CHAPTER 4

Curvature

4.1 Introduction

The curvature of a plane curve is the measure of change of the direction of the curve. Assuming the curve parametrized by arc-length and expressing this direction as a unit tangent vector along the curve exhibits the (unsigned) curvature as the modulus of the second derivative of the curve. In the case of a surface in $\mathbf{R}^3$, Gauss had already shown how to measure curvature: this is the rate of change of the normal direction of the surface. Locally, one chooses a unit normal vector field and differentiates it at a point as a map into the unit sphere. Since the surface is two-dimensional, the result is now a map, namely a linear endomorphism of the tangent space at that point. This turns out to be symmetric, hence diagonalizable over $\mathbf{R}$. Its eigenvalues are called the principal curvatures λ_1 and λ_2. They represent the extreme values of the curvatures of the plane curves given by the normal sections to the surface. Equivalently, one can look at $2H = \lambda_1 + \lambda_2$ and $K = \lambda_1\lambda_2$. The second expression is called the Gaussian curvature and, according to Gauss' celebrated *theorema egregium*, has an intrinsic meaning in the sense that it can be expressed solely in terms of the coefficients of the metric in a coordinate system.

Riemann generalized Gauss' results and explained how to define the curvature of a Riemannian manifold M. Here the dimension of M is at least two, so we start by selecting a 2-plane E contained in T_pM. Exponentiating a small neighborhood of 0_p in E gives a piece of surface S through p contained in M. The curvature of M at E is defined to be the Gaussian curvature of S at p. This gives the sectional curvature function.

As it is, this definition cannot be very useful: it is difficult to compute and, especially, it does not reflect relations between the sectional curvatures of neighboring planes. After Riemann, the matter took a few decades more of study to be settled, until tensor calculus entered the scene.

Throughout this chapter, (M, g) denotes a Riemannian manifold and ∇ denotes its Levi-Cività connection.

4.2 The Riemann-Christoffel curvature tensor

The *curvature tensor* is the tri-linear map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

It is an easy consequence of the Leibniz rule for ∇ that R is $C^\infty(M)$-linear on each argument. As in the case of connections, this suffices to show that the value of $R(X, Y)Z$ at p depends only on X_p, Y_p, and Z_p. Hence we have a tri-linear map

$$R_p : T_pM \times T_pM \times T_pM \to T_pM.$$

The following are the fundamental symmetries of this map.

4.2.1 Proposition (algebraic properties of the curvature tensor) *We have that*
a. $R(X,Y)Z = -R(Y,X)Z$
b. $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$
c. $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$
d. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(first Bianchi identity)*
for every X, Y, Z, $W \in \Gamma(TM)$.

Proof. (a) This is clear from the definition.

(b) We compute

$$\begin{aligned}
\langle R(X,Y)Z,Z\rangle &= \langle\nabla_X\nabla_Y Z,Z\rangle - \langle\nabla_Y\nabla_X Z,Z\rangle - \langle\nabla_{[X,Y]}Z,Z\rangle \\
&= X\langle\nabla_Y Z,Z\rangle - \langle\nabla_Y Z,\nabla_X Z\rangle \\
&\qquad - \big(Y\langle\nabla_X Z,Z\rangle - \langle\nabla_X Z,\nabla_Y Z\rangle\big) - \frac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= \frac{1}{2}XY\langle Z,Z\rangle - \frac{1}{2}YX\langle Z,Z\rangle - \frac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= 0,
\end{aligned}$$

where we have used several times the compatibility of the Levi-Cività connection with the metric. The identity follows.

(d) We compute

$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\
&\qquad + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\
&\qquad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\
&= \nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_{[X,Y]}Z \\
&\qquad + \nabla_Y(\nabla_Z X - \nabla_X Z) - \nabla_{[Y,Z]}X \\
&\qquad + \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[Z,X]}Y \\
&= \nabla_X[Y,Z] - \nabla_{[Y,Z]}X \\
&\qquad + \nabla_Y[Z,X] - \nabla_{[Z,X]}Y \\
&\qquad + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
&= 0,
\end{aligned}$$

where we have used the fact that the Levi-Cività connection is torsionless several times, and the Jacobi identity in the last line.

(c) We use (a), (b) and (d) to compute

$$\begin{aligned}
\langle R(X,Y)Z,W\rangle &= -\langle R(Y,Z)X,W\rangle - \langle R(Z,X)Y,W\rangle \\
&= \langle R(Y,Z)W,X\rangle + \langle R(Z,X)W,Y\rangle \\
&= -\langle R(Z,W)Y,X\rangle - \langle R(W,Y)Z,X\rangle - \langle R(X,W)Z,Y\rangle - \langle R(W,Z)X,Y\rangle \\
&= 2\langle R(Z,W)X,Y\rangle + \langle R(W,Y)X + R(X,W)Y,Z\rangle \\
&= 2\langle R(Z,W)X,Y\rangle - \langle R(Y,X)W,Z\rangle \\
&= 2\langle R(Z,W)X,Y\rangle - \langle R(X,Y)Z,W\rangle,
\end{aligned}$$

which gives the result. $\square$

Let $p \in M$ and let $E \subset T_pM$ be a 2-plane. The *sectional curvature* of M at E is defined to be

$$K(E) = K(x,y) = \frac{-\langle R_p(x,y)x, y\rangle}{||x||^2||y||^2 - \langle x, y\rangle^2},$$

where $\{x, y\}$ is a basis of E. One checks that this expression does not depend on the choice of basis of E as follows. It is very easy to see that $K(y,x)$, $K(\lambda x, y)$ $(\lambda \neq 0)$, $K(x+y, y)$ are all equal to $K(x,y)$. But one can get from $\{x, y\}$ to any other basis of E by performing a number of times the simple transformations

$$\left\{\begin{array}{l} x \mapsto y \\ y \mapsto x \end{array}\right., \quad \left\{\begin{array}{l} x \mapsto \lambda x \\ y \mapsto y \end{array}\right., \quad \left\{\begin{array}{l} x \mapsto x + y \\ y \mapsto y \end{array}\right..$$

4.2.2 Proposition *We have the following identity*

$$\begin{aligned} &\langle R_p(x,y)z, w\rangle \\ &= \frac{1}{6}\frac{\partial^2}{\partial\alpha\partial\beta}\left(\langle R_p(x+\alpha z, y+\beta w)(x+\alpha z), y+\beta w\rangle - \langle R_p(x+\alpha w, y+\beta z)(x+\alpha w), y+\beta z\rangle\right), \end{aligned}$$

where x, y, z, $w \in T_pM$.

Proof. By direct computation. $\square$

It is important to remark that the identity in the preceding proposition is proved using only the algebraic properties of the curvature tensor. Of course, the next corollary is of an algebraic nature as well.

4.2.3 Corollary *The sectional curvature function $E \mapsto K(E)$ and the metric at a point p determine the curvature tensor at p.*

A Riemannian manifold (M, g) of dimension $n \geq 2$ is said to have *constant curvature* κ if for every point $p \in M$ and every 2-plane $E \subset T_pM$, the sectional curvature at E equals κ. A Riemannian manifold (M, g) of dimension $n \geq 2$ is called *flat* if it has constant curvature κ and $\kappa = 0$. This terminology is consistent with the one introduced in section 1.3: since local isometries must preserve the sectional curvature (see end of this section), a Riemannian manifold locally isometric to Euclidean space must have vanishing sectional curvatures; conversely, we will see in chapter 6 that a Riemannian manifold with vanishing sectional curvatures is locally isometric to Euclidean space. A one-dimensional Riemannian manifold is also called flat, although its tangent spaces do not contain 2-planes, since in this case we have $R \equiv 0$ by Proposition 4.2.1(a). A Riemannian manifold is said to have *positive curvature* (resp. *negative curvature*) if the sectional curvature function is positive (resp. negative) everywhere.

If $\dim M = 2$, then a 2-plane E must coincide with T_pM, and then we have a scalar-valued function $K(p) = K(T_pM)$, which can be shown to coincide with the Gaussian curvature of M in the case in which M is a surface in $\mathbf{R}^3$ equipped with the induced metric (cf. Add. notes §2).

Next, suppose that $\dim M \geq 3$. In this case, we say that M has *isotropic curvature at a point* p if $K(E) = \kappa_p$ for every 2-plane $E \subset T_pM$, where κ_p is a real constant. From the definition of sectional curvature, we have that

$$\langle R_p(x,y)x, y\rangle = -\kappa_p\left(||x||^2||y||^2 - \langle x, y\rangle^2\right)$$

for all $p \in M$ and $x, y \in T_pM$. Set

$$\langle R^0_p(x,y)z, w\rangle = -\langle x,z\rangle\langle y,w\rangle + \langle x,w\rangle\langle y,z\rangle,$$

where $p \in M$ and $x, y, z, w \in T_pM$. Then R^0 is a tensor that has the same symmetries as R. Corollary 4.2.3 implies that

$$R_p = \kappa_p R^0_p. \tag{4.2.4}$$

Obviously, a Riemannian manifold with constant curvature has isotropic curvature at all points. It is a result due to Schur that the converse is true in dimensions at least 3.

4.2.5 Lemma (Schur) *Let M be a connected Riemannian manifold. If M has isotropic curvature at all points and* $\dim M \geq 3$, *then it has constant curvature.*

We will prove the above lemma in section 4.4. Note that the curvature tensor of a Riemannian manifold of constant curvature satisfies identity (4.2.4) where κ_p does not depend on p. We also remark that local isometries must preserve the curvature tensor in the following sense, as is easily seen by using arguments from section 2.5. If $f : M \to N$ is a local isometry between two Riemannian manifolds, then

$$R_{f(p)}(df_p(X_p), df(Y_p))df_p(Z_p) = R_p(X_p, Y_p)Z_p \tag{4.2.6}$$

for every $p \in M$ and every $X, Y, Z \in \Gamma(TM)$. Of course, it also follows that $K(df(E)) = K(E)$ for every 2-plane E contained in T_pM and every $p \in M$.

4.2.7 Remark Let $\varphi : N \to M$ be a smooth map, let $X, Y \in \Gamma(TN)$ be vector fields in N and let $U \in \Gamma(\varphi^*TM)$ be a vector field along φ. Recall the induced connection along φ that was introduced in Proposition 2.6.1. Then one can check that the following identity holds:

$$R(\varphi_*X, \varphi_*Y)U = \nabla^\varphi_X\nabla^\varphi_Y U - \nabla^\varphi_Y\nabla^\varphi_X U - \nabla^\varphi_{[X,Y]}U.$$

4.3 The Ricci tensor and scalar curvature

One can say that the Riemann curvature tensor contains so much information about the Riemannian manifold that it makes sense to consider also some simpler tensors derived from it, and these are the Ricci tensor and the scalar curvature.

The *Ricci tensor* Ric at a point $p \in M$ is the bilinear map $\mathrm{Ric}_p : T_pM \times T_pM \to \mathbf{R}$ given by

$$\mathrm{Ric}_p(x,y) = \mathrm{trace}\,(v \mapsto -R_p(x,v)y),$$

where $x, y \in T_pM$. Note that the Ricci tensor is defined directly in terms of the curvature tensor without involving the metric. It follows immediately from the symmetries of the curvature tensor given by Proposition 4.2.1 that Ric is symmetric, namely,

$$\mathrm{Ric}_p(x,y) = \mathrm{Ric}_p(y,x)$$

for $x, y \in T_pM$ and $p \in M$. So the Ricci tensor is of the same type as the metric tensor g, and it makes sense to compare the two. An *Einstein manifold* is a Riemannian manifold whose Ricci tensor is proportional to the metric. If $\dim M \geq 3$, it follows from Exercise 4 that the constant of proportionality is independent of the point, and hence the condition is that there exists $\lambda \in \mathbf{R}$ such that

$$\mathrm{Ric} = \lambda\, g.$$

Riemannian manifolds satisfying $\mathrm{Ric} = 0$ are called *Ricci-flat*. Of course, a Riemannian manifold of constant sectional curvature is Einstein, and a flat Riemannian manifold is Ricci-flat.

We can also use the metric to view the Ricci tensor at $p \in M$ as a linear map $T_pM \to T_pM$ by setting

$$\langle \mathrm{Ric}(x), y\rangle = \mathrm{Ric}(x, y).$$

for x, $y \in T_pM$. Then it makes sense to take the trace of Ric: the *scalar curvature* is the smooth function $\mathrm{scal} : M \to \mathbf{R}$ given by

$$\mathrm{scal}(p) = \operatorname{trace} \mathrm{Ric}_p,$$

where $p \in M$.

Fix a point $p \in M$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM. Then

$$\mathrm{Ric}_p(x, y) = -\sum_{j=1}^{n} \langle R(x, e_j)y, e_j\rangle,$$

where x, $y \in T_pM$. In particular, if x is a unit vector, we can assume that $e_1 = x$ and then

$$\mathrm{Ric}_p(x, x) = \sum_{j=2}^{n} K(x, e_j). \tag{4.3.1}$$

The quadratic form (4.3.1) is sometimes called the *Ricci curvature*; of course, its values on the unit sphere of T_pM completely determine the Ricci tensor at p, and (4.3.1) shows that $\mathrm{Ric}_p(x, x)$ is the (unnormalized) average of the sectional curvatures of the 2-planes containing x. We also have that

$$\mathrm{scal}(p) = \sum_{i=1}^{n} \mathrm{Ric}_p(e_i, e_i) = \sum_{i \neq j} K(e_i, e_j) = 2 \sum_{i<j} K(e_i, e_j),$$

and this equation shows that the scalar curvature at p is the (unnormalized) average of the sectional curvatures of the 2-planes in T_pM.

4.4 Covariant derivative of tensors ★

At this juncture, we feel like it is time to discuss how to differentiate tensors on a manifold. If M is a Riemannian manifold, there is a canonical way of differentiating smooth vector fields on M, namely, this is given by the Levi-Cività connection ∇. Viewing vector fields as tensor fields of type $(1, 0)$, we can prove that ∇ naturally extends to connections on all tensor bundles $T^{(r,s)}M$. Denote by $c : T^{(r,s)}M \to T^{(r-1,s-1)}M$ an arbitrary contraction.

4.4.1 Proposition *There is a unique family of connections on the tensor bundles $T^{(r,s)}M$ for r, $s \geq 0$, still denoted by ∇, such that the following conditions hold for $X \in \Gamma(TM)$:*

a. $\nabla_X f = Xf$ *for* $f \in C^\infty(M) = \Gamma(T^{(0,0)}M)$*;*

b. $\nabla_X Y$ *for* $Y \in \Gamma(TM)$ *is the covariant derivative associated to the Levi-Cività connection;*

c. ∇_X *commutes with contractions, that is,* $\nabla_X c(T) = c(\nabla_X T)$ *for* $T \in \Gamma(T^{(r,s)}M)$ *with* r, $s > 0$*;*

d. ∇_X *is a derivation, that is,* $\nabla_X(T \otimes T') = \nabla_X T \otimes T' + T \otimes \nabla_X T'$ *for* $T \in \Gamma(T^{(r,s)}M)$ *and* $T' \in \Gamma(T^{(r',s')}M)$.

Proof. One first proves uniqueness, as follows. Let $X \in \Gamma(TM)$ and assume ∇_X is defined and satisfies the conditions in the statement. Using the same argument as in Subsection 2.2, for an open subset U of M we see that if two tensor fields T, $T' \in \Gamma(T^{(r,s)}M)$ coincide on U then $\nabla_X T$ and $\nabla_X T'$ also coincide on U.

It is now enough to show that $\nabla_X(T|_U)$ is uniquely defined. Write T is a coordinate system $(U, x^1, \dots, x^n)$ as

$$T|_U = \sum a^{i_1\cdots i_r}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots dx^{j_s},$$

where $a^{i_1\cdots i_r}_{j_1\cdots j_s} \in C^\infty(U)$. The Leibniz rule (d) then gives a formula for $\nabla_X(T|_U)$ is terms of the action of ∇_X on functions, vector fields and 1-forms; the first two cases are taken care by (a) and (b), so we need only show that $\nabla_X\omega$ is uniquely defined for a 1-form ω on M. For that purpose, let $Y \in \Gamma(TM)$ and compute, using (a), (b), (c) and (d):

$$\begin{aligned} \nabla_X\omega(Y) &= c(\nabla_X\omega \otimes Y) \\ &= c(\nabla_X(\omega \otimes Y) - \omega \otimes \nabla_X Y) \\ &= \nabla_X c(\omega \otimes Y) - \omega(\nabla_X Y) \\ &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y), \end{aligned}$$

where c denotes the obvious contraction. Since the last line of this equation is $C^\infty(M)$-linear with respect to Y, yields $\nabla_X\omega$ as a 1-form.

For the existence, one first defines for $\omega \in \Gamma(T^{(0,s)}M)$

$$\begin{aligned} &\nabla_X\omega(X_1, \dots, X_s) \\ &= X(\omega(X_1, \dots, X_s)) - \sum_{i=1}^s \omega(X_1, \dots, \nabla_X X_i, \dots, X_s). \end{aligned}$$

Next, for $T \in \Gamma(T^{(r,s)}M)$, note that $T(\omega_1, \dots, \omega_r) \in \Gamma(T^{(0,s)}M)$ for $\omega_i, \dots, \omega_r \in \Gamma(T^*M)$, so we can define

$$\begin{aligned} &\nabla_X T(\omega_1, \dots, \omega_r) \\ &= \nabla_X(T(\omega_1, \dots, \omega_r)) - \sum_{i=1}^r T(\omega_1, \dots, \nabla_X\omega_i, \dots, \omega_s). \end{aligned}$$

We leave to the reader to check that this definition satisfies (c) and (d). □

As a first application of Proposition 4.4.1, we view g as a tensor field of type $(0, 2)$ and note that the condition that the Levi-Cività connection be compatible with the metric (Proposition 2.2.5(b)) can be restated as simply saying that $\nabla g = 0$, since

$$\nabla_X g(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

This is referred to as the *parallelism of the metric*.

As another application of the Proposition 4.4.1, we prove the *second Bianchi identity* in Proposition 4.4.3 below. Since R is $C^\infty(M)$-linear in each variable, we can view it as a tensor field of type $(1, 3)$, namely,

$$\begin{array}{rcl} \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(T^*M) & \to & \mathbf{R} \\ (X, Y, Z, \omega) & \to & \omega(R(X, Y)Z). \end{array}$$

Conversely, $\nabla_X R$, as a tensor of type $(1,3)$, can be viewed as a map

$$\Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM).$$

It now follows from the definition of ∇_X acting on $\Gamma(T^{(1,3)}M)$ that we have

$$\nabla_X R(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W. \tag{4.4.2}$$

4.4.3 Proposition (Second Bianchi identity) *We have that*

$$\nabla_X R(Y,Z)W + \nabla_Y R(Z,X)W + \nabla_Z R(X,Y)W = 0 \tag{4.4.4}$$

for every X, Y, Z, $W \in \Gamma(TM)$.

Proof. Dropping the W in (4.4.2) and using the identity $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we get

$$\begin{aligned} \nabla_X R(Y,Z) &= [\nabla_X, R(Y,Z)] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z) \\ &= [\nabla_X, [\nabla_Y, \nabla_Z]] - [\nabla_X, \nabla_{[Y,Z]}] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z) \\ &= [\nabla_X, [\nabla_Y, \nabla_Z]] - \nabla_{[X,[Y,Z]]} - R(X, [Y,Z]) - R(\nabla_X Y, Z) - R(Y, \nabla_X Z). \end{aligned}$$

Summing this formula with the other two obtained by cyclic permutation of (X,Y,Z), we see that the first two terms on the right hand side cancel out because of the Jacobi identity, and invoking the relation $\nabla_X Y - \nabla_Y X = [X,Y]$ also makes remaining terms also disappear. The identity is proved. □

Finally, we use the second Bianchi identity to prove Lemma 4.2.5.

Proof of Lemma 4.2.5. We view $\kappa_p = \kappa(p)$ as a function on M. Note that formula (4.2.4) implies that this function is smooth. We use that formula to get

$$\nabla_X R(Y,Z)W = (X\kappa)R^0(Y,Z)W + \kappa \nabla_X R^0(Y,Z)W.$$

Summing over the cyclic permutations of (X,Y,Z), we have

$$(X\kappa)R^0(Y,Z)W + (Y\kappa)R^0(Z,X)W + (Z\kappa)R^0(X,Y)W = 0$$

by an application of the second Bianchi identity (4.4.4) to R and R^0. Let X be an arbitrary unit vector field. As $\dim M \geq 3$, we can select Y, Z so that $\{X,Y,Z\}$ is orthonormal. Also, put $W = Y$. Then

$$X\kappa = 0.$$

The connectedness of M implies that κ is constant, as desired. □

4.4.5 Remark The *musical isomorphisms* are defined as follows. For each vector field X on the Riemannian manifold (M,g), one can define the differential 1-form ω given by $\omega(Y) = g(X,Y)$. Note that smoothness of g implies that ω is indeed smooth, and non-degeneracy of g at each point implies that this defines an isomorphism between spaces of sections $\flat : \Gamma(TM) \to \Gamma(T^*M)$, the *flat*, so that $\omega = X^\flat$. The inverse isomorphism is naturally called the *sharp*, denoted $\sharp$, so that $X = \omega^\sharp$. The flat and sharp isomorphisms extend to define isomorphisms $\Gamma(T^{(r,s)}M) \to \Gamma(T^{(r',s')}M)$ for $r + s = r' + s'$ and, as is easily seen, the parallelism of the metric implies that these isomorphisms commute with the covariant derivatives on $\Gamma(T^{(r,s)}M)$ and $\Gamma(T^{(r',s')}M)$. As an example, the curvature tensor R can be viewed as a $(0,4)$ tensor, namely, $R(X,Y,Z,W) = W^\flat(R(X,Y)Z) = g(R(X,Y)Z,W)$.

4.5 Examples

Flat manifolds

Euclidean space is flat, since

$$R(X,Y)Z = X(Y(Z)) - Y(X(Z)) - [X,Y](Z) = 0.$$

Since local isometries must preserve the curvature, it follows that the tori $\mathbf{R}^n/\Gamma$ are also flat.

S^n and $\mathbf{R}P^n$

Since S^n is a Riemannian submanifold of $\mathbf{R}^{n+1}$, for its Levi-Cività connection we have that

$$\nabla_X Y = X(Y) - \langle X(Y), \mathbf{p}\rangle \mathbf{p}, \tag{4.5.1}$$

where X, $Y \in \Gamma(TS^n)$ and we have denoted by $\mathbf{p}$ the position vector. It follows that

$$\begin{aligned} \nabla_X\nabla_Y Z &= X(\nabla_Y Z) - \langle X(\nabla_Y Z), \mathbf{p}\rangle\mathbf{p} \\ &= XY(Z) - \langle XY(Z), \mathbf{p}\rangle\mathbf{p} - \langle Y(Z), X\rangle\mathbf{p} - \langle Y(Z), \mathbf{p}\rangle X \\ &\qquad -\langle XY(Z), \mathbf{p}\rangle\mathbf{p} + \langle XY(Z), \mathbf{p}\rangle\mathbf{p} + \langle Y(Z), X\rangle\mathbf{p} \\ &= XY(Z) - \langle XY(Z), \mathbf{p}\rangle\mathbf{p} + \langle Z, Y\rangle X \end{aligned}$$

where we have used that $\langle Y(Z), \mathbf{p}\rangle = -\langle Z, Y\mathbf{p}\rangle = -\langle Z, Y\rangle$ since $\langle Z, \mathbf{p}\rangle = 0$. Therefore,

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y. \tag{4.5.2}$$

Comparing with (4.2.4) shows we have proved that S^n has constant curvature 1. Since $\mathbf{R}P^n$ is isometrically covered by S^n, it also has constant curvature 1.

$\mathbf{R}H^n$

Consider the hyperboloid model of $\mathbf{R}H^n$ sitting inside the Lorentzian space $\mathbf{R}^{1,n}$. Although the metric in the ambient space is now Lorentzian, the Levi-Cività connection of $\mathbf{R}H^n$ is given by a formula very similar to (4.5.1), namely, the tangential component of the ambient derivative:

$$\nabla_X Y = X(Y) + \langle X(Y), \mathbf{p}\rangle\mathbf{p}.$$

Indeed, one cheks easily that this formula specifies a connection on $\mathbf{R}H^n$ that satisfies the defining conditions for the Levi-Cività connection. A computation very similar to that in the case of S^n thus gives that

$$R(X,Y)Z = -\langle Y, Z\rangle X + \langle X, Z\rangle Y. \tag{4.5.3}$$

Hence $\mathbf{R}H^n$ has constant curvature -1.

Riemannian products

Let $(M,g) = (M_1,g_1) \times (M_2,g_2)$ be a Riemannian product. It follows immediately from the description of the Levi-Cività connection on M for decomposable vector fields (2.8.1) that the curvature tensor of M is given by

$$R_p(x,y)z = R^1_{p_1}(x_1,y_1)z_1 + R^2_{p_2}(x_2,y_2)z_2,$$

where x, y, $z \in T_pM$ for $p = (p_1, p_2) \in M_1 \times M_2$, $x = x_1 + x_2$, $y = y_1 + y_2$, $z = z_1 + z_2$ are the decompositions relative to the splitting $T_pM = T_{p_1}M_1 \oplus T_{p_2}M_2$, and R^i denotes the curvature tensor of M^i.

In particular,

$$g(R_p(x_1, y_2)x_1, y_2) = g_1(R^1_{p_1}(x_1, 0)x_1, 0) + g_2(R^2_{p_2}(0, y_2)0, y_2) = 0.$$

This shows that a *mixed plane* in M, i.e. a plane with nonzero components in both M_1 and M_2, has sectional curvature equal to zero. It also shows that the product of two positively curved Riemannian manifolds has non-negative curvature.

Riemannian submersions and $\mathbf{C}P^n$

Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion and consider the splitting $T\tilde{M} = \mathcal{H} \oplus \mathcal{V}$ into the horizontal and vertical distributions. A vector field $\tilde{X}$ on $\tilde{M}$ is called:

- *horizontal* if $\tilde{X}_{\tilde{p}} \in \mathcal{H}_{\tilde{p}}$ for all $\tilde{p} \in \tilde{M}$;
- *vertical* if $\tilde{X}_{\tilde{p}} \in \mathcal{V}_{\tilde{p}}$ for all $\tilde{p} \in \tilde{M}$;
- *projectable* if, for fixed $p \in M$, $d\pi(\tilde{X}_{\tilde{p}})$ is independent of $\tilde{p} \in \pi^{-1}(p)$;
- *basic* if it is horizontal and projectable.

Note that if $\tilde{X}$ is a smooth projectable vector field on $\tilde{M}$, then it defines a smooth vector field X on M by setting $X_p = d\pi(X_{\tilde{p}})$ for any $\tilde{p} \in \pi^{-1}(p)$; in this case, $\tilde{X}$ and X are π-related. It also follows from the definitions that a vertical vector field is projectable and, indeed, a vector field on $\tilde{M}$ is vertical if and only if it is π-related to 0.

If X is a smooth vector field on M, it is clear that there exists a unique basic vector field $\tilde{X}$ on $\tilde{M}$ such that $\tilde{X}$ and X are π-related; the vector field $\tilde{X}$ is necessarily smooth and it is called the *horizontal lift* of X.

4.5.4 Lemma *Let $\tilde{X}$, $\tilde{Y}$ be horizontal lifts of X, $Y \in \Gamma(TM)$, resp., and let $U \in \Gamma(T\tilde{M})$ be a vertical vector field. Then the vector fields $[\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}$ and $[U, \tilde{X}]$ are vertical.*

Proof. Since U is π-related to 0 and $\tilde{X}$ is π-related to X, we have that $[U, \tilde{X}]$ is π-related to $[0, X] = 0$. A similar argument proves the other assertion. □

The next proposition describes the Levi-Cività connection $\tilde{\nabla}$ of $\tilde{M}$ in terms of the Levi-Cività connection ∇ of M. Denote by $(\cdot)^v$ the vertical component of a vector field on $\tilde{M}$.

4.5.5 Proposition *Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. If X, $Y \in \Gamma(TM)$ with horizontal lifts $\tilde{X}$, $\tilde{Y} \in \Gamma(T\tilde{M})$, then*

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^v.$$

Proof. Apply the Koszul formula (2.2.6) to $\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z})$, where $\tilde{Z}$ is the horizontal lift of $Z \in \Gamma(TM)$. Since $d\pi$ restricted to each $\mathcal{H}_{\tilde{p}}$ is a linear isometry onto T_pM for $p = \pi(\tilde{p})$,

$$\tilde{X}_{\tilde{p}}\tilde{g}(\tilde{Y}, \tilde{Z}) = X_p g(Y, Z).$$

Also, by the first assertion of Lemma 4.5.4,

$$\tilde{g}_{\tilde{p}}([\tilde{X}, \tilde{Y}], \tilde{Z}) = g_p([X, Y], Z).$$

Hence

$$\tilde g_{\tilde p}(\tilde\nabla_{\tilde X}\tilde Y,\tilde Z)=g_p(\nabla_X Y,Z)=\tilde g_{\tilde p}(\widetilde{\nabla_X Y},\tilde Z). \tag{4.5.6}$$

Next, apply the Koszul formula to $\tilde g(\tilde\nabla_{\tilde X}\tilde Y,U)$, where $U\in\Gamma(T\tilde M)$ is vertical. Since $\tilde g(\tilde X,\tilde Y)$ is constant along the fibers of π, $U\tilde g(\tilde X,\tilde Y)=0$. Using the second assertion of Lemma 4.5.4 yields that

$$\tilde g(\tilde\nabla_{\tilde X}\tilde Y,U)=\frac{1}{2}\tilde g([\tilde X,\tilde Y],U). \tag{4.5.7}$$

The desired result is equivalent to (4.5.6) and (4.5.7). □

The next proposition relates the sectional curvatures of M and $\tilde M$.

4.5.8 Proposition *Let $\pi:(\tilde M,\tilde g)\to(M,g)$ be a Riemannian submersion. If X, $Y\in\Gamma(TM)$ is an orthonormal pair with horizontal lifts $\tilde X$, $\tilde Y\in\Gamma(T\tilde M)$, then*

$$K(X,Y)=\tilde K(\tilde X,\tilde Y)+\frac{3}{4}||[\tilde X,\tilde Y]^v||^2.$$

Proof. We start by observing that for a vertical vector field U on $\tilde M$,

$$\tilde g(\tilde\nabla_{\tilde X}U,\tilde Y)=-\tilde g(U,\tilde\nabla_{\tilde X}\tilde Y)=-\frac{1}{2}\tilde g(U,[\tilde X,\tilde Y]^v)$$

by Proposition 4.5.5, and

$$\tilde g(\tilde\nabla_U\tilde X,\tilde Y)=\tilde g(\tilde\nabla_{\tilde X}U,\tilde Y)+\tilde g([U,\tilde X],\tilde Y)=\tilde g(\tilde\nabla_{\tilde X}U,\tilde Y),$$

by Lemma 4.5.4. Using these identities and (4.5.5) a few times, we have

$$\begin{aligned}\tilde\nabla_{\tilde X}\tilde\nabla_{\tilde Y}\tilde X &= \tilde\nabla_{\tilde X}\left(\widetilde{\nabla_Y X}\right)+\frac{1}{2}\tilde\nabla_{\tilde X}\left([\tilde Y,\tilde X]^v\right)\\ &= \widetilde{\nabla_X\nabla_Y X}+\frac{1}{2}[\tilde X,\widetilde{\nabla_Y X}]^v-\frac{1}{2}\tilde\nabla_{\tilde X}\left([\tilde X,\tilde Y]^v\right),\end{aligned}$$

and

$$\begin{aligned}\tilde g(\tilde\nabla_{\tilde X}\tilde\nabla_{\tilde Y}\tilde X,\tilde Y) &= \tilde g(\widetilde{\nabla_X\nabla_Y X},\tilde Y)-\frac{1}{2}\tilde g(\tilde\nabla_{\tilde X}[\tilde X,\tilde Y]^v,\tilde Y)\\ &= g(\nabla_X\nabla_Y X,Y)+\frac{1}{4}||[\tilde X,\tilde Y]^v||^2\end{aligned}$$

Similarly

$$\tilde g(\tilde\nabla_{\tilde Y}\tilde\nabla_{\tilde X}\tilde X,\tilde Y)=\tilde g(\tilde\nabla_{\tilde Y}\widetilde{\nabla_X X},\tilde Y)=g(\nabla_Y\nabla_X X,Y),$$

and

$$\begin{aligned}\tilde g(\tilde\nabla_{[\tilde X,\tilde Y]}\tilde X,\tilde Y) &= \tilde g(\tilde\nabla_{\widetilde{[X,Y]}}\tilde X,\tilde Y)+\tilde g(\tilde\nabla_{[\tilde X,\tilde Y]^v}\tilde X,\tilde Y)\\ &= g(\nabla_{[X,Y]}X,Y)-\frac{1}{2}||[\tilde X,\tilde Y]^v||^2.\end{aligned}$$

It follows that

$$\tilde g(\tilde R(\tilde X,\tilde Y)\tilde X,\tilde Y)=g(R(X,Y)X,Y)-\frac{3}{4}||[\tilde X,\tilde Y]^v||^2,$$

and this clearly implies the desired formula. $\square$

We now apply the above results to the question of computing the sectional curvature of $\mathbf{C}P^n$. Consider as usual the Riemannian submersion $\pi : \tilde{M} = S^{2n+1} \to M = \mathbf{C}P^n$. We will first define a complex structure on each tangent space to M.[1] Since the horizontal space $\mathcal{H}_{\tilde{p}} \subset T_{\tilde{p}}S^{2n+1}$, for $\tilde{p} \in S^{2n+1}$, is the orthogonal complement of $\mathbf{R}\{\tilde{p}, i\tilde{p}\} = \mathbf{C}\tilde{p}$ in $\mathbf{C}^{2n+1}$, it follows that $\mathcal{H}_{\tilde{p}}$ is a complex vector subspace of $\mathbf{C}^{n+1}$. We transfer the complex structure of $\mathcal{H}_{\tilde{p}}$ to T_pM, where $p = \pi(\tilde{p})$, by conjugation with the isometry $d\pi_{\tilde{p}}|_{\mathcal{H}_{\tilde{p}}} : \mathcal{H}_{\tilde{p}} \to T_pM$, namely we set

$$J_p v = d\pi_{\tilde{p}} \circ J_0 \circ (d\pi_{\tilde{p}}|_{\mathcal{H}_{\tilde{p}}})^{-1}(v) = d\pi(i\tilde{v}),$$

where $J_0 : \mathbf{R}^{2n+2} \to \mathbf{R}^{2n+2}$ is the standard complex structure on $\mathbf{R}^{2n+2}$ that allows us to identify $\mathbf{R}^{2n+2} \cong \mathbf{C}^{n+1}$, and $\tilde{v}$ is the horizontal lift of v at $\tilde{p}$. Let us check that J_p is well defined in the sense that if we had started with a different point $\tilde{p}' \in \pi^{-1}(p)$, we would have gotten the same result. Indeed $\tilde{p}' = z\tilde{p}$ for some $z \in S^1$. Denote by $\varphi_z : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ the multiplication by z. Then $\pi \circ \varphi_z = \pi$ which, via the chain rule, yields that $d\pi_{\tilde{p}'} \circ \varphi_z = d\pi_{\tilde{p}}$ and hence

$$\begin{aligned} d\pi_{\tilde{p}'} \circ J_0 \circ (d\pi_{\tilde{p}'}|_{\mathcal{H}_{\tilde{p}'}})^{-1} &= d\pi_{\tilde{p}} \circ \varphi_z \circ J_0 \circ \varphi_{z^{-1}} \circ (d\pi_{\tilde{p}}|_{\mathcal{H}_{\tilde{p}}})^{-1} \\ &= d\pi_{\tilde{p}} \circ J_0 \circ (d\pi_{\tilde{p}}|_{\mathcal{H}_{\tilde{p}}})^{-1}, \end{aligned}$$

since φ_z maps $\mathcal{H}_{\tilde{p}}$ onto $\mathcal{H}_{\tilde{p}'}$. Next, it is clear that

$$J_p^2 = -\mathrm{id}_{T_pM},$$

so J_p introduces on T_pM the structure of a complex vector space. It is also easy to see that J_p is a linear isometry because

$$g(J_pv, J_pw) = \tilde{g}(i\tilde{v}, i\tilde{w}) = \tilde{g}(\tilde{v}, \tilde{w}) = g(v, w),$$

where v, $w \in T_pM$ and $\tilde{v}$, $\tilde{w} \in \mathcal{H}_{\tilde{p}}$ are their corresponding lifts, and we have used the fact that multiplication by i is an isometry of $\mathbf{C}^{n+1}$. Now consider J_p for varying $p \in \mathbf{C}P^n$. If X is a smooth vector field on $\mathbf{C}P^n$, then, plainly, $JX = d\pi(i\tilde{X})$, and this implies that also JX is a smooth vector field on $\mathbf{C}P^n$. Hence J is a smooth tensor field of type $(1, 1)$ on $\mathbf{C}P^n$. Next, we introduce the vertical vector field ξ by putting

$$\xi(\tilde{p}) = \frac{d}{d\theta}\Big|_{\theta=0}(e^{i\theta}\tilde{p}) = i\tilde{p} = J_0(\tilde{p}). \tag{4.5.9}$$

Note that ξ is a smooth, unit vector field on S^{2n+1}. Then $\tilde{X}(\xi) = J_0(\tilde{X}) = i\tilde{X}$, so using the expression of the Levi-Cività connection in S^{2n+1} (4.5.1), we have

$$\begin{aligned} \tilde{\nabla}_{\tilde{X}}\xi &= \tilde{X}(\xi) - \langle \tilde{X}(\xi), \mathbf{p}\rangle \mathbf{p} \\ &= i\tilde{X} - \langle i\tilde{X}, \mathbf{p}\rangle \mathbf{p} \\ &= i\tilde{X}, \end{aligned}$$

[1] For a real vector space V, a *complex structure* is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. A complex structure J on V allows one to view V as a complex vector space with half the real dimension of V, namely, one puts $(a + ib)v = av + bJv$ for all a, $b \in \mathbf{R}$, $v \in V$. A complex structure on V can exist only if the dimension of V is even (since $(\det J)^2 = (-1)^{\dim V}$), in which case there are many such structures, for the general linear group of V acts on the set of complex strutures by conjugation. Finally, if V is an Euclidean space, a complex structure J on V is called *orthogonal* if J is an orthogonal transformation. The standard complex structure of $\mathbf{R}^{2n}$ is given by $J_0(x, y) = (-y, x)$ for all x, $y \in \mathbf{R}^n$, so that the complex vector space $(\mathbf{R}^{2n}, J_0)$ is isomorphic to $\mathbf{C}^n$ via $(x, y) \mapsto x + iy$.

as $i\tilde{X}$ is tangent to the sphere. Therefore

$$\begin{aligned}\tilde{g}(\xi,[\tilde{X},\tilde{Y}]^v) &= 2\tilde{g}(\xi,\tilde{\nabla}_{\tilde{X}}\tilde{Y}) && \text{(by Proposition 4.5.5)}\\ &= -2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\xi,\tilde{Y})\\ &= -2\tilde{g}(i\tilde{X},\tilde{Y})\\ &= -2g(JX,Y).\end{aligned}$$

Since ξ is a unit vector field, in view of Proposition 4.5.8, we finally have that

$$K(X,Y) = 1 + 3\langle JX, Y\rangle^2. \tag{4.5.10}$$

In particular, the sectional curvatures of $\mathbf{C}P^n$ lie between 1 and 4. Further, the sectional curvature of a 2-plane E is 4 (resp. 1) if and only if E is complex (resp. totally real).[■2■] On the other hand, if we change the metric on $\mathbf{C}P^n$ to the quotient metric coming from the Riemannian submersion $\pi : S^{2n+1}(2) \to \mathbf{C}P^n$ where $S^{2n+1}(2)$ denotes the sphere of radius 2, then its sectional curvatures will lie between $\frac{1}{4}$ and 1 (cf. exercise 2).

For a general even-dimensonal smooth manifold M, a smooth tensor field J of type $(1,1)$ satisfying $J_p^2 = -\mathrm{id}_{T_pM}$ for all $p \in M$ is called an *almost complex structure.* If J is an almost complex structure on M, a Riemannian metric g on M is called a *Hermitian metric* if J_p is a linear isometry of T_pM with respect to g_p for all $p \in M$. If, in addition, J is parallel ($\nabla J \equiv 0$) with respect to the Levi-Cività connection of (M,g), then (M,g,J) is called an *almost Kähler manifold.*

A *complex manifold* is an even dimensional smooth manifold M admitting a *holomorphic atlas*, namely, an atlas whose transition maps are holomorphic maps between open sets of $\mathbf{C}^n$, after identifying $\mathbf{R}^{2n} \cong \mathbf{C}^n$. It is easy to see that a holomorphic atlas allows one to transfer the complex structure of $\mathbf{R}^{2n}$ to the tangent spaces of M so that a complex manifold automatically inherits a canonical almost complex structure. Not all almost complex structures on a smooth manifold are obtained from a holomorphic atlas in this way and the ones that do are called *integrable.* The celebrated Newlander-Nirenberg theorem supplies a criterium for the integrability of almost complex structures, similar to the Frobenius theorem. An almost Kähler manifold with integrable complex structure is called a *Kähler manifold.* An introduction to the theory of complex manifolds is [Wel08].

We come back to the Riemannian submersion $\pi : S^{2n+1} \to \mathbf{C}P^n$ and the almost complex structure J on $\mathbf{C}P^n$. Note first that $\mathbf{C}^n$ is obviously a complex manifold and indeed a Kähler manifold: for vector fields X, $Y : \mathbf{C}^n \to \mathbf{C}^n$ the Levi-Cività connection $\nabla^{\mathbf{C}^n}_X Y = dY(X)$, so the chain rule yields

$$\nabla^{\mathbf{C}^n}_X(J_0Y) = d(J_0 \circ Y)(X) = dJ_0 \circ dY(X) = J_0\nabla^{\mathbf{C}^n}_X Y$$

and hence $\nabla^{\mathbf{C}^n} J_0 = 0$. Now J_0 restricts to an endomorphism of $\mathcal{H}$ and the Levi-Cività connection of S^{2n+1} is obtained from $\nabla^{\mathbf{C}^n}$ by orthogonal projection, so

$$\tilde{\nabla}_{\tilde{X}}(J_0\tilde{Y}) = J_0\tilde{\nabla}_{\tilde{X}}\tilde{Y}$$

from which it follows that

$$\nabla_X(JY) = J\nabla_X Y,$$

for all X, $Y \in \Gamma(T\mathbf{C}P^n)$. This proves that the almost complex structure of $\mathbf{C}P^n$ is parallel. That $\mathbf{C}P^n$ is a Kähler manifold finally follows from the fact that the transition maps (1.3.4) of the smooth atlas constructed in chapter 1 are holomorphic.

[■2■] A subspace E of an Euclidean vector space V with orthogonal complex structure J is called *totally real* (resp. *complex*) if $J(E) \perp E$ (resp. $J(E) \subset E$).

Lie groups

Let G be a Lie group equipped with a bi-invariant metric. In this example, we will compute the sectional curvatures of G. Denote by $\mathfrak{g}$ the Lie algebra of G. Any 2-plane E contained in T_gG, $g \in G$, is spanned by X_g, Y_g for some X, $Y \in \mathfrak{g}$, so $K(E) = K(X_g, Y_g)$. Further, since left-translations are isometries, we can write $K(X_g, Y_g) = K(X, Y)$ unambiguously. Next, recall the formula (2.8.8) for the covariant derivative. It yields

$$\nabla_X \nabla_Y X = \frac{1}{2}[X, \nabla_Y X] = \frac{1}{4}[X, [Y, X]] = \frac{1}{4}[[X, Y], X],$$
$$\nabla_Y \nabla_X X = 0,$$
$$\nabla_{[X,Y]} X = \frac{1}{2}[[X, Y], X],$$

hence

$$R(X, Y)X = -\frac{1}{4}[[X, Y], X].$$

Assuming that $\{X, Y\}$ is orthonormal and using (2.8.7), we finally get that

$$K(X, Y) = \frac{1}{4}||[X, Y]||^2.$$

We conclude that G has nonnegative curvature. Let $X \in \mathfrak{g}$ be a unit vector and let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{g}$ with $E_1 = X$. Due to (4.3.1), we also have

$$\mathrm{Ric}(X, X) = \sum_{j=2}^{n} K(X, E_j) = \frac{1}{4}\sum_{j=2}^{n} ||[X, E_j]||^2.$$

It follows that G has positive Ricci curvature in case its center is discrete. We can also rewrite the preceding equation as

$$\mathrm{Ric}(X, X) = -\frac{1}{4}\sum_{j=2}^{n} g([[X, [X, E_j]], E_j) = -\frac{1}{4}\sum_{j=2}^{n} g(\mathrm{ad}_X^2 E_j, E_j) = -\frac{1}{4}\,\mathrm{trace}\,(\mathrm{ad}_X^2).$$

Thus, by bilinearity and polarization,

$$-4\mathrm{Ric}(X, Y) = \mathrm{trace}\,(\mathrm{ad}X \circ \mathrm{ad}Y) \tag{4.5.11}$$

for every X, $Y \in \mathfrak{g}$.

For a general Lie group G, the right-hand side of equation (4.5.11) defines a bilinear symmetric form $B_{\mathfrak{g}}$ on $\mathfrak{g}$ called the *Killing form* (or *Cartan-Killing form*) of $\mathfrak{g}$, and one easily checks that

$$B_{\mathfrak{g}}(\mathrm{ad}_Z X, Y) + B_{\mathfrak{g}}(X, \mathrm{ad}_Z Y) = 0$$

for every X, Y, $Z \in \mathfrak{g}$. If, in addition, G is compact and the center of $\mathfrak{g}$ is trivial, then one shows that $-B_{\mathfrak{g}}$ is also positive definite [Hel78, Prop. 6.6]. Assuming further that G is connected, it follows by Proposition 2.8.5 and the discussion in chapter 1 that $-B_{\mathfrak{g}}$ induces a bi-invariant metric on G. Hence, in the special case in which the bi-invariant metric on G comes from the Killing form, equation (4.5.11) shows that the Ricci tensor is a multiple of the metric tensor, and G is thus an Einstein manifold.

4.6 Additional notes

§1 We make a small digression into the classical theory of surfaces in $\mathbf{R}^3$, see e.g. [Car76], and prove the following proposition.

4.6.1 Proposition *Let M be a regular surface in $\mathbf{R}^3$ equipped with the induced metric. Then the sectional curvature and the Gaussian curvature of M coincide at each point $p \in M$.*

Proof. Let $\mathbf{x} : U \to M$ be a parametrization, where U is an open subset of $\mathbf{R}^2$. We have that $\{\mathbf{x}_u, \mathbf{x}_v\}$ span the tangent plane to M at each point. The smooth functions $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, $G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$ are the coefficients of the first fundamental form of M (the induced Riemannian metric). The unit normal vector field is given by

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}.$$

This defines the Gauss map $N : M \to S^2$. Its differential at $p \in M$ is a symmetric linear map $dN_p : T_pM \to T_pM$ which is represented in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ by the matrix

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Using the Christoffel symbols, we can write

$$\begin{aligned} \mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + eN \\ \mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + fN \\ \mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + gN \end{aligned}$$

The sectional curvature of M is given by

$$\begin{aligned} K(\mathbf{x}_u, \mathbf{x}_v) &= \frac{-\langle R(\mathbf{x}_u, \mathbf{x}_v)\mathbf{x}_u, \mathbf{x}_v \rangle}{||\mathbf{x}_u||^2 ||\mathbf{x}_v||^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2} \\ &= -\frac{\langle \nabla_{\mathbf{x}_u} \nabla_{\mathbf{x}_v} \mathbf{x}_u - \nabla_{\mathbf{x}_v} \nabla_{\mathbf{x}_u} \mathbf{x}_u, \mathbf{x}_v \rangle}{EG - F^2}, \end{aligned}$$

since $[\mathbf{x}_u, \mathbf{x}_v] = 0$. The Levi-Cività connection ∇ is just the tangential component of the derivative in $\mathbf{R}^3$, so $\nabla_{\mathbf{x}_v} \mathbf{x}_u = (\mathbf{x}_{vu})^\top = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v$ and

$$\begin{aligned} \nabla_{\mathbf{x}_u} \nabla_{\mathbf{x}_v} \mathbf{x}_u &= \left((\Gamma_{12}^1)_u \mathbf{x}_u + \Gamma_{12}^1 \mathbf{x}_{uu} + (\Gamma_{12}^2)_u \mathbf{x}_v + \Gamma_{12}^2 \mathbf{x}_{uv} \right)^\top \\ &= \left((\Gamma_{12}^1)_u + \Gamma_{12}^1 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{12}^1 \right) \mathbf{x}_u + \left((\Gamma_{12}^2)_u + \Gamma_{12}^1 \Gamma_{11}^2 + (\Gamma_{12}^2)^2 \right) \mathbf{x}_v. \end{aligned}$$

Similarly, one computes that

$$\nabla_{\mathbf{x}_v} \nabla_{\mathbf{x}_u} \mathbf{x}_u = \left((\Gamma_{11}^1)_v + \Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{11}^2 \Gamma_{22}^1 \right) \mathbf{x}_u + \left((\Gamma_{11}^2)_v + \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 \right) \mathbf{x}_v.$$

It follows from formulas (5) and (5a) in [Car76, section 4.3] that $K(\mathbf{x}_u, \mathbf{x}_v)$ equals the Gaussian curvature of M. We realize that this proof is really a restatement of the proof of the *Theorema Egregium*. In chapter 7, we will present an alternative way of proving this proposition. $\square$

§2 Curvature, in any of its manifestations, is the single most important invariant in Riemannian geometry. It is a local invariant that severely restricts the possibilities for local isometries of a

Riemannian manifold; this is partially reflected in the fact that the group of global isometries of a Riemannian manifold is a finite-dimensional Lie group. At the same time, it is really the presence of curvature that gives rise to the huge variety of non-equivalent Riemannian metrics on a given smooth manifold that we can see. The curvature tensor and its covariant derivatives are indeed the only Riemannian invariants if one demands that they be algebraic invariants stemming from the connection. However, if one requires only tensors that are invariant under isometries — the so-called *natural tensors* — then there is not even hope of achieving a classification without imposing further restrictions [Eps75].

§3 Does the curvature determine the metric? This is a very natural question, and an interesting result of Kulkarni [Kul70] asserts that diffeomorphisms preserving the sectional curvature are isometries if the sectional curvature is not constant and the dimension is bigger than 3. On the other hand, it is important to realize that the curvature tensor, in general, does *not* determine the metric, even given that for $n > 3$ the dimension of the space of (pointwise) curvature tensors $\frac{n^2(n^2-1)}{12}$ is much larger than the dimension of the (pointwise) metric tensors $\frac{n(n-1)}{2}$. Indeed, there are many examples of nonisometric Riemannian manifolds admitting diffeomorphisms that preserve the respective curvature tensors. Of course, the difference between the curvature tensor and the sectional curvature is that the latter involves the metric.

4.7 Exercises

1 Let M be an n-dimensional Riemannian manifold of constant curvature κ. Compute that

$$\mathrm{Ric} = (n-1)\kappa g \quad \text{and} \quad \mathrm{scal} = n(n-1)\kappa.$$

2 Let g and $\bar{g}$ be two Riemannian metrics in the smooth manifold M such that $\bar{g} = \lambda g$ for a constant $\lambda > 0$. Show that the curvature tensor, the sectional curvature, the Ricci tensor and the scalar curvature of the Riemannian manifolds $(M, \bar{g})$ and (M, g) are related by the following equations:

$$\overline{R} = R, \quad \overline{K} = \lambda^{-1}K, \quad \overline{\mathrm{Ric}} = \mathrm{Ric} \quad \text{and} \quad \overline{\mathrm{scal}} = \lambda^{-1}\mathrm{scal}.$$

3 Use the symmetries of the curvature tensor to show that the Ricci tensor determines the curvature tensor in a Riemannian manifold of dimension 3.

4 Let M be a connected Einstein manifold of dimension at least 3. Prove that the constant of proportionality is independent of the point. Deduce Lemma 4.2.5 from this result.

5 Let M be a Riemannian manifold with the property that for any two points p, $q \in M$, the parallel transport map from p to q along a piecewise smooth curve γ joining p to q does not depend on γ. Prove that M must be flat.

6 As a partial converse to the previous exercise, suppose M is a flat manifold, p, $q \in M$, and γ_0, γ_1 are two smooth curves joining p to q. Prove that if γ_0 and γ_1 are smoothly homotopic with the endpoints fixed, then the parallel transport maps from p to q along γ_0 and along γ_1 coincide.

7 Prove that the curvature tensor of $\mathbf{C}P^n$ is

$$R(X,Y)Z = -\langle X, Z\rangle Y + \langle Y, Z\rangle X + \langle X, JZ\rangle JY - \langle Y, JZ\rangle JX + 2\langle X, JY\rangle JZ$$

for vector fields X, Y, Z on $\mathbf{C}P^n$. (Hint: Use formula (4.5.10).)

8 Prove that the curvature tensor and the Ricci tensor of a Kähler manifold (M, g, J) satisfy the following identities:

$$R(X,Y)J = JR(X,Y), \quad R(JX,JY) = R(X,Y) \quad \text{and} \quad \mathrm{Ric}(JX,JY) = \mathrm{Ric}(X,Y),$$

for all vector fields X and Y on M.

9 Prove that the curvature tensor of a Riemannian manifold satisfies the following identities:

a. For tangent vectors x, y, z and w, we have

$$\begin{aligned} 6\langle R(x,y)z,w\rangle &= \langle R(x,y+z)(y+z),w\rangle - \langle R(x,y-z)(y-z),w\rangle \\ &\quad + \langle R(y,x-z)(x-z),w\rangle - \langle R(y,x+z)(x+z),w\rangle \end{aligned}$$

b. For tangent vectors a, b, c, we have

$$4\langle R(a,b)a,c\rangle = \langle R(a,b+c)a,b+c\rangle - \langle R(a,b-c)a,b-c\rangle$$

Deduce an alternative proof of Corollary 4.2.3.

10 Extend the notion of parallel transport along a curve to tensors of type (r, s).

11 Let $\varphi : N \to M$ be a smooth map, let X, $Y \in \Gamma(TN)$ be vector fields in N and let U, $V \in \Gamma(\varphi^*TM)$ be vector fiels along φ. Prove that

$$R(\varphi_*X, \varphi_*Y)U = \nabla^\varphi_X \nabla^\varphi_Y U - \nabla^\varphi_X \nabla^\varphi_Y U - \nabla^\varphi_{[X,Y]} U$$

where R denotes the curvature tensor of M and ∇^φ denotes the induced connection along φ. (Hint: Imitate the argument in the proof of Proposition 2.6.2.)

12 (Riemannian volume) Let (M, g) be an oriented Riemannian manifold of dimension n. Let $\mathcal{E} = (E_1, \ldots, E_n)$ be a positively oriented orthonormal frame on an open subset U (that is, $E_1, \ldots, E_n$ are smooth vector fields defined on U which are orthonormal at each point), and let $(\theta^1, \ldots, \theta^n)$ be the dual coframe of 1-forms on U. Define the n-form $\omega_\mathcal{E} = \theta^1 \wedge \cdots \wedge \theta^n$ on U.

a. Prove that for another positively oriented orthonormal frame $\mathcal{E}'$ defined on U' we have $\omega_\mathcal{E} = \omega_{\mathcal{E}'}$ on $U \cap U'$. Deduce that there exists a smooth differential form vol_M of degree n on M such that

$$(\mathrm{vol}_M)_p(e_1, \ldots, e_n) = 1$$

for every positively oriented orthonormal basis $e_1, \ldots, e_n$ of T_pM and all $p \in M$. The n-form vol_M is called the *volume form* of (M, g) and the associated measure is called the *Riemannian measure* on M associated to g.

b. Show that for a positively oriented basis $v_1, \ldots, v_n$ of T_pM, we have

$$(\mathrm{vol}_M)_p(v_1, \ldots, v_n) = \sqrt{\det\left(g_p(v_i, v_j)\right)}.$$

Deduce that, in local coordinates $(U, \varphi = (x^1, \ldots, x^n))$,

$$\mathrm{vol}_M = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

13 Let (M, g) be an n-dimensional Riemannian manifold.

a. For any smooth function $f : M \to \mathbf{R}$, the *gradient* of f is the smooth vector field grad$f$ defined by $g((\text{grad}f)_p, v) = df_p(v)$ for all $v \in T_pM$ and all $p \in M$. Prove that

$$\text{grad}(f_1 + f_2) = \text{grad}f_1 + \text{grad}f_2 \quad \text{and} \quad \text{grad}(f_1 f_2) = f_1\,\text{grad}f_2 + f_2\,\text{grad}f_1$$

for all smooth functions f_1, f_2 on M.

b. For any smooth vector field X on M, the *divergence* of X is the smooth function $\text{div}\, X = \text{trace}\,(v \mapsto \nabla_v X)$. Prove that

$$\text{div}\,(X + Y) = \text{div}\, X + \text{div}\, Y \quad \text{and} \quad \text{div}\,(fX) = \langle \text{grad}f, X \rangle + f\,\text{div}\, X$$

for all smooth fuctions f and smooth vector fields X, Y on M.

c. For any smooth function f on M, the *Laplacian* of f is the smooth function $\Delta f = \text{div}\,\text{grad}f$. The function f is called *harmonic* is $\Delta f = 0$. Prove that

$$\Delta(f_1 f_2) = f_1 \Delta f_2 + 2\langle \text{grad}f_1, \text{grad}f_2 \rangle + f_2 \Delta f_1$$

for all smooth functions f_1, f_2 on M.

d. For any smooth function f on M, the *Hessian* of f is the $(0, 2)$-tensor $\text{Hess}(f) = \nabla df$. Prove that

$$\text{Hess}(f)(X, Y) = X(Yf) - (\nabla_X Y)f$$

and

$$\text{Hess}(f)(X, Y) = \text{Hess}(f)(Y, X)$$

for all smooth vector fields X, Y on M. Show also that the trace of the Hessian coincides with the Laplacian.

14 (Divergence theorem) Let M be an oriented Riemannian manifold.

a. Prove that for any smooth vector field

$$L_X(dV) = (\text{div}X)\, dV$$

where dV denotes the volume form vol_M. A vector field is called *incompressible* if it is divergence-free. Deduce that a vector field is incompressible if and only if its local flows are volume preserving.

b. Suppose now Ω is a domain in M with smooth boundary and let $\partial\Omega$ be oriented by the outward unit normal ν. Denote the Riemannian volume form of $\partial\Omega$ by dS. Use Stokes' theorem to show that for any compactly supported smooth vector field X on M we have

$$\int_\Omega \text{div}\, X\, dV = \int_{\partial\Omega} \langle X, \nu \rangle\, dS$$

15 (Green identities) Let M be an oriented Riemannian manifold and let Ω be a domain in M as in exercise 14.

a. Prove the "integration by parts formula"

$$\int_\Omega f_1 \Delta f_2\, dV + \int_\Omega \langle \text{grad}\, f_1, \text{grad}\, f_2 \rangle\, dV = \int_{\partial\Omega} f_1 \frac{\partial f_2}{\partial \nu}\, dS$$

for any compactly supported smooth functions f_1, f_2 on M. Deduce the *weak maximum principle*: if f is compactly supported and sub- or super-harmonic (i.e. $\Delta f \geq 0$ or $\Delta f \leq 0$) then f is constant. (Hint: first show $\Delta f = 0$; then apply integration by parts to $f = f_1 = f_2$ and $\Omega = M$.)

b. Prove that

$$\int_\Omega (f_1 \Delta f_2 - f_2 \Delta f_1)\, dV = \int_{\partial\Omega} \left(f_1 \frac{\partial f_2}{\partial \nu} - f_2 \frac{\partial f_1}{\partial \nu} \right) dS$$

for any compactly supported smooth functions f_1, f_2 on M. Deduce that if f_1 and f_2 are two eigenfunctions of the Laplacian on a compact oriented Riemannian manifold M associated to different eigenvalues λ_1, λ_2, resp., then f_1 and f_2 are orthogonal in the sense that $\int_M f_1 f_2\, dV = 0$.