_a^b \langle Y, \overline{\nabla}_{\frac{\partial}{\partial s}}\gamma' \rangle \, ds. \tag{5.3.3}$$

Proof. Consider first the case in which γ and H are smooth. Then the integrand of

$$E(\gamma_t) = \frac{1}{2}\int_a^b \langle \gamma_t', \gamma_t' \rangle \, ds = \frac{1}{2}\int_a^b \langle \frac{\bar\partial}{\partial s}, \frac{\bar\partial}{\partial s} \rangle \, ds$$

is smooth and we can compute $\frac{d}{dt}E(\gamma_t)$ by differentiation under the integral sign, namely,

$$\begin{aligned} \frac{d}{dt}E(\gamma_t) &= \frac{1}{2}\int_a^b \frac{\partial}{\partial t}\langle \frac{\bar\partial}{\partial s}, \frac{\bar\partial}{\partial s} \rangle \, ds \\ &= \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s}, \frac{\bar\partial}{\partial s} \rangle \, ds \\ &= \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t}, \frac{\bar\partial}{\partial s} \rangle \, ds \\ &= \int_a^b \frac{\partial}{\partial s}\langle \frac{\bar\partial}{\partial t}, \frac{\bar\partial}{\partial s} \rangle - \langle \frac{\bar\partial}{\partial t}, \overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial s} \rangle \, ds. \end{aligned} \tag{5.3.4}$$

Here we have used that $\bar\nabla_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s}-\bar\nabla_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t} = H_*[\frac{\partial}{\partial t},\frac{\partial}{\partial s}] = 0$, according to Proposition 2.6.2. Evaluating the above formula at $t=0$ gives the desired formula in the case in which γ and H are smooth:

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = \langle Y,\gamma'\rangle\Big|_{a^+}^{b^-} - \int_a^b \langle Y, \bar\nabla_{\frac{\partial}{\partial s}}\gamma'\rangle\, ds.$$

The formula in the general case is obtained from this one by observing that the energy is additive over a union of subintervals. □

5.3.5 Proposition (Critical points of E) *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve. We have that*

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = 0$$

for every variation with fixed endpoints if and only if γ is a geodesic.

Proof. In the class of variations with fixed endpoints, we have that $Y(a) = Y(b) = 0$, so formula (5.3.3) can be rewritten as

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = -\sum_{i=1}^{n-1}\langle Y,\gamma'\rangle\Big|_{s_i^-}^{s_i^+} - \int_a^b \langle Y, \bar\nabla_{\frac{\partial}{\partial s}}\gamma'\rangle\, ds. \tag{5.3.6}$$

If γ is a geodesic, then $\bar\nabla_{\frac{\partial}{\partial s}}\gamma' = 0$ and γ' is continuous, so both terms in (5.3.6) vanish proving one direction of the proposition.

Conversely, suppose that $0 = \frac{d}{dt}\big|_{t=0}E(\gamma_t) = 0$ for every variation with fixed endpoints. Let $f : [a,b] \to \mathbf{R}$ be a smooth function such that $f(s) > 0$ if $s \neq s_i$ and $f(s_i) = 0$ for $i = 0,\ldots,n$, and set $Y = f\bar\nabla_{\frac{\partial}{\partial s}}\gamma'$. Then Y is a piecewise smooth vector field along γ (note that Y is indeed continuous at s_i) with $Y(a) = Y(b) = 0$, and so it defines via Lemma 5.3.1 a variation $\{\gamma_t\}$ with fixed endpoints for which (5.3.6) gives that $0 = -\int_a^b f||\bar\nabla_{\frac{\partial}{\partial s}}\gamma'||^2\, ds$. This already implies that γ is a geodesic on (s_{i-1}, s_i) for $i = 1,\ldots,n$. Since $\gamma|_{[s_{i-1},s_i]}$ is smooth by assumption, it follows that $\bar\nabla_{\frac{\partial}{\partial s}}\gamma'|_{s_i} = 0$ in the sense of side derivatives.

Next, we take Y to be a smooth vector field along γ satisfying $Y(a) = Y(b) = 0$ and $Y(s_i) = \gamma'(s_i^+) - \gamma'(s_i^-)$ for $i = 2,\ldots,n-1$. Substituting into (5.3.6) now gives that $0 = -\sum_{i=2}^{n-1}||\gamma'(s_i^+) - \gamma'(s_i^-)||^2$. This of course implies that γ is of class C^1. Since we already know that $\gamma|_{[s_{i-1},s_i]}$ is a geodesic for $i = 1,\ldots,n$, this implies that these restrictions are segments of the same geodesic γ defined on $[a,b]$ by the uniqueness result (Proposition 2.4.3). □

5.3.7 Corollary (Critical points of L) *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve. We have that*

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t) = 0$$

for every variation with fixed endpoints if and only if γ is a geodesic, up to reparametrization.

Proof. Let $\tilde\gamma = \gamma\circ\varphi$ be a reparametrization of γ with constant speed, where $\varphi : [a,b] \to [a,b]$ is an orientation-preserving diffeomorphism. Given a variation H with fixed endpoints of γ, we define a variation $\tilde H$ of $\tilde\gamma$ by setting $\tilde H(s,t) = H(\varphi(s),t)$, and we denote $\tilde\gamma_t(s) = \tilde H(s,t) = (\gamma_t\circ\varphi)(s)$. Of course $L(\gamma_t) = L(\tilde\gamma_t)$, so we may assume without loss of generality that γ is parametrized with constant speed from the outset. Now

$$\frac{d}{dt}L(\gamma_t) = \int_a^b \frac{\partial}{\partial t}\langle\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial s}\rangle^{1/2}\, ds = \frac{1}{2}\int_a^b \langle\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial s}\rangle^{-1/2}\frac{\partial}{\partial t}\langle\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial s}\rangle\, ds.$$

Evaluating at $t = 0$ and using that $||\gamma'||$ is a constant $k \neq 0$ gives that

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t) = \frac{1}{2k}\int_a^b \frac{\partial}{\partial t}\Big|_{t=0}\langle\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial s}\rangle\, ds = \frac{1}{k}\frac{d}{dt}\Big|_{t=0} E(\gamma_t).$$

This shows that L and E have the same critical points, up to reparametrization. Thus the desired result is an immediate consequence of Proposition 5.3.5. □

5.3.8 Proposition (Second variation of energy) *Let $\gamma : [a,b] \to M$ be a geodesic, and let H be a piecewise smooth variation of γ with associated variational vector field Y. Then*

$$\frac{d^2}{dt^2}\Big|_{t=0} E(\gamma_t) = \langle \overline{\nabla}_{\frac{\partial}{\partial t}} \frac{\bar\partial}{\partial t}\Big|_{t=0}, \gamma'\rangle\Big|_a^b + \int_a^b ||Y'||^2 + \langle R(\gamma', Y)\gamma', Y\rangle\, ds, \tag{5.3.9}$$

where $Y' = \frac{\nabla Y}{ds}$.

Proof. Starting with formula (5.3.4), we compute that

$$\begin{aligned}
\frac{d^2}{dt^2}E(\gamma_t) &= \int_a^b \frac{\partial}{\partial t}\langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial s}\rangle\, ds\\
&= \int_a^b \frac{\partial}{\partial t}\langle \overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s}\rangle\, ds\\
&= \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s}\rangle + \langle \overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t},\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s}\rangle\, ds\\
&= \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s}\rangle + \langle R(\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s})\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s}\rangle + \left|\left|\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t}\right|\right|^2 ds\\
&= \int_a^b \frac{\partial}{\partial s}\langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s}\rangle - \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t},\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial s}\rangle + \langle R(\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial t})\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial t}\rangle + \left|\left|\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t}\right|\right|^2 ds
\end{aligned}$$

In the fourth equality, we used that $\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t} - \overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t} = R(\frac{\bar\partial}{\partial t},\frac{\bar\partial}{\partial s})\frac{\bar\partial}{\partial t}$, according to exercise 11 of chapter 4. Evaluating this formula at $t = 0$ yields that

$$\frac{d^2}{dt^2}\Big|_{t=0}E(\gamma_t) = \int_a^b \frac{\partial}{\partial s}\langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t}\Big|_{t=0},\gamma'\rangle - \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t}\Big|_{t=0},\gamma''\rangle + \langle R(\gamma',Y)\gamma',Y\rangle + ||Y'||^2\, ds$$

Since γ' and $\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial t}$ are continuous and $\gamma'' = 0$, this proves the desired formula. □

5.4 Jacobi fields

Throughout this section, we fix a geodesic $\gamma : [0,\ell] \to M$. The second variation formula (5.3.9) defines a quadratic form on the space of piecewise smooth vector fields along γ vanishing at 0 and ℓ whose associated symmetric bilinear form I is called the *index form* and is clearly given by

$$I(X,Y) = \int_0^\ell \langle X', Y'\rangle + \langle R(\gamma', X)\gamma', Y\rangle\, ds,$$

where $X' = \frac{\nabla X}{ds}$, $Y' = \frac{\nabla Y}{ds}$. Let $0 = s_0 < s_1 < \cdots < s_n = \ell$ be a subdivision of $[0,\ell]$ such that X and Y are smooth on $[s_{i-1}, s_i]$ for $i = 1,\ldots,n$. Since $\langle X', Y'\rangle = \langle X, Y'\rangle' - \langle X, Y''\rangle$ on each

$[s_{i-1}, s_i]$, we can write

$$
\begin{aligned}
I(X,Y) &= \sum_{i=1}^{n}\int_{s_{i-1}}^{s_i} \langle X, Y'\rangle'\,ds + \int_0^\ell -\langle X, Y''\rangle + \langle R(\gamma', Y)\gamma', X\rangle\,ds \\
&= \sum_{i=1}^{n}\langle X, Y'\rangle\Big|_{s_{i-1}^+}^{s_i^-} + \int_0^\ell \langle -Y'' + R(\gamma', Y)\gamma', X\rangle\,ds \\
&= -\sum_{i=1}^{n-1}\langle Y'(s_i^+) - Y'(s_i^-), X\rangle + \int_0^\ell \langle -Y'' + R(\gamma', Y)\gamma', X\rangle\,ds \qquad (5.4.1)
\end{aligned}
$$

A *Jacobi field along* γ is a smooth vector field Y along γ (not necessarily vanishing at the endpoints of γ) such that

$$
-Y'' + R(\gamma', Y)\gamma' = 0. \qquad (5.4.2)
$$

Hence the space of Jacobi fields along γ vanishing at the endpoints of γ is contained in the kernel of I as a bilinear form; it is easy to show that these spaces in fact coincide by using ideas very similar to the ones in the proof of Proposition 5.3.5 (cf. exercise 2). Equation (5.4.2) is called the *Jacobi equation along* γ.

Next, denote by $\mathcal{J}$ the space of all Jacobi fields along γ. It is obvious that $\mathcal{J}$ is a vector space. It is also a very simple matter to check that the smooth vector fields along γ given by $Y_0(s) = \gamma'(s)$ and $Y_1(s) = s\gamma'(s)$ belong to $\mathcal{J}$. The next proposition shows that a Jacobi field Y along γ, being a solution of a second-order linear ordinary differential equation, is completely determined by its initial conditions $Y(0) \in T_pM$ and $Y'(0) \in T_pM$. It follows that $\mathcal{J}$ is a finite-dimensional vector space and $\dim \mathcal{J} = 2\dim M$.

5.4.3 Proposition *Let $\gamma : [0,\ell] \to M$ be a geodesic, and put $\gamma(0) = p$.*

a. *Given u, $v \in T_pM$, there exists a unique Jacobi field $Y \in \mathcal{J}$ such that $Y(0) = u$ and $Y'(0) = v$.*
b. *If X, $Y \in \mathcal{J}$, then the function $\langle X', Y\rangle - \langle X, Y'\rangle$ is constant on $[0,\ell]$. It follows that $\langle \gamma'(s), Y(s)\rangle = as + b$ for some constants a, $b \in \mathbf{R}$ and $s \in [0,\ell]$.*

Proof. (a) Select an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM with $e_1 = \gamma'(0)$ and extend it to an orthonormal frame $\{E_1,\ldots,E_n\}$ of parallel vector fields along γ; since γ is a geodesic, $E_1 = \gamma'$. Let Y be a smooth vector field along γ. Then we can write $Y = \sum_{i=1}^n f_iE_i$, where $f_i : [0,\ell] \to \mathbf{R}$ are smooth functions. In these terms, the Jacobi equation (5.4.2) is

$$
\sum_{i=1}^{n} -f_i''E_i + f_iR(\gamma', E_i)\gamma' = 0.
$$

Taking the inner product of the left-hand side with E_j yields that

$$
-f_j'' + \sum_{i=2}^{n}\langle R(\gamma', E_i)\gamma', E_j\rangle f_i = 0
$$

for $j = 1,\ldots,n$. This is a system of second-order ordinary linear differential equations for which the standard theorems of existence and uniqueness of solutions apply, hence the result.

(b) In order to prove the constancy of the function, it suffices to differentiate it along γ:

$$\begin{aligned}(\langle X',Y\rangle - \langle X,Y'\rangle)' &= (\langle X'',Y\rangle + \langle X',Y'\rangle) - (\langle X',Y'\rangle + \langle X,Y''\rangle)\\ &= \langle R(\gamma',X)\gamma',Y\rangle - \langle X,R(\gamma',Y)\gamma'\rangle\\ &= 0,\end{aligned}$$

where we have used the Jacobi equation (5.4.2) and the symmetry of R (Proposition 4.2.1(c)).

Finally, in order to get the last assertion, take $X=\gamma'$ in the function. Then $\langle\gamma',Y'\rangle = \langle\gamma',Y\rangle'$ is a constant. It follows that $\langle\gamma',Y\rangle$ has the required form. □

Proposition 5.4.3(b) shows that $Y\in\mathcal{J}$ satisfies $\langle\gamma'(s),Y(s)\rangle = as+b$ for all $s\in[0,\ell]$ where $a=\langle\gamma'(0),Y'(0)\rangle$ and $b=\langle\gamma'(0),Y(0)\rangle$. Writing

$$Y = (Y - aY_1 - bY_0) + bY_0 + aY_1$$

shows that there exists a splitting

$$\mathcal{J} = \mathcal{J}^\perp \oplus \mathbf{R}Y_0 \oplus \mathbf{R}Y_1,$$

where $\mathcal{J}^\perp$ is the subspace of Jacobi fields along γ that are always orthogonal to γ', namely,

$$\mathcal{J}^\perp = \{\, Y\in\mathcal{J} \mid \langle Y(s),\gamma'(s)\rangle = 0 \quad \text{for all } s\in[0,\ell]\,\}.$$

Since Y_0 and Y_1 *always* belong to $\mathcal{J}$, it is the subspace $\mathcal{J}^\perp$ that can give us effective information about the geodesic γ, if any.

The next proposition refines the information of Lemma 5.3.1. It also points out the fact that the Jacobi fields along a geodesic somehow control the behaviour of the nearby geodesics.

5.4.4 Proposition *Let $\gamma:[0,\ell]\to M$ be a geodesic. If H is a smooth variation of γ through geodesics, then the associated variational vector field Y is a Jacobi field along γ. On the other hand, every Jacobi field Y along γ is the variational vector field associated to a variation H of γ through geodesics.*

Proof. Suppose first that H is a smooth variation of γ through geodesics and let $Y=\frac{\bar\partial}{\partial t}\big|_{t=0}$ be the associated variational vector field. Then, $\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial s}=0$, so using exercise 11 of chapter 4,

$$\overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t} = \overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s} = \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial s} + R(\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial t})\frac{\bar\partial}{\partial s} = R(\frac{\bar\partial}{\partial s},\frac{\bar\partial}{\partial t})\frac{\bar\partial}{\partial s}.$$

Evaluating this formula at $t=0$ gives that $Y''=R(\gamma',Y)\gamma'$, and hence, Y is a Jacobi field.

Suppose now that Y is a Jacobi field along γ. We construct a variation H of γ as follows. Take any smooth curve η satisfying $\eta(0)=\gamma(0)$ and $\eta'(0)=Y(0)$. Let X_0 and X_1 be the parallel vector fields along η such that $X_0(0)=\gamma'(0)$ and $X_1(0)=Y'(0)$, and let $X(t)=X_0(t)+tX_1(t)$. Finally, set $H(s,t)=\exp_{\eta(t)}(sX(t))$.

By construction, H is a variation through geodesics, so $\frac{\bar\partial}{\partial t}\big|_{t=0} = dH(\frac{\partial}{\partial t})\big|_{t=0}$ is a Jacobi field along γ by the first part of this proof. Let us compute the initial conditions of $\frac{\bar\partial}{\partial t}\big|_{t=0}$ at $s=0$. Since $H(0,t)=\eta(t)$, we have

$$\frac{\bar\partial}{\partial t}\Big|_{\substack{t=0\\ s=0}} = \eta'(0) = Y(0).$$

Moreover,

$$\frac{\bar\partial}{\partial s}\Big|_{s=0} = d(\exp_{\eta(t)})_{0_{\eta(t)}}(X(t)) = X(t),$$

so

$$\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\bar\partial}{\partial t}\Big|_{\substack{t=0\\s=0}} = \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\bar\partial}{\partial s}\Big|_{\substack{t=0\\s=0}} = X'(0) = X_1(0) = Y'(0).$$

Since $\frac{\bar\partial}{\partial t}\big|_{t=0}$ and Y are Jacobi fields along γ having the same initial conditions at $s=0$, they are equal, and this finishes the proof of the proposition. □

5.4.5 Scholium *Consider a point $p \in M$ and two tangent vectors u, $v \in T_pM$. Let γ be the geodesic $\gamma(s) = \exp_p(sv)$, and let Y be the Jacobi field along γ satisfying $Y(0) = 0$ and $Y'(0) = u$. Then*

$$Y(s) = d(\exp_p)_{sv}(su)$$

for all s in the domain of γ.

Proof. This proof is contained in the proof of second assertion in the statement of Proposition 5.4.4. Indeed, using the notation from that proof, η is the constant curve at p, X_0 is the constant vector field $\gamma'(0) = v$ and X_1 is the constant vector field $Y'(0) = u$, so $H(s,t) = \exp_p(s(v+tu))$ and

$$Y(s) = \frac{\bar\partial}{\partial t}\Big|_{(s,0)} = d(\exp_p)_{sv}(su),$$

as desired. □

5.4.6 Example In special cases, knowledge of the Jacobi fields can be used to compute the sectional curvature. Recall the surface of revolution in $\mathbf{R}^3$ as in Example 1.2.2(b). Note that the meridians $\theta =$ const. are geodesics by the reflection argument used in the case of S^n (cf. page 57). By rotational symmetry, it suffices to compute the sectional curvature along the meridian $\gamma(s) = \varphi(s,0)$. We produce a variation of γ by using nearby meridians, namely $H(s,t) = \mathbf{x}(s,t)$. In this case the Jacobi field is $Y(s) = \frac{\bar\partial}{\partial t}\big|_{(s,0)} = \mathbf{x}_\theta(s,0) = f(s)\frac{\partial}{\partial y}$. Note that $\{\gamma', \frac{\partial}{\partial y}\}$ is a parallel orthonormal frame along γ. Therefore the Jacobi equation (5.4.2) is $-f''(s) - K(s)f(s) = 0$, where K is the Gaussian curvature along the parallel $\mathbf{x}(s,\cdot)$. Hence $K = -f''/f$.

5.5 Conjugate points

Let $\gamma(s) = \exp_p(sv)$ be a geodesic in M, where $p \in M$ and $v \in T_pM$. A point $\gamma(s_0)$, where $s_0 > 0$, is called a *point conjugate to p along γ* or a *conjugate point of p along γ* if there exists a nontrivial Jacobi field Y along γ such that $Y(0) = 0$ and $Y(s_0) = 0$; the parameter value s_0 is called a *conjugate value*. In this case, we also have that p is conjugate to $\gamma(s_0)$ along γ^{-1}, so we sometimes say that p and $\gamma(s_0)$ are *conjugate points along γ*. A point $q \in M$ is called a *point conjugate to p* if q is conjugate to p along some geodesic emanating from p. The set of all points of M conjugate to p is called the *conjugate locus of p*.

If $q = \gamma(s_0)$ is conjugate to p along $\gamma(s) = \exp_p(sv)$, and Y is a Jacobi field along γ such that $Y(0) = 0$ and $Y(s_0) = 0$, then Y is everywhere perpendicular to γ' by Proposition 5.4.3(b). Even more interesting, $Y'(0)$ lies in the kernel of the map $d(\exp_p)_{s_0v}$ as it follows from Scholium 5.4.5. Hence, the points conjugate to p are exactly the critical values of $\exp_p$. The *multiplicity* of q as a point conjugate to p along γ is the dimension of the kernel of $d(\exp_p)_{s_0v}$.

Intuitively speaking, the meaning of q being a conjugate point of p along a geodesic γ is that some nearby geodesics emanating from p must meet γ at q *at least in the infinitesimal sense.* Before proceeding with the main result of this section, we prove two lemmas.

5.5.1 Lemma (Gauss, global version) *Consider a point $p \in M$, two tangent vectors u, $v \in T_pM$, and the geodesic $\gamma(s) = \exp_p(sv)$. Then*

$$g_{\gamma(s)}\big(\,d(\exp_p)_{sv}(u), d(\exp_p)_{sv}(v)\,\big) = g_p(u, v).$$

Proof. Note the right-hand-side in the formula is the value at $s = 0$ of the left-hand-side of it. Note also that $d(\exp_p)_{sv}(v) = \gamma'(s)$. Next, let Y denote the Jacobi field along γ with initial conditions $Y(0) = 0$ and $Y'(0) = u$. On the one hand, we know from Scholium 5.4.5 that $d(\exp_p)_{sv}(u) = \frac{1}{s}Y(s)$ for $s \neq 0$. On the other hand, decompose $u = \lambda v + u_1$, where u_1 is perpendicular to v, and let Y_0, Y_1 be the Jacobi fields along γ vanishing at $s = 0$ such that $Y_0'(0) = \lambda v$ and $Y_1'(0) = u_1$. Then $Y_0(s) = \lambda s\gamma'(s)$ and $Y(s) = Y_0(s) + Y_1(s) = \lambda s\gamma'(s) + Y_1(s)$, so, if $s \neq 0$,

$$\begin{aligned} g_{\gamma(s)}\big(\,d(\exp_p)_{sv}(u), d(\exp_p)_{sv}(v)\,\big) &= g_{\gamma(s)}\big(\,\frac{1}{s}Y(s), \gamma'(s)\,\big) \\ &= \lambda g_{\gamma(s)}\big(\,\gamma'(s), \gamma'(s)\,\big) + \frac{1}{s}g_{\gamma(s)}\big(\,Y_1(s), \gamma'(s)\,\big). \end{aligned}$$

The first term in the last line of the above calculation is $\lambda g_p(v, v) = g_p(u, v)$, since the length of the tangent vector of a geodesic is constant. The second term in there is zero by Proposition 5.4.3(b) because $Y_1(0)$ and $Y_1'(0)$ are perpendicular to $\gamma'(0)$, and this proves the formula. □

5.5.2 Lemma *Consider a point $p \in M$, and a tangent vector $v \in T_pM$. Let $\varphi : [0, 1] \to T_pM$ denote the radial segment $\varphi(s) = sv$, and let $\psi : [0, 1] \to T_pM$ be an arbitrary piecewise smooth curve joining the origin 0 to v. Then*

$$L(\exp_p \circ\psi) \geq L(\exp_p \circ\varphi) = ||v||.$$

Proof. Without loss of generality, we may assume that $\psi(s) \neq 0$ for $s > 0$. In the case in which ψ is smooth, write $\psi(s) = r(s)u(s)$ where $r : (0, 1] \to (0, +\infty)$ and $u : (0, 1] \to S^{n-1}$ are smooth, and S^{n-1} denotes the unit sphere of (T_pM, g_p). Then

$$\psi'(s) = r'(s)u(s) + r(s)u'(s)$$

with $\langle u(s), u'(s)\rangle = 0$. Applying Gauss lemma 5.5.1 twice in the following computation,

$$\begin{aligned} ||(\exp_p \circ\psi)'(s)||^2 &= ||d(\exp_p)_{\psi(s)}(\psi'(s))||^2 \\ &= (r'(s))^2 \underbrace{||d(\exp_p)_{\psi(s)}(u(s))||^2}_{=||u(s)||^2=1} + (r(s))^2||d(\exp_p)_{\psi(s)}(u'(s))||^2 \\ &\geq (r'(s))^2, \end{aligned}$$

we get that

$$L(\exp_p \circ\psi) \geq \int_0^1 |r'(s)|\, ds \geq |r(1) - \lim_{s\to 0+} r(s)| = ||v||.$$

In the general case, we repeat the argument above over each subinterval where ψ is smooth and add up the estimates. □

Next, we prove the main result of this chapter. It gives a sufficient condition and a necessary condition for a geodesic segment to be locally minimizing is the space of curves with the same endpoints.

5.5.3 Theorem (Jacobi-Darboux) *Let $\gamma : [0, \ell] \to M$ be a geodesic segment parametrized with unit speed and with endpoints $\gamma(0) = p$ and $\gamma(\ell) = q$.*

a. If there are no points conjugate to p along γ, then there exists a neighborhood V of γ in the C^0-topology (or uniform topology) in the space of piecewise smooth curves parametrized on $[0, \ell]$ and joining p to q such that $E(\eta) \geq E(\gamma)$ and $L(\eta) \geq L(\gamma)$ for every $\eta \in V$. Moreover, if $L(\eta) = L(\gamma)$ for some $\eta \in V$, then η and γ differ by a reparametrization.

b. If $\gamma(s_0)$ is conjugate to p along γ for some $s_0 \in (0, \ell)$, then there exists a variation $\{\gamma_t\}$ of γ with fixed endpoints such that $E(\gamma_t) < E(\gamma)$ and $L(\gamma_t) < L(\gamma)$ for sufficiently small t.

Proof. Put $\gamma'(0) = v$ and define $\varphi : [0, \ell] \to T_pM$ by $\varphi(s) = sv$. By assumption, $\varphi(s)$ is a regular point of $\exp_p$ for $s \in [0, \ell]$. Since $\varphi([0, \ell])$ is compact, we can cover it by a union $\cup_{i=1}^k W_i$ of open balls $W_i \subset T_pM$ such that $\exp_p$ is a diffeomorphism of W_i onto an open subset $U_i \subset M$. Choose a subdivision $0 = s_0 < s_1 < \ldots < s_k = \ell$ such that $\varphi([s_{i-1}, s_i]) \subset W_i$ for all i. Let V be the open ball centered at γ of radius $\epsilon > 0$, namely, V consists of the piecewise smooth curves $\eta : [0, \ell] \to M$ joining p to q and satisfying $d(\eta(s), \gamma(s)) < \epsilon$ for $s \in [0, \ell]$. We take ϵ so that $\eta([s_{i-1}, s_i]) \subset U_i$ for $\eta \in V$ and $i = 1, \ldots, k$. Note that $\exp_p(W_{i-1} \cap W_i)$ is an open neighborhood of $\gamma(s_{i-1})$ contained in $U_{i-1} \cap U_i$. We further decrease ϵ, if necessary, so as to obtain that $\eta(s_{i-1}) \in \exp_p(W_{i-1} \cap W_i)$ for $\eta \in V$ and $i = 2, \ldots, k$.

For each $\eta \in V$, we lift η to a piecewise smooth curve ψ in T_pM as follows. Define

$$\psi(s) = (\exp_p|_{W_1})^{-1}(\eta(s)) \quad \text{for } s \in [0, s_1].$$

Note that $\psi(0) = 0$. Assume that ψ has already been defined on $[0, s_{i-1}]$ for some $2 \leq i \leq k$ such that it satisfies $\exp_p(\psi(s)) = \eta(s)$ for $s \in [0, s_{i-1}]$ and $\psi(s_{i-1}) \in W_{i-1}$. Note that these conditions imply that

$$\exp_p(\psi(s_{i-1})) = \eta(s_{i-1}) \in \exp_p(W_{i-1} \cap W_i),$$

so $\psi(s_{i-1}) \in W_i$. Hence

$$\psi(s) = (\exp_p|_{W_i})^{-1}(\eta(s)) \quad \text{for } s \in [s_{i-1}, s_i]$$

continuously extends ψ to $[0, s_i]$. This completes the induction step and shows that ψ can be defined on $[0, \ell]$. Since $\eta(\ell) \in W_k$, we have $\psi(\ell) = \ell v$. By Lemma 5.5.2,

$$L(\eta) = L(\exp_p \circ \psi) \geq L(\exp_p \circ \varphi) = L(\gamma).$$

Moreover, since $d(\exp_p)_{\psi(s)}$ is injective for $s \in [0, \ell]$, the proof of the lemma shows that the inequality is sharp unless u is constant and r' is nonnegative in the notation of that proof, that is, η coincides with γ up to reparametrization. As for the assertion concerning the energy, we observe that

$$E(\eta) \geq \frac{1}{2\ell} L(\eta)^2 \geq \frac{1}{2\ell} L(\gamma)^2 = E(\gamma)$$

by the Cauchy-Schwarz inequality (5.2.2). This proves part (a).

(b) By assumption, there exists a nontrivial Jacobi field Y along γ such that $Y(0) = Y(s_0) = 0$. Owing to the non-triviality of Y, $Y'(s_0) \neq 0$. Let Z_1 be the parallel vector field along γ with $Z_1(s_0) = -Y'(s_0)$, construct a smooth function $\theta : [0, \ell] \to \mathbf{R}$ such that $\theta(0) = \theta(\ell) = 0$ and $\theta(s_0) = 1$, and set $Z(s) = \theta(s) Z_1(s)$. Also, extend Y to a piecewise smooth vector field on $[0, \ell]$ by putting $Y|_{[s_0, \ell]} = 0$, and set $Y_\alpha(s) = Y(s) + \alpha Z(s)$ for $s \in [0, \ell]$ and $\alpha \in \mathbf{R}$.

Now Y_α is a piecewise smooth vector field along γ which is everywhere normal to γ' and vanishes at 0 and ℓ. Consider a variation with fixed endpoints $\{\gamma_t\}$ with associated variational vector field Y_α. Then

$$\begin{aligned} I(Y_\alpha, Y_\alpha) &= I(Y,Y) + 2\alpha I(Y,Z) + \alpha^2 I(Z,Z) \\ &= -2\alpha\langle Y'(s_0^+) - Y'(s_0^-), Z(s_0)\rangle + \alpha^2 I(Z,Z) \\ &= -2\alpha||Y'(s_0^-)||^2 + \alpha^2 I(Z,Z) \\ &< 0, \end{aligned}$$

where $\alpha > 0$ is chosen sufficiently small so as to ensure the last inequality. Hence $E(\gamma_t) < E(\gamma)$ for sufficiently small t. Also,

$$L(\gamma_t)^2 \leq 2\ell E(\gamma_t) < 2\ell E(\gamma) = L(\gamma)^2,$$

and this completes the proof. □

As a corollary of the theorem of Jacobi-Darboux 5.5.3, we have the following refinement of Proposition 3.4.3.

5.5.4 Corollary *Let M be a complete Riemannian manifold. Then, for each $p \in M$, the exponential map*

$$\exp_p : \mathrm{D}_p \to M \setminus \mathrm{Cut}(p)$$

is a diffeomorphism.

Proof. We have already seen that $\exp_p(\mathrm{D}_p) = M \setminus \mathrm{Cut}(p)$. Theorem 5.5.3 implies that a geodesic $\gamma_v : [0, +\infty) \to M$, where $v \in T_pM$ and $||v|| = 1$, does not minimize L past its first conjugate point, so a conjugate point along γ_v, if existing, must occur at a parameter value $s_0 \geq \rho(v)$. It follows that $\exp_p$ is a local diffeomorphism at sv for $s \in [0, \rho(v))$. Since v is an arbitrary unit tangent vector at p, this shows that $\exp_p$ is a local diffeomorphism on D_p. It remains only to check that $\exp_p$ is injective on D_p. But this is clear since any point in $\exp_p(\mathrm{D}_p)$ can be joined to p by a unique minimal geodesic as was already observed right after the proof of Proposition 3.4.3. □

The *first conjugate point* along a geodesic $\gamma(s) = \exp_p(sv)$, where $p \in M$ and $v \in T_pM$, is the smallest parameter value $s_0 > 0$ such that $\gamma(s_0)$ is conjugate to p along γ. It also follows from the theorem of Jacobi-Darboux 5.5.3 that the first conjugate point to p along γ cannot occur before the cut point; in particular, the conjugate locus of a point is empty if its cut locus is empty. The following proposition gives more information.

5.5.5 Proposition *Let M be a complete Riemannian manifold, and let $p \in M$. Then a point q belongs to the cut locus* $\mathrm{Cut}(p)$ *if and only if one of the following non-mutually exclusive assertions is true:*

a. There exist at least two distinct minimizing geodesics joining p to q.

b. The point q is the first conjugate point to p along a minimizing geodesic.

In particular, $q \in \mathrm{Cut}(p)$ if and only if $p \in \mathrm{Cut}(q)$.

Proof. By Lemma 3.4.1 and Theorem 5.5.3, we already know that the conditions in the statement are sufficient for q to belong to $\mathrm{Cut}(p)$. Conversely, suppose that $q \in \mathrm{Cut}(p)$. Then we can write $q = \exp_p(\rho(v)v)$ for some unit vector $v \in T_pM$ with $\rho(v) < +\infty$. In particular, $\gamma(s) = \exp_p(sv)$, where $0 \leq s \leq \rho(v)$, is a minimal geodesic joining p to q. Choose a sequence (s_j) of real numbers such that $s_j \searrow \rho(v)$. For each j, there exists a minimal geodesic γ_j joining p to $\gamma(s_j)$, say $\gamma_j(s) =$

$\exp_p(sw_j)$, where $w_j \in T_pM$ and $||w_j|| = 1$. Let $d_j = d(p, \gamma(s_j))$, so that $\gamma_j(d_j) = \gamma(s_j)$. Since $s_j > \rho(v)$, we have that $\gamma|_{[0,s_j]}$ is not minimal so that $d_j < s_j$.

Next, by compactness of the unit sphere in T_pM and by passing to a subsequence if necessary, we may assume that (w_j) converges to a unit vector $w \in T_pM$. Since the distance d is continuous, $d_j = d(p, \gamma(s_j)) \to d(p, \gamma(\rho(v))) = \rho(v)$. By taking the limit as $j \to +\infty$ in $\gamma(s_j) = \gamma_j(d_j) = \exp_p(d_j w_j)$, we get that $q = \exp_p(\rho(v)w)$. Now there are two cases to be considered.

If $w \neq v$, then $\eta(s) = \exp_p(sw)$ is a minimizing geodesic joining p to q and $\eta \neq \gamma$, so we are in situation (a). On the other hand, if $w = v$, then we already have that $\exp_p(d_j w_j) = \gamma(s_j) = \exp_p(s_j v)$ for all j, where $d_j w_j \to \rho(v)v$ and $s_j v \to \rho(v)v$. It follows that $\exp_p$ is not locally injective at $\rho(v)v$, so $\rho(v)v$ is a singular point of $\exp_p$. Hence $q = \exp_p(\rho(v)v)$ is conjugate to p along γ. Since γ is minimizing on $[0, \rho(v)]$, q must be the first conjugate point to p along γ, and we are in situation (b).

For the last assertion, one needs to note that conditions (a) and (b) are symmetric in p and q. This is clear for (a) and follows from Theorem 5.5.3(b) for (b). □

All possibilities given by Proposition 5.5.5 for a point $q \in \mathrm{Cut}(p)$ can indeed occur: both (a) and (b); (a) and not (b); (b) and not (a). Comparing the examples in the sequel with the examples of section 3.5, one immediately finds situations in which the first two possibilities occur. However, the third possibility — in which q is the first conjugate point along a minimizing geodesic γ and there is no other minimizing geodesic from p to q — is not so easy to detect. The Heisenberg group (consisting of upper triangular real matrices of size 3 with 1's along the diagonal) equipped with some left-invariant metric provides such an example [Wal97, p. 352].

5.6 Examples

Flat manifolds

For a flat manifold, $R \equiv 0$, so the Jacobi equation is $Y'' = 0$. Hence Jacobi fields along a geodesic γ have the form $Y(s) = sE_1(s) + E_2(s)$, where E_1 and E_2 are parallel vector fields along γ. For instance, a Jacobi field Y along a geodesic γ in Euclidean space $\mathbf{R}^n$ is of the form $Y(s) = u + sv$, where u, $v \in \mathbf{R}^n$. If T^n is a flat torus and $\pi : \mathbf{R}^n \to T^n$ denotes the corresponding Riemannian covering, then a Jacobi field along the geodesic $\pi \circ \gamma$ in T^n is of the form $\bar{Y}(s) = d\pi_{\gamma(s)}(Y(s)) = d\pi_{\gamma(s)}(u) + sd\pi_{\gamma(s)}(v)$.

In particular, in a flat manifold there are no conjugate points, so any geodesic segment is a local minimum for L. Note that in a flat torus there are infinitely many geodesics with given endpoints p and q, and generically (meaning the case in which $q \notin \mathrm{Cut}(p)$) only one of them is a global minimum.

Manifolds of nonzero constant curvature

Consider first the unit sphere S^n. If γ is a unit speed geodesic and Y is a Jacobi field along γ which is everywhere perpendicular to γ', then formula (4.5.2) says that $R(\gamma', Y)\gamma' = -Y$, so the Jacobi equation is $Y'' = -Y$. It follows that $Y(s) = \cos s E_1(s) + \sin s E_2(s)$, where E_1 and E_2 are parallel vector fields along γ which are perpendicular to γ' (Note that a parallel vector field along γ which is perpendicular to γ' is nothing but a constant vector field on the surrounding $\mathbf{R}^{n+1}$ which is perpendicular to the 2-plane spanned by $\gamma(0)$ and $\gamma'(0)$.) In particular, if Y vanishes at $s = 0$, then $E_1 = 0$. Assuming Y is nontrivial, that is, $E_2 \neq 0$, then the conjugate values are $s = \pi$, 2π, $3\pi, \ldots$. Therefore the first conjugate point of $p = \gamma(0)$ along γ is $-p$, so that the first conjugate locus coincides with the cut locus; since $Y'(0)$ can be any vector perpendicular to $\gamma'(0)$, the multiplicity of $-p$ is $n-1$. Note also that p is conjugate to itself along γ.

Consider now $\mathbf{R}P^n$. Since it has the same curvature tensor as S^n, it has also the same Jacobi equation, the same Jacobi fields and the same conjugate values. However, the difference to S^n is that now the first conjugate point $\gamma(\pi)$ along a geodesic γ coincides with $\gamma(0)$, so the first conjugate point occurs after the cut point $\gamma(\frac{\pi}{2})$. In particular, a geodesic of length $\frac{\pi}{2}+\epsilon$, $\epsilon > 0$ small, is a local minimum for L, but not a global one.

The case of $\mathbf{R}H^n$ is similar to that of S^n. By (4.5.3), the Jacobi equation is $Y'' = Y$, so the Jacobi fields along a geodesic γ have the form $Y(s) = \cosh s E_1(s) + \sinh s E_2(s)$, where E_1 and E_2 are parallel vector fields along γ which are perpendicular to γ'. In particular, if Y vanishes at $s=0$, then $E_1 = 0$. Assuming Y is nontrivial, that is, $E_2 \neq 0$, there are no conjugate values. Hence the conjugate locus of a point is empty. Of course, this result is in line with the remark after the proof of Corollary 5.5.4 since we already knew that the cut locus of $\mathbf{R}H^n$ is empty.

$\mathbf{C}P^n$

Owing to Proposition 3.5.1, the geodesics of $\mathbf{C}P^n$ are the projections of the horizontal geodesics of S^{2n+1} with respect to the Riemannian submersion $\pi : S^{2n+1} \to \mathbf{C}P^n$. Let $\tilde\gamma(s) = \cos s\tilde p + \sin s\tilde v$ be a horizontal geodesic of S^{2n+1}, where $\tilde p \in S^{2n+1}$ and $\tilde v \in \mathcal{H}_{\tilde p}$ is a unit vector, and consider the geodesic $\gamma = \pi\circ\tilde\gamma$ of $\mathbf{C}P^n$. It follows that the Jacobi fields along γ are projections of some Jacobi fields along $\tilde\gamma$. Note that whereas a Jacobi field along γ is associated to a variation of $\tilde\gamma$ through horizontal geodesics, this does not imply that the associated Jacobi field along $\tilde\gamma$ must be horizontal. In the following, we want to describe the conjugate points along γ, so we need to describe the Jacobi fields along γ that vanish at $s=0$ and are everywhere orthogonal to γ'.

Consider first the variation through horizontal geodesics

$$\tilde H_0(s,t) = e^{it}\cdot\tilde\gamma(s) = \cos s(\cos t + \sin t(i\tilde p)) + \sin s(\cos t + \sin t(i\tilde v)).$$

The associated Jacobi field is

$$\tilde Y_0(s) = i\tilde\gamma(s),$$

and it coincides with the restriction of the vertical vector field (4.5.9) along $\tilde\gamma$. Of course, the corresponding variation of γ is trivial and, accordingly, $\tilde Y_0$ projects down to a trivial Jacobi field along γ.

Next, consider an arbitrary Jacobi field $\tilde Y$ along $\tilde\gamma$ associated to a variation through horizontal geodesics and with the property that it projects down to a Jacobi field Y along γ such that $Y(0)=0$ and $\langle Y,\gamma'\rangle \equiv 0$. We already know that $\tilde Y(s) = \cos s\tilde E_1(s) + \sin s\tilde E_2(s)$ for some parallel vector fields E_1, E_2 along $\tilde\gamma$. The condition that $0 = Y(0) = d\pi_{\tilde p}(\tilde Y(0))$ imposes that $\tilde Y(0)$ must be vertical, namely, a multiple of ip. Since $\tilde Y_0$ projects down to zero and the Jacobi fields along a geodesic form a vector space, we can add a suitable multiple of $\tilde Y_0$ to $\tilde Y$ and assume that $\tilde Y(0) = 0$. Now $\tilde E_1 = 0$ and $\tilde Y(s) = \sin s\tilde E_2(s)$. We must have $\langle \tilde Y, \tilde\gamma'\rangle \equiv 0$, so $\tilde E_2(s)$ is a constant vector $\tilde u \in \mathbf{R}^{n+1}$ orthogonal to $\tilde p$ and $\tilde v$. A variation associated to $\tilde Y$ is

$$\tilde H(s,t) = \cos s\,\tilde p + \sin s(\cos t\,\tilde v + \sin t\,\tilde u).$$

Note that $\tilde\gamma_t$ is horizontal if and only if $\tilde\gamma_t'(0) = \cos t\tilde v + \sin t\tilde u$ is orthogonal to $i\tilde p$ if and only if $\tilde u \perp i\tilde p$. We compute

$$\begin{aligned}\langle \tilde Y(s), i\tilde\gamma(s)\rangle &= \langle \sin s\,\tilde u, \cos s(i\tilde p) + \sin s(i\tilde v)\rangle \\ &= \sin^2 s\langle \tilde u, i\tilde v\rangle.\end{aligned}$$

Now there are two cases. If $\tilde{u} \perp i\tilde{v}$, then $\tilde{Y}$ is a horizontal vector field and the corresponding Jacobi field is $Y(s) = \sin s U(s)$, where $U(s)$ is the parallel vector field along γ with $U(0) = d\pi_{\tilde{p}}(\tilde{u})$; the space of such Jacobi fields is $2n-2$-dimensional and the associated conjugate values are multiples of π. On the other hand, if $\tilde{u} = i\tilde{v}$, then the horizontal component of $\tilde{Y}$ is

$$\begin{aligned}\tilde{Y}(s) - \sin^2 s(i\tilde{\gamma}(s)) &= \sin s(i\tilde{v}) - \sin^2 s(\cos s(i\tilde{p}) + \sin s(i\tilde{v})) \\ &= \sin s(\cos s^2(i\tilde{v}) - \sin s \cos s(i\tilde{p})) \\ &= \sin s \cos s(i\tilde{\gamma}'(s)).\end{aligned}$$

In this case, $Y(s) = \sin s \cos s(J\gamma'(s)) = \frac{1}{2}\sin 2s(J\gamma'(s))$; the space of such Jacobi fields is one-dimensional and the associated conjugate values are multiples of $\pi/2$. Finally, it follows from our considerations that the first conjugate locus of a point coincides with the cut locus.

Lie groups

Let G be a Lie group equipped with a bi-invariant metric. In this example, we will describe the conjugate locus of a point in G. By homogeneity, it suffices to compute the conjugate locus of the identity. Denote by $\mathfrak{g}$ the Lie algebra of G. Any geodesic through 1 has the form $\gamma(t) = \exp tX$ for some $X \in \mathfrak{g}$. Let $\{E_1, \ldots, E_n\}$ be a basis of $\mathfrak{g}$. Consider the Jacobi equation $-Y'' + R(\gamma', Y)\gamma' = 0$ along γ. Write $Y(t) = \sum_{i=1}^n y_i(t)E_i$ where y_i are smooth functions on $\mathbf{R}$. Note that $\gamma'(t) = d(L_{\gamma(t)})_1\gamma'(0) = X_{\gamma(t)}$. Then

$$Y'' = \sum_i y_i'' E_i + 2y_i' \nabla_X E_i + y_i \nabla_X \nabla_X E_i$$

and

$$R(\gamma', Y)\gamma' = R(X, Y)X = \sum_i y_i \left(\nabla_X \nabla_{E_i} X - \nabla_{[X, E_i]} X\right).$$

A simple calculation using the formula (2.8.8) for the Levi-Cività connection yields that the Jacobi equation along γ has the form

$$\frac{d^2}{dt^2}Y + \mathrm{ad}_X \frac{d}{dt}Y = 0. \tag{5.6.1}$$

Recall that ad_X is a skew-symmetric endomorphism of $\mathfrak{g} \cong T_1G$ with respect to the metric at the identity, so there exists an ad_X-invariant orthogonal decomposition

$$\mathfrak{g} = V_0 \oplus \bigoplus_{j=1}^r V_j$$

where V_0 is the kernel of ad_X and for $j = 1, \ldots, r$ we have $\dim V_j$ is even and the eigenvalues of ad_X on V_j are $\pm i\lambda_j$, $\lambda_j \neq 0$. Now the general solution of (5.6.1) has the form

$$Y(t) = C + Y_0 t + \sum_{j=1}^r \cos(\lambda_j t)Y_j + \frac{\sin(\lambda_j t)}{\lambda_j}\mathrm{ad}_X Y_j \tag{5.6.2}$$

where $Y_j \in V_j$ for $j = 0, \ldots, r$ and $C \in \mathfrak{g}$. Therefore the space of Jacobi fields vanishing at $t = 0$ is spanned by

$$Y_0 t - Y_j + \cos(\lambda_j t)Y_j + \frac{\sin(\lambda_j t)}{\lambda_j}\mathrm{ad}_X Y_j$$

where $Y_j \in V_j$ for $j = 1, \ldots, r$. This Jacobi field can vanish again only if $Y_0 = 0$; in this case, it is periodic and vanishes exactly when t is a multiple of $2\pi/\lambda_j$. We finally deduce that the points conjugate to 1 along γ are $\gamma(2\pi k/\lambda_j)$, where $k \in \mathbf{Z}$, with multiplicity $\dim V_j$. In particular, the multiplicity of a conjugate point is always even.

5.7 Additional notes

§1 One can recover the results of this chapter by replacing variational calculus by standard calculus on infinite-dimensional smooth manifolds as follows. To begin with, it is necessary to consider a larger class of curves to work with, namely, the absolutely continuous curves $\gamma : [a, b] \to M$ joining p to q with square-integrable $||\gamma'||$. This is a metric space with respect to the distance

$$d(\gamma_1, \gamma_2) = \sup_{t\in[a,b]} d(\gamma_1(t), \gamma_2(t)) + \left(\int_a^b ||\gamma_1'(s) - \gamma_2'(s)||^2\, ds\right)^{1/2}.$$

Plainly, E and L are continuous functions with respect to this distance. Next, there is a natural way of endowing this space with the structure of a smooth Hilbert manifold. We will not discuss the details of this construction, for which the interested reader is referred to [Kli95, § 2.3] or [PT88, ch. 11]. It turns out that E becomes a smooth function and the first and second variation formulas correspond to its first two derivatives. The main results of this chapter can then be fashioned in the context of Morse theory in Hilbert spaces.

§2 In 1921-30, in the three editions of Blaschke's book [Bla30], it was discussed the problem of whether it is true that a closed surface in $\mathbf{R}^3$ with the property that the first conjugate locus of any point reduces to a single point must be isometric to S^2; he called surfaces with this property *wiedersehens* surfaces. Blaschke studied a number of features of these surfaces and showed, among other things, that: they can be equivalently defined by requiring that the first conjugate point always occurs at the same distance; all of their geodesics are closed and of the same length (hence their name in German); they are homeomorphic to S^2. Of course, if we admit abstract 2-dimensional Riemannian manifolds, then $\mathbf{R}P^2$ also shares this property. In 1963, L. Green [Gre63] proved that S^2 and $\mathbf{R}P^2$ are indeed the only examples. Later, the work of Weinstein [Wei74], Berger-Kazdan [BK80] and Yang [Yan80] extended this result to all dimensions proving that a simply-connected n-dimensional wiedersehens manifold is isometric to S^n.

§3 More generally, it is natural to ask to which extent the conjugate locus structure restricts the topological, differentiable or metric structure of a n-dimensional Riemannian manifold M [War67]. The case of empty conjugate locus will be discussed in the additional notes of chapter 6. The case in which the first tangential conjugate locus of every point $p \in M$ is a round hypersphere in (T_pM, g_p) of the same radius is exactly the subject of §2 above. Consider now the case in which the first tangential conjugate locus of every p is a round sphere in T_pM of the same radius but the multiplicity of the corresponding conjugate points is possibly less than maximal. Namely, we assume that there exists a number $\ell > 0$ and an integer k between 1 and $n-1$ such that, for every $p \in M$ and every geodesic starting at p, the first conjugate point of p occurs at distance ℓ and has multiplicity k; such a manifold is called an *Allamigeon-Warner manifold* [Bes78, chap. 5]. We have already seen that S^n and $\mathbf{C}P^n$ are examples of simply-connected Allamigeon-Warner manifolds; other examples are the quaternionic projective spaces $\mathbf{H}P^n$ and the Cayley projective plane $\mathbf{Ca}P^2$, manifolds that we will discuss later in this book (indeed, we will see that the spheres S^n and the compact projective spaces $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathbf{Ca}P^2$ are collectively known as the *compact rank one symmetric spaces*). Non-simply-connected examples are given by quotients of those; for instance, $\mathbf{R}P^n$ and lens spaces.

§4 A somehow more specialized condition on a manifold is requiring that the cut-locus structure of each point be similar to that of a compact rank one symmetric space; see [Bes78, chap. 5]. Namely, for distinct points p and q in a complete Riemannian manifold M, the *link from p to q* is the subset $\Lambda(p,q)$ of the unit sphere U_qM of T_qM comprised of vectors of the form $-\gamma'(d(p,q)) \in T_qM$, where $\gamma : [0, d(q,p)] \to M$ is a unit speed minimizing geodesic joining p to q. A compact Riemannian manifold M is called a *Blaschke manifold* if for every $p \in M$ and $q \in \mathrm{Cut}(p)$, the link $\Lambda(p,q)$ is a great sphere of U_qM; here it is not required that the tangential cut-locus at a point is a round sphere, but this follows from the definition. It is known that a Blaschke manifold is Allamigeon-Warner, and both concepts are equivalent in the simply-connected case. Note that $\Lambda(p,q)$ equals U_qM for S^n, it consists of two antipodal points of U_qM for $\mathbf{R}P^n$, and it consists of a great circle of U_qM for $\mathbf{C}P^n$. One sees that $\Lambda(p,q)$ is a great 3-sphere of U_qM for $\mathbf{H}P^n$ and a great 7-sphere of U_qM for $\mathbf{Ca}P^2$. The *Blaschke conjecture* asserts that every Blaschke manifold is isometric to a compact rank one symmetric space. This is one of the famous yet open problems in geometry, with many partial results proved. The book [Bes78] contains a discussion of this conjecture as well as more general discussions of Riemannian manifolds all of whose geodesics are closed; see [Rez94] for a more recent bibliography.

5.8 Exercises

1 Let $\gamma : [a,b] \to M$ be a geodesic parametrized with unit speed in a Riemannian manifold M, and let H be a piecewise smooth variation of γ with associated variational vector field Y. Show that

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} L(\gamma_t) &= \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\Big|_{t=0}, \gamma' \rangle \Big|_a^b + \int_a^b ||Y'||^2 + \langle R(\gamma', Y)\gamma', Y\rangle - \langle Y', \gamma'\rangle^2 \, ds \\ &= \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\Big|_{t=0}, \gamma' \rangle \Big|_a^b + \int_a^b ||Y_\perp'||^2 + \langle R(\gamma', Y_\perp)\gamma', Y_\perp\rangle \, ds, \end{aligned}$$

where $Y_\perp = Y - \langle Y, \gamma'\rangle\gamma'$ is the normal component of Y.

2 Let $\gamma : [0,\ell] \to M$ be a geodesic in a Riemannian manifold M. Consider the index form I on the space of piecewise smooth vector fields along γ vanishing at 0 and ℓ. Prove that the kernel of I consists precisely of the Jacobi fields along γ vanishing at 0 and ℓ. (Hint: Use the formula (5.4.1), and for a given element Y in the kernel of I, choose suitable elements X as it was done in the proof of Proposition 5.3.5).

3 Let $\gamma : [0,\ell] \to M$ be a geodesic in a Riemannian manifold M. Extend the definition of the index form I to the space of piecewise smooth vector fields along γ non-necessarily vanishing at the endpoints. Prove that if γ is a minimizing geodesic, X is a smooth vector field along γ, and Y is a Jacobi vector field along γ with the same values as X at the endpoints, then $I(X,X) \geq I(Y,Y)$.

4 Let N_1 and N_2 be two closed submanifolds of a complete Riemannian manifold M. Assume that one of N_1, N_2 is compact.

a. Prove that there exist points $p_1 \in N_1$ and $p_2 \in N_2$ such that $d(N_1,N_2) = d(p_1,p_2)$.

b. Prove that there exists a geodesic γ of M joining p_1 and p_2 and that $L(\gamma) = d(N_1,N_2)$.

c. Prove that γ is perpendicular to N_1 (resp. N_2) at p_1 (resp. p_2). (Hint: Use the first variation formula.)

5 Let $\gamma : [a, b] \to M$ be a geodesic in a Riemannian manifold, and let $\gamma(a) = p$ and $\gamma(b) = q$. Prove that if p and q are not conjugate along γ, then given $u \in T_pM$ and $v \in T_qM$, there exists a unique Jacobi field J along γ such that $J(a) = u$ and $J(b) = v$.

6 Let M be a Riemannian manifold, and let X be a Killing field on M.

a. If γ is a geodesic in M, prove that the restriction $J = X \circ \gamma$ of X to a vector field along γ is a Jacobi field.

b. If M is complete and $p \in M$, prove that X is completely determined by the values of $X(p) \in T_pM$ and $(\nabla X)_p \in \mathrm{End}(T_pM)$.

c. Deduce from part (b) that the dimension of the Lie algebra of Killing fields on M is bounded by $\frac{1}{2}n(n+1)$, where $n = \dim M$.

7 Let M be a Riemannian manifold and let X be a Killing field on M. Prove that

$$\nabla_U \nabla_V X - \nabla_{\nabla_U V} X + R(X, U)V = 0$$

for all smooth vector fields U and V on M. (Hint: Use Exercise 6(a).)

8 Let (M, g) be a Riemannian manifold, fix $p \in M$ and choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM. Let $\epsilon > 0$ be such that $\exp_p : B(0_p, \epsilon) \subset T_pM \to M$ is a diffeomorphism onto its image U, and use it to define a local coordinates $x^1, \ldots, x^n$ around p. Let $v \in T_pM$ be a unit vector and consider the geodesic $t \mapsto \exp_p(tv)$. Show that the coefficients of the metric in this chart admit expansions

$$g_{ij}(\exp_p tv) = \delta_{ij} + \langle R(v, e_i)v, e_j \rangle \frac{t^2}{3} + \mathrm{O}(t^3),$$

where $1 \le i, j \le n$, $0 < t < \epsilon$, and $\mathrm{O}(t^3)$ denotes a term such that $\mathrm{O}(t^3)/t^2 \to 0$ as $t \to 0$. (Hint: Use the result of Scholium 5.4.5.)

9 Let (M, g) be a compact Riemannian manifold.

a. Prove that if the Ricci tensor of M is negative definite everywhere, then the isometry group $\mathrm{Iso}(M, g)$ is finite. (Hint: Use exercise 7 and the divergence theorem (exercise 14 in chapter 4) to show that there are no nontrivial Killing fields on M.)

b. Prove that if the Ricci tensor of M is negative semi-definite everywhere, then any Killing field is parallel.

10 Let G be a Lie group equipped with a bi-invariant metric. Use exercise 12 of chapter 2 and exercise 6(a) above to show that the restriction of a left-invariant or right-invariant vector field along a geodesic γ is a Jacobi field. Deduce that a general Jacobi field along γ has the form $J_1 + J_2$, where $J_1 = X_1 \circ \gamma$, $J_2 = X_2 \circ \gamma$, X_1 is left-invariant and X_2 is right-invariant. Reconcile this result with formula (5.6.2).

11 Let M be a Riemannian manifold.

a. Prove that the "cut-distance" function $\rho : UM \to (0, +\infty]$ is upper semi-continuous. (Hint: For $v_i \to v$, prove that $\limsup \rho(v_i) \le \rho(v)$ using the continuity of the distance function d on $M \times M$).

b. Assume now that M is complete and prove that ρ is continuous. (Hint: for $v_i \to v$, prove that $\liminf \rho(v_i) \ge \rho(v)$ using ideas from the proof of Proposition 5.5.5.)

c. Deduce from part (b) that the injectivity radius inj_p depends continously on p.