# Completeness

## 3.1 Introduction

Geodesics of Riemannian manifolds were defined in section 2.4 as solutions to a second order ordinary diferential equation that, in a sense, means that they have acceleration zero, or, so to say, that they are the "straightest" curves in the manifold. On the other hand, the geodesics of Euclidean space are the lines, and it is known that line segments are the shortest curves between its endpoints. One of the goals of this chapter is to propose an alternative characterization of geodesics in Riemannian manifolds as the "shortest" curves in the manifold. As we will see soon, in a general Riemannian manifold we cannot expect this property to hold globally, but only locally.

To begin with, we prove the Gauss lemma and use it to introduce a metric space structure in the Riemannian manifold in order to be able to talk about distances and curves that minimize distance. The proposed characterization as the locally minimizing curves then follows easily from some results of section 2.4. Next, a natural question is how far a geodesic can minimize distance. The appropriate category of Riemannian manifolds in which to consider this question is that of complete Riemannian manifolds, namely, Riemannian manifolds whose geodesics can be extended indefinitely. In this context, we prove our first global result which is the fundamental Hopf-Rinow theorem. Finally, the question of how far a geodesic can minimize distance brings us to the notion of cut-locus.

Throughout this chapter, we let $(M, g)$ denote a *connected* Riemannian manifold.

## 3.2 The metric space structure

As a preparation for the introduction of the metric space structure, we prove the Gauss lemma and use it to show that the radial geodesics emanating from a point and contained in a normal neighborhood are the shortest curves among the piecewise smooth curves with the same endpoints.

So fix a point $p \in M$. By Proposition 2.4.4, there exist $\epsilon > 0$ and an open neighborhood $U$ of $p$ in $M$ such that $\exp_p : B(0_p, \epsilon) \to U$ is a diffeomorphism. Then we have a diffeomorphism

$$f : (0, \epsilon) \times S^{n-1} \to U \setminus \{p\}, \qquad f(r, v) = \exp_p(rv),$$

where $S^{n-1}$ denotes the unit sphere of $(T_pM, g_p)$. Note that $\gamma_v(t) = f(t, v)$ if $|t| < \epsilon$.

**3.2.1 Lemma (Gauss, local version)** *The radial geodesic $\gamma_v$ is perpendicular to the hyperspheres $f(\{r\} \times S^{n-1})$ for $0 < r < \epsilon$. It follows that*

$$f^*g = dr^2 + h_{(r,v)}$$

*where $h_{(r,v)}$ is the metric induced on $S^{n-1}$ from $f : \{r\} \times S^{n-1} \to M$.*

*Proof.* For a smooth vector field $X$ on $S^{n-1}$, we denote by $\tilde{X} = f_*X$ the induced vector field on $U$. Also, we denote by $\frac{\partial}{\partial r}$ the coordinate vector field on $(0, \epsilon)$ and set $\frac{\tilde{\partial}}{\partial r} = f_* \frac{\partial}{\partial r}$. Next, note that $\gamma_v'(r) = \frac{\tilde{\partial}}{\partial r}|_{f(r,v)}$ and that every vector tangent to $S(p, r) := f(\{r\} \times S^{n-1})$ at $f(r, v)$ is of the form $\tilde{X}|_{f(r,v)}$ for some smooth vector field $X$ on $S^{n-1}$. In view of that, the problem is reduced to proving that $g(\tilde{X}, \frac{\tilde{\partial}}{\partial r}) = 0$ at $f(r, v)$. With this is mind, we start computing

$$\begin{aligned}
\frac{d}{dr} g\left(\tilde{X}, \frac{\tilde{\partial}}{\partial r}\right) &= g\left(\nabla_{\frac{\tilde{\partial}}{\partial r}} \tilde{X}, \frac{\tilde{\partial}}{\partial r}\right) + g\left(\tilde{X}, \nabla_{\frac{\tilde{\partial}}{\partial r}} \frac{\tilde{\partial}}{\partial r}\right) \\
&= g\left(\nabla_{\tilde{X}} \frac{\tilde{\partial}}{\partial r}, \frac{\tilde{\partial}}{\partial r}\right) \\
&= \frac{1}{2} \tilde{X}\, g\left(\frac{\tilde{\partial}}{\partial r}, \frac{\tilde{\partial}}{\partial r}\right) \\
&= 0,
\end{aligned}$$

where we have used the following facts: the compatibility of $\nabla$ with $g$, $\nabla_{\frac{\tilde{\partial}}{\partial r}} \frac{\tilde{\partial}}{\partial r} = 0$ since $\gamma_v$ is a geodesic, $\nabla_{\frac{\tilde{\partial}}{\partial r}} \tilde{X} - \nabla_{\tilde{X}} \frac{\tilde{\partial}}{\partial r} = [\nabla_{\frac{\tilde{\partial}}{\partial r}}, \tilde{X}] = f_*[\frac{\partial}{\partial r}, X] = 0$ and $g\left(\frac{\tilde{\partial}}{\partial r}, \frac{\tilde{\partial}}{\partial r}\right) = 1$. Now we have that $g(\tilde{X}, \frac{\tilde{\partial}}{\partial r}) = 0$ is constant as a function of $r \in (0, \epsilon)$. Hence

$$g\left(\tilde{X}, \frac{\tilde{\partial}}{\partial r}\right)\Big|_{f(r,v)} = \lim_{r \to 0} g\left(\tilde{X}, \frac{\tilde{\partial}}{\partial r}\right)\Big|_{f(r,v)} = 0$$

due to the fact that $\tilde{X}|_{f(r,v)} = d(\exp_p)_{rv}(rX_v)$ goes to 0 as $r \to 0$.

Regarding the last assertion in the statement, the above result shows that in the expression of $f^*g$ there are no mixed terms, namely, no terms involving both $dr$ and coordinates on $S^{n-1}$, and $g\left(\frac{\tilde{\partial}}{\partial r}, \frac{\tilde{\partial}}{\partial r}\right) = 1$ shows that the coefficient of $dr^2$ is 1. ☐

**3.2.2 Proposition** *Let $p \in M$, and let $\epsilon > 0$ be such that $U = \exp_p(B(0_p, \epsilon))$ is a normal neighborhood of $p$. Then, for any $x \in U$, there exists a unique geodesic $\gamma$ of length less than $\epsilon$ joining $p$ and $x$. Moreover, $\gamma$ is the shortest piecewise smooth curve in $M$ joining $p$ to $x$, and any other piecewise smooth curve joining $p$ to $x$ with the same length as $\gamma$ must coincide with it, up to reparametrization.*

*Proof.* We already know that there exists a unique $v \in T_pM$ with $g_p(v, v)^{1/2} < \epsilon$ and $\exp_p v = x$. Taking $\gamma$ to be $\gamma_v : [0, 1] \to M$, it is clear that the length of $\gamma$ is less than $\epsilon$.

Next, let $\eta$ be another piecewise curve joining $x$ to $y$. We need to prove that $L(\gamma) \leq L(\eta)$, where the equality holds if and only if $\eta$ and $\gamma$ coincide, up to reparametrization. Without loss of generality, we may assume that $\eta$ is defined on $[0, 1]$ and that $\eta(t) \neq p$ for $t > 0$. There are two cases:

(a) If $\eta$ is entirely contained in $U$, then we can write $\eta(t) = f(r(t), v(t))$ for $t > 0$. In this case, due to the Gauss lemma 3.2.1:

$$\begin{aligned} L(\eta) &= \int_0^1 g_{\eta(t)}(\eta'(t), \eta'(t))^{1/2}\, dt \\ &= \int_0^1 \left( r'(t)^2 + h_{(r(t),v(t))}(v'(t), v'(t)) \right)^{1/2} dt \\ &\geq \int_0^1 |r'(t)|\, dt \\ &\geq |r(1) - \lim_{t\to 0} r(t)| \\ &= L(\gamma). \end{aligned}$$

(b) If $\eta$ is not contained in $U$, let

$$t_0 = \inf\{\, t \mid \eta(t) \notin U \,\}.$$

Then, using again the Gauss lemma:

$$L(\eta) \geq L(\eta|_{[0,t_0]}) \geq \int_0^{t_0} |r'(t)|\, dt \geq r(t_0) = \epsilon > L(\gamma).$$

In any case, we have $L(\eta) \geq L(\gamma)$. If $L(\eta) = L(\gamma)$, then we are in the first case and $r'(t) > 0$, $v'(t) = 0$ for all $t$, so $\eta$ is a radial geodesic, up to reparametrization. □

For points $x$, $y \in M$, define

$$d(x, y) = \inf\{\, L(\gamma) \mid \gamma \text{ is a piecewise smooth curve joining } x \text{ and } y \,\}.$$

Note that the infimum in general need not be attained. This happens for instance in the case in which $M = \mathbf{R}^2 \setminus \{(0,0)\}$ and we take $x = (-1, 0)$, $y = (1, 0)$; here $d(x, y) = 2$, but there is no curve of length 2 joining these points.

**3.2.3 Proposition** *We have that $d$ is a distance on $M$, and it induces the manifold topology in $M$.*

*Proof.* First notice that the distance of any two points is finite. In fact, since a manifold is locally Euclidean, the set of points of $M$ that can be joined to a given point by a piecewise smooth curve is open. This gives a partition of $M$ into open sets. By connectedness, there must be only one such set.

Next, we remark that $d(x, y) = d(y, x)$, since any curve can be reparametrized backwards. Also, the triangular inequality $d(x, y) \leq d(x, z) + d(z, y)$ holds by juxtaposition of curves, and $d(x, x) = 0$ holds by using a constant curve.

In order to have that $d$ is a distance, it only remains to prove that $d(x, y) > 0$ for $x \neq y$. Choose $\epsilon > 0$ such that $U = \exp_x(B(0_x, \epsilon))$ is a normal neighborhood of $x$; since $\exp_x : B(0_x, \epsilon) \to U$ is diffeomorphism, by decreasing $\epsilon$, if necessary, we may assume that $y \notin U$. If $\gamma$ is any piecewise smooth curve joining $x$ to $y$, and $t_0 = \inf\{\, t \mid \gamma(t) \notin U \,\}$, then $L(\gamma) \geq L(\gamma|_{[0,t_0]}) \geq \epsilon$, where the second inequality is a consequence of Proposition 3.2.2. It follows that $d(x, y) > 0$.

Now that we have $d$ is a distance, we remark that the same Proposition 3.2.2 indeed implies that, in the normal neighborhood $U$ of $x$, namely for $0 < r < \epsilon$, the distance spheres

$$S(x, r) := \{\, z \in M \mid d(z, x) = r \,\}$$

coincide with the geodesic spheres

$$\{\exp_x(v) \mid g_x(v,v)^{1/2} = r\}.$$

In particular, the distance balls

$$B(x,r) := \{ z \in M \mid d(z,x) < r \}$$

coincide with the geodesic balls $\exp_x(B(0_x, r))$. Since the former make up a system of fundamental neighborhoods of $x$ for the topology of $(M, d)$, and the latter make up a system of fundamental neighborhoods of $x$ for the manifold topology of $M$, and $x \in M$ is arbitrary, it follows that the topology induced by $d$ coincides with the manifold topology of $M$. □

Combining results of Propositions 2.4.7 and 3.2.2, we now have the following proposition.

**3.2.4 Proposition** *Let $p \in M$, and let $\epsilon > 0$ be such that $U$ is an $\epsilon$-totally normal neighborhood of $p$ as in Proposition 2.4.7. Then, for any $x$, $y \in U$, there exists a unique geodesic $\gamma$ of length less than $\epsilon$ joining $x$ and $y$; moreover, $\gamma$ depends smoothly on $x$ and $y$. Finally, the length of $\gamma$ is equal to the distance between $x$ and $y$, and $\gamma$ is the only piecewise smooth curve in $M$ with this property, up to reparametrization.*

*Proof.* The first part of the statement is just a paraphrase of Proposition 2.4.7. The second one follows from Proposition 3.2.2. □

We say that a piecewise smooth curve $\gamma : [a, b] \to M$ is *minimizing* if $L(\gamma) = d(\gamma(a), \gamma(b))$.

**3.2.5 Lemma** *Let $\gamma : [a, b] \to M$ be a minimizing curve. Then the restriction $\gamma|_{[c,d]}$ to any subinterval $[c, d] \subset [a, b]$ is also minimizing.*

*Proof.* Suppose, on the contrary, that $\gamma$ is not minimizing on $[c, d]$. This means that there is a piecewise smooth curve $\eta$ from $\gamma(c)$ to $\gamma(d)$ that is shorter than $\gamma|_{[c,d]}$. Consider the piecewise smooth curve $\zeta : [a, b] \to M$ constructed by replacing $\gamma|_{[c,d]}$ by $\eta$, namely,

$$\zeta(t) = \begin{cases} \gamma(t) & \text{if } t \in [a, c], \\ \eta(t) & \text{if } t \in [c, d], \\ \gamma(t) & \text{if } t \in [d, b]. \end{cases}$$

Then $\zeta$ is a piecewise smooth curve from $\gamma(a)$ to $\gamma(b)$ and it is clear that $\zeta$ is shorter than $\gamma$, which is a contradiction. Hence, $\gamma$ is minimizing on $[c, d]$. □

We can now state the promised characterization of geodesics as the locally minimizing curves.

**3.2.6 Theorem (Geodesics are the locally minimizing curves)** *A piecewise smooth curve $\gamma : [a, b] \to M$ is a geodesic up to reparametrization if and only if every sufficiently small arc of it is a minimizing curve.*

*Proof.* Just by continuity, every sufficiently small arc of $\gamma$ is contained in a $\epsilon$-totally normal neighborhood $U$ of some point of $M$. But the length of a curve in $U$ of realizes the distance between the endpoints of the curve if and only if that curve is a geodesic, up to reparametrization, by Proposition 3.2.4. Since being a geodesic is a local property, the result is proved. □

Since geodesics are smooth, it follows from Lemma 3.2.5 and Theorem 3.2.6 that a minimizing curve must be smooth.

## 3.3 Geodesic completeness and the Hopf-Rinow theorem

A Riemannian manifold $M$ is called *geodesically complete* if every geodesic of $M$ can be extended to a geodesic defined on all of $\mathbf{R}$. For instance, $\mathbf{R}^n$ satisfies this condition since its geodesics are lines, but $\mathbf{R}^n$ minus one point does not. A more interesting example is the upper half-plane:

$$\{(x,y) \in \mathbf{R}^2 \,|\, y > 0\}.$$

This manifold is not geodesically complete with respect to the Euclidean metric $dx^2 + dy^2$, but it is so with respect to the hyperbolic metric $\frac{1}{y^2}(dx^2 + dy^2)$ (cf. example 2.4.8 of chapter 2). Of course, an equivalent way of rephrasing this definition is to say that $M$ is geodesically complete if and only if $\exp_p$ is defined on all of $T_pM$, for all $p \in M$.

We will use the following lemma twice in the proof of the Hopf-Rinow theorem.

**3.3.1 Lemma** *Let $(M,g)$ be a connected Riemannian manifold. Let $x$, $y \in M$ be distinct points and let $S$ be the geodesic sphere of radius $\delta$ and center $x$ in $(M,d)$. Then, for sufficiently small $\delta > 0$, there exists $z \in S$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

*Proof.* If $\delta > 0$ is sufficiently small so that the ball $B(0_x, \delta)$ is contained in an open set where $\exp_x$ is a diffeomorphism onto its image, then $S = \exp_x(S(0_x, \delta))$, where $S(0_x, \delta)$ is the sphere of center $0_x$ and radius $\delta$ in $(T_xM, g_x)$. It will also be convenient to assume that $\delta < d(x,y)$. Since $S$ is compact, there exists a point $z \in S$ such that $d(y,S) = d(y,z)$.

If $\gamma$ is a piecewise smooth curve from $x$ to $y$ parametrized on $[0,1]$, as $d(x,y) > \delta$, we have that $\gamma$ meets $S$ at a point $\gamma(t)$, and then

$$\begin{aligned} L(\gamma) &= L(\gamma|_{[0,t]}) + L(\gamma|_{[t,1]}) \\ &\geq d(x,\gamma(t)) + d(\gamma(t),y) \\ &\geq d(x,z) + d(z,y). \end{aligned}$$

This implies that $d(x,y) \geq d(x,z) + d(z,y)$. The thesis now follows from the triangle inequality. □

Historically speaking, it is interesting to notice that the celebrated Hopf-Rinow theorem was only proved in 1931 [HR31]. For ease of presentation, we divide its statement into two parts. The proof of (3.3.2) presented below is due to de Rham [dR73] and is different from the original argument in [HR31].

**3.3.2 Theorem (Hopf-Rinow)** *Let $(M,g)$ be a connected Riemannian manifold.*

*a. Let $p \in M$. If $\exp_p$ is defined on all of $T_pM$, then any point of $M$ can be joined to $p$ by a minimizing geodesic.*

*b. If $M$ is geodesically complete, then any two points of $M$ can be joined a minimizing geodesic.*

The converse of item (b) in the theorem is false, as can be seen simply by taking $M$ to be an open ball (or any convex subset) of $\mathbf{R}^n$ with the induced metric.

*Proof of Theorem 3.3.2.* Plainly, it is enough to prove assertion (a) as this assertion implies the other one. So we assume that $\exp_p$ is defined on all of $T_pM$, and we want to produce a minimizing geodesic from $p$ to a given point $q \in M$. Roughly speaking, the idea of the proof is to start from $p$ with a geodesic in the "right direction", and then to prove that this geodesic eventually reaches $q$.

By Lemma 3.3.1, for sufficiently small $\delta > 0$, there exists $p_0$ in a normal neighborhood of $p$ such that

$$d(p, p_0) = \delta \quad \text{and} \quad d(p, p_0) + d(p_0, q) = d(p, q).$$

Let $v \in T_pM$ be the unit vector such that $\exp_p(\delta v) = p_0$, and consider $\gamma(t) = \exp_p(tv)$. We have that $\gamma$ is a geodesic defined on all of $\mathbf{R}$. We will prove that $\gamma(d(p, q)) = q$.

Let $I = \{ t \in [0, d(p, q)] \mid d(p, q) = t + d(\gamma(t), q) \}$. We already know that $0, \delta \in I$, so $I$ is nonempty. Let $T = \sup I$. Since the distance $d : M \times M \to \mathbf{R}$ is a continuous function, $I$ is a closed set, and thus contains $T$. Note that the result will follow if we can prove that $T = d(p, q)$. So suppose that $T < d(p, q)$. Then we can apply Lemma 3.3.1 to the points $\gamma(T)$ and $q$ to find $\epsilon > 0$ and $q_0 \in M$ such that

$$d(\gamma(T), q_0) = \epsilon \quad \text{and} \quad d(\gamma(T), q_0) + d(q_0, q) = d(\gamma(T), q). \tag{3.3.3}$$

Hence

$$\begin{aligned} d(p, q_0) &\geq d(p, q) - d(q_0, q) \\ &= d(p, q) - \big( d(\gamma(T), q) - d(\gamma(T), q_0) \big) \\ &= \big( d(p, q) - d(\gamma(T), q) \big) + d(\gamma(T), q_0) \\ &= T + \epsilon, \end{aligned} \tag{3.3.4}$$

since $T \in I$. Let $\eta$ be the unique unit speed minimizing geodesic from $\gamma(T)$ to $q_0$. Since the concatenation of $\gamma|_{[0,T]}$ and $\eta$ is a piecewise smooth curve of length $T + \epsilon$ joining $p$ to $q_0$, it follows from estimate (3.3.4) that $d(p, q_0) = T + \epsilon$. Now the concatenation is a minimizing curve, so by Lemma 3.2.5 and Theorem 3.2.6 it must be a geodesic, thence, smooth. Due to the uniqueness of geodesics with given initial conditions, $\eta$ must extend $\gamma|_{[0,T]}$ as a geodesic, and therefore $\gamma(T+\epsilon) = \eta(\epsilon) = q_0$. Using this and equations (3.3.3), we finally get that

$$d(q, \gamma(T + \epsilon)) + T + \epsilon = d(q, q_0) + d(\gamma(T), q_0) + T = d(\gamma(T), q) + T = d(p, q),$$

and this implies that $T + \epsilon \in I$, which is a contradiction. Hence the supposition that $T < d(p, q)$ is wrong and the result follows. □

**3.3.5 Theorem (Hopf-Rinow)** *Let $(M, g)$ be a connected Riemannian manifold. Then the following assertions are equivalent:*

a. *$(M, g)$ is geodesically complete.*
b. *For every $p \in M$, $\exp_p$ is defined on all of $T_pM$.*
c. *For some $p \in M$, $\exp_p$ is defined on all of $T_pM$.*
d. *Every closed and bounded subset of $(M, d)$ is compact.*
e. *$(M, d)$ is complete as a metric space.*

*Proof.* The assertions that (a) implies (b) and (b) implies (c) are obvious. We start the proof showing that (c) implies (d). Let $K$ be a closed and bounded subset of $M$. Since $K$ is bounded, there exists $R > 0$ such that $\sup_{x \in K}\{d(p, x)\} < R$. For every $q \in K$, there exists a minimizing geodesic $\gamma$ from $p$ to $q$ by assumption and the first part of Theorem 3.3.2. Note that $L(\gamma) = d(p, q) < R$. This shows that $K \subset \exp_p(\overline{B(0_p, R)})$. Now $K$ is a closed subset of compact set and thus compact itself.

The proof that (d) implies (e) is a general argument in the theory of complete metric spaces. In fact, any Cauchy sequence in $(M, d)$ is bounded, hence contained in a closed ball, which must be compact by (d). Therefore the sequence admits a convergent subsequence, and thus it is convergent itself, proving (e).

Finally, let us show that (e) implies (a). This is maybe the most relevant part of the proof of this corollary. So assume that $\gamma$ is a geodesic of $(M,g)$ parametrized with unit speed. The maximal interval of definition of $\gamma$ is open by Theorem 2.4.2 on the local existence and uniqueness of solutions of second order differential equations; let it be $(a,b)$, where $a \in \mathbf{R}\cup\{-\infty\}$ and $b \in \mathbf{R}\cup\{+\infty\}$.

We claim that $\gamma$ is defined on all of $\mathbf{R}$. Suppose, on the contrary, that $b < +\infty$. Choose a sequence $(t_n)$ in $(a,b)$ such that $t_n \nearrow b$. Since

$$d(\gamma(t_m),\gamma(t_n)) \leq L(\gamma|_{[t_m,t_n]}) = t_n - t_m$$

for $n > m$, the sequence $(\gamma(t_n))$ is a Cauchy sequence and thus converges to a point $p \in M$ by (e). Let $U$ be a totally normal neighborhood of $p$ given by Proposition 2.4.7 such that every geodesic starting at a point in $U$ is defined at least on the interval $(-\epsilon,\epsilon)$, for some $\epsilon > 0$. Choose $n$ so that $|t_n - b| < \frac{\epsilon}{2}$ and $\gamma(t_n) \in U$. Then $t_n + \epsilon > b + \frac{\epsilon}{2}$ and the geodesic $\gamma$ can be extended to $(a, t_n+\epsilon)$, which is a contradiction. Hence $b = +\infty$. Similarly, one shows that $a = -\infty$, and this finishes the proof of the corollary. □

We call the attention of the reader to the equivalence of statements (a) and (e) in Theorem 3.3.5. Because of it, hereafter we can say unambiguously that a Riemannian manifold is *complete* if it satisfies either one of assertions (a) or (e). The following are immediate corollaries of the Hopf-Rinow theorem.

**3.3.6 Corollary** *A compact Riemannian manifold is complete.*

Recall that the *diameter* of a metric space $(M,d)$ is defined to be

$$\mathrm{diam}(M) = \sup\{\, d(x,y) \mid x,\ y \in M \,\}$$

**3.3.7 Corollary** *A complete Riemannian manifold of bounded diameter is compact.*

As an application of the concept of completeness, we prove the following proposition which will be used in Chapter 6.

**3.3.8 Proposition** *Let $\pi : (\tilde{M},\tilde{g}) \to (M,g)$ be a local isometry.*

*a. If $\pi$ is a Riemannian covering map and $(M,g)$ is complete, then $(\tilde{M},\tilde{g})$ is also complete.*

*b. If $(\tilde{M},\tilde{g})$ is complete, then $\pi$ is a Riemannian covering map and $(M,g)$ is also complete.*

*Proof.* (a) Let $\tilde{\gamma}$ be a geodesic in $\tilde{M}$. Then the curve $\gamma$ in $M$ defined by $\gamma = \pi\circ\tilde{\gamma}$ is a geodesic of $M$ by Proposition 2.8.3. In view of the completeness of $M$, $\gamma$ is defined on all of $\mathbf{R}$. Again by Proposition 2.8.3, $\tilde{\gamma}$ is a lifting of $\gamma$, so $\tilde{\gamma}$ can be extended to be defined on all of $\mathbf{R}$, proving that $\tilde{M}$ is geodesically complete.

(b) Let $p \in M$. We need to construct an evenly covered neighborhood $p$ in $M$. Suppose that $\pi^{-1}(p) = \{\, \tilde{p}_i \in \tilde{M} \mid i \in I \,\}$, where $I$ is some index set. We can choose $r > 0$ such that $\exp_p : B(0_p,r) \to B(p,r)$ is a diffeomorphism, where $B(p,r)$ denotes the open ball in $M$ of center $p$ and radius $r$. Set $U = B(p,\frac{r}{2})$ and $\tilde{U}_i = B(\tilde{p}_i,\frac{r}{2})$; these are open sets in $M$, $\tilde{M}$, respectively. Since $\pi$ is a local isometry by assumption, we have that the diagram

(3.3.9)
$$\begin{array}{ccc} B(0_{\tilde{p}_i},\frac{r}{2}) & \xrightarrow{\exp_{\tilde{p}_i}} & \tilde{U}_i \\ {\scriptstyle d\pi_{\tilde{p}_i}}\downarrow & & \downarrow{\scriptstyle \pi} \\ B(0_p,\frac{r}{2}) & \xrightarrow[\exp_p]{} & U \end{array}$$

is commutative for all $i$. Next, we use the assumption that $(\tilde{M}, \tilde{g})$ is geodesically complete for the first time (it will be used again below). It implies via the Theorem of Hopf-Rinow that any point in $\tilde{U}_i$ can be joined to $\tilde{p}_i$ by a minimizing geodesic, and hence

$$\exp_{\tilde{p}_i}\left(B\left(0_{\tilde{p}_i}, \frac{r}{2}\right)\right) = \tilde{U}_i \tag{3.3.10}$$

for all $i$ (note that the direct inclusion is always valid, so we actually used the assumption only to get the reverse inclusion). This, put together with (3.3.9), gives that $\pi(\tilde{U}_i) = U$ for all $i$, hence

$$\bigcup_{i\in I} \tilde{U}_i \subset \pi^{-1}(U).$$

Since $\exp_p \circ\, d\pi_{\tilde{p}_i} : B(0_{\tilde{p}_i}, \frac{r}{2}) \to U$ is injective, (3.3.9) and (3.3.10) indeed imply that

$$\pi : \tilde{U}_i \to U$$

is injective; as it is already surjective and a local diffeomorphism, this implies that it is a diffeomorphism. We also claim that the $\tilde{U}_i$ for $i \in I$ are pairwise disjoint. Indeed, if there is a point $q \in \tilde{U}_i \cap \tilde{U}_j$, then

$$d(\tilde{p}_i, \tilde{p}_j) \leq d(\tilde{p}_i, q) + d(q, \tilde{p}_j) < \frac{r}{2} + \frac{r}{2} = r,$$

so $\tilde{p}_j \in B(\tilde{p}_i, r)$. But one sees that $\pi$ is injective on $B(\tilde{p}_i, r)$ in the same way as we saw that $\pi$ is injective on $\tilde{U}_i$. It follows that $\tilde{p}_i = \tilde{p}_j$ and hence $i = j$.

It remains to show that $\pi^{-1}(U) \subset \cup_{i\in I}\tilde{U}_i$. Let $\tilde{q} \in \pi^{-1}(U)$. Set $\pi(\tilde{q}) = q \in U$. By our choice of $r$, there is a unique $v \in T_qM$ such that $||v|| < \frac{r}{2}$ and $p = \exp_q v$. Let $\tilde{v} = (d\pi_{\tilde{q}})^{-1}(v) \in T_{\tilde{q}}\tilde{M}$. The geodesic $\tilde{\gamma}(t) = \exp_{\tilde{q}}(t\tilde{v})$ is defined on $\mathbf{R}$ since $(\tilde{M}, \tilde{g})$ is complete. Now

$$\pi \circ \gamma(1) = \pi \circ \exp_{\tilde{q}}(\tilde{v}) = \exp_{\pi(\tilde{q})}((d\pi)_{\tilde{q}}(\tilde{v})) = \exp_q v = p,$$

so $\tilde{\gamma}(1) = \tilde{p}_{i_0}$ for some $i_0 \in I$. Since $||\tilde{v}|| < \frac{r}{2}$, we have that $\tilde{q} = \tilde{\gamma}(0) \in B(\tilde{p}_{i_0}, \frac{r}{2}) = \tilde{U}_{i_0}$, as desired.

Now that we know that $\pi$ is a Riemannian covering, the completeness of $M$ follows from that of $\tilde{M}$ and Proposition 2.8.3. □

We close this section by proving that Killing fields on complete Riemannian manifolds are complete.

**3.3.11 Proposition** *Let $M$ be a complete Riemannian manifold. Then any Killing field on $M$ is complete as a vector field. It follows that the Lie algebra of Killing fields on $M$ is isomorphic to the Lie algebra of the isometry group of $M$.*

*Proof.* Let $X$ be a Killing field on $M$, and let $\gamma : (a, b) \to M$ be an integral curve of $X$. In order to prove that $X$ is complete, it suffices to show that $\gamma$ can be extended to $(a, b]$. In fact formula (2.5.1) implies that $Xg(X, X) = 0$, whence $||\gamma'||$ is a constant $c$. Therefore for $t_1, t_2 \in (a, b)$, we have

$$d(\gamma(t_1), \gamma(t_2)) \leq L(\gamma|_{[t_1,t_2]}) = c(t_2 - t_1).$$

Then it follows from the completeness of $M$ that $\lim_{t\to b-} \gamma(t)$ exists, as desired.

We have proved that Killing fields are infinitesimal generators of (global) one-parameter subgroups of isometries of $M$. The second assertion follows. □

## 3.4 Cut locus

Consider the following facts that we have already discussed: every geodesic is locally minimizing (Theorem 3.2.6); a minimizing geodesic remains minimizing when restricted to a subinterval of its domain (Lemma 3.2.5); in a complete Riemannian manifold, the domain of any geodesic can be extended to all of $\mathbf{R}$. In view of this, a natural question can be posed now: how far is a geodesic in a complete Riemannian manifold minimizing? This is the motivation to introduce the concept of cut locus. We start with a lemma.

**3.4.1 Lemma** *Let $M$ be a connected Riemannian manifold. Let $\gamma : I \to \mathbf{R}$ be a geodesic, where $I$ is an open interval, and let $[a,b] \subset I$.*

a. *If there exists another geodesic $\eta$ of the same length as $\gamma|_{[a,b]}$ from $\gamma(a)$ to $\gamma(b)$, then $\gamma$ is not minimizing on $[a, b+\epsilon]$ for any $\epsilon > 0$.*

b. *If $(M,g)$ is complete and no geodesic from $\gamma(a)$ to $\gamma(b)$ is shorter than $\gamma|_{[a,b]}$, then $\gamma$ is minimizing on $[a,b]$.*

*Proof.* (a) Consider the piecewise smooth curve $\zeta : [a, b+\epsilon] \to M$ defined by

$$\zeta(t) = \begin{cases} \eta(t) & \text{if } t \in [a,b], \\ \gamma(t) & \text{if } t \in [b, b+\epsilon]. \end{cases}$$

Since $\eta$ and $\gamma$ are distinct geodesics, $\zeta$ is not smooth at $t = b$. It follows that $\zeta$ is not minimizing on $[a, b+\epsilon]$. Since $\gamma$ and $\zeta$ have the same length on $[a, b+\epsilon]$, this implies that neither $\gamma$ is minimizing on this interval.

(b) If $M$ is complete, there exists a minimizing geodesic $\zeta$ from $\gamma(a)$ to $\gamma(b)$ by the Hopf-Rinow theorem. Since no geodesic from $\gamma(a)$ to $\gamma(b)$ is shorter than $\gamma$, $\zeta$ and $\gamma$ have the same length, so $\gamma$ is also minimizing. □

Henceforth, in this section, we assume that $M$ is a complete Riemannian manifold. Fix a point $p \in M$. For each unit tangent vector $v \in T_pM$, we define

$$\rho(v) = \sup\{\, t > 0 \mid d(p, \gamma_v(t)) = t \,\}. \tag{3.4.2}$$

Of course, $\rho(v)$ can be infinite. Notice that the set in the right hand side is a closed interval. It is immediate from the definition that $\gamma_v$ is minimizing on $[0,t]$ if $0 < t \leq \rho(v)$, and $\gamma_v$ is not minimizing on $[0,t]$ if $t > \rho(v)$. It follows from Lemma 3.4.1 that $\gamma_v$ is the unique minimizing geodesic from $p$ to $\gamma_v(t)$ if $0 < t < \rho(v)$.

One proves that $\rho$ is a continuous function from the unit tangent bundle $UM$ of $M$ into $(0, +\infty]$ (see exercise 11 in chapter 5); as usual, the topology we are considering in $(0,+\infty]$ is such that a system of local neighborhoods of the point $+\infty$ is given by the complements in $(0,+\infty]$ of the compact subsets of $(0,+\infty)$. By compactness of the unit sphere $U_pM$ of $T_pM$, it follows that there exists $v_0 \in U_pM$ such that $\rho(v_0) = \inf_{v \in U_pM} \rho(v)$, but it can happen that $\rho(v_0) = +\infty$.

The *injectivity radius at $p$* is defined to be

$$\mathrm{inj}_p(M) = \{\, \inf \rho(v) \mid v \in T_pM,\ ||v|| = 1 \,\}.$$

It follows that $\mathrm{inj}_p(M) \in (0, +\infty]$. Also, the *injectivity radius* of $M$ is defined to be

$$\mathrm{inj}(M) = \inf_{p \in M} \mathrm{inj}_p(M).$$

One can show that $p \in M \mapsto \mathrm{inj}_p(M) \in (0,+\infty]$ is a continuous function (see exercise 11 of chapter 5).

In the case in which $M$ is compact, its diameter is finite, so no geodesic can be minimizing past $t = \mathrm{diam}(M)$. Hence $\rho(v)$ is finite for every unit vector $v \in T_pM$, and it follows that $\rho$ is bounded and $\mathrm{inj}(M)$ is finite and positive.

The *tangential cut locus* of $M$ at $p$ is defined as the subset of $T_pM$ given by

$$\mathrm{C}_p = \{\, \rho(v)v \in T_pM \mid v \in T_pM,\ ||v|| = 1 \,\}.$$

The *cut locus* of $M$ at $p$ is defined as the subset of $M$ given by

$$\mathrm{Cut}(p) = \exp_p \mathrm{C}_p = \{\, \gamma_v(\rho(v)) \mid v \in T_pM,\ ||v|| = 1 \,\}.$$

We will also consider the star-shaped open subset of $T_pM$ given by

$$\mathrm{D}_p = \{\, tv \in T_pM \mid 0 \leq t < \rho(v),\ v \in T_pM\ , ||v|| = 1 \,\}.$$

Notice that $\partial \mathrm{D}_p = \mathrm{C}_p$ and $\mathrm{inj}_p(M) = d(p, \mathrm{Cut}(p))$ (possibly infinite).

**3.4.3 Proposition** *Let $M$ be a complete Riemannian manifold. Then, for every $p \in M$, we have a disjoint union*

$$M = \exp_p(\mathrm{D}_p) \dot\cup \mathrm{Cut}(p).$$

*Proof.* Given $x \in M$, by the Hopf-Rinow theorem there exists a minimizing unit speed geodesic $\gamma_v$ joining $p$ to $x$, where $v \in T_pM$ and $||v|| = 1$. As $\gamma_v$ is minimizing on $[0, d(p,x)]$, we have that $\rho(v) \geq d(p,x)$. This implies that $d(p,x)v \in \mathrm{D}_p \cup \mathrm{C}_p$, thence $x = \exp_p(d(p,x)v) \in \exp_p(\mathrm{D}_p) \cup \mathrm{Cut}(p)$ proving that $M = \exp_p(\mathrm{D}_p) \cup \mathrm{Cut}(p)$.

On the other hand, suppose that $x \in \exp_p(\mathrm{D}_p) \cap \mathrm{Cut}(p)$. Then $x \in \exp_p(\mathrm{D}_p)$ means that there exists a minimizing unit speed geodesic $\gamma : [0, a] \to M$ with $\gamma(0) = p$, $\gamma(a) = x$ and $\gamma$ is minimizing on $[0, a + \epsilon]$ for some $\epsilon > 0$. On the other hand, $x \in \mathrm{Cut}(p)$ means that there exists a minimizing unit speed geodesic $\eta : [0, b] \to M$ with $\eta(0) = p$, $\eta(b) = x$ and $\eta$ is not minimizing past $b$. It follows that $\gamma$ and $\eta$ are distinct. We reach a contradiction by noting that $\gamma$ cannot be minimizing past $a$ by Lemma 3.4.1(a). Hence such an $x$ cannot exist, namely, $\exp_p(\mathrm{D}_p) \cap \mathrm{Cut}(p) = \varnothing$. □

We already know that $\exp_p$ is injective on $\mathrm{D}_p$. We will see in Corollary 5.5.4 that $\exp_p$ is a diffeomorphism of $\mathrm{D}_p$ onto its image. It follows that, if $M$ is compact, $\exp_p(\mathrm{D}_p)$ is homeomorphic to an open ball in $\mathbf{R}^n$, and $M$ is obtained from $\mathrm{Cut}(p)$ by attaching an $n$-dimensional cell via the map $\exp_p : \mathrm{C}_p \to \mathrm{Cut}(p)$. In particular, $\mathrm{Cut}(p)$ is a strong deformation retract of $M \setminus \{p\}$:[1] one simply pushes $M \setminus \{p\}$ out to $\mathrm{Cut}(p)$ along the geodesics emanating from $p$.

## 3.5 Examples

### Empty cut-locus

In the case of $\mathbf{R}^n$ and $\mathbf{R}H^n$, we already know that the geodesics are defined on $\mathbf{R}$, so these Riemannian manifolds are complete (see exercise 7 of chapter 2 for the geodesics of $\mathbf{R}H^n$). We also know that there is a unique geodesic segment joining two given distinct points; since by the Hopf-Rinow theorem there must be a minimizing geodesic joining those two points, that geodesic segment must be the minimizing one. It follows that any geodesic segment is minimizing and hence the cut-locus of any point is empty. This situation will be generalized in chapter 6 (cf. Corollary 6.5.3).

[1] Mention implications for the topology of $M$.

### $S^n$ and $\mathbf{R}P^n$

In the case of $S^n$, the geodesics are the great circles, so they are defined on $\mathbf{R}$, even if they are all periodic. Therefore $S^n$ is complete. Let $p \in S^n$. A unit speed geodesic $\gamma$ starting at $\gamma(0) = p$ is minimizing before it reaches the antipodal point $\gamma(\pi) = -p$ because $\gamma$ is the only geodesic joining $p$ to $\gamma(t)$ for $t \in (0, \pi)$. If $t = \pi + \epsilon$ for some small $\epsilon > 0$, then there is a shorter geodesic $\eta$ joining $p$ to $\gamma(t)$ which has $\eta'(0) = -\gamma'(0)$. It follows that $\mathrm{Cut}(p) = \{-p\}$.

In the case of $\mathbf{R}P^n$, the geodesics are the projections of the geodesics of $S^n$ under the double covering $\pi : S^n \to \mathbf{R}P^n$. Let $\bar{p} = \pi(p)$. Given two distinct unit speed geodesics $\gamma_1$, $\gamma_2$ in $S^n$ starting at $p$, the smallest $t > 0$ for which we can have $\gamma_1(t) = -\gamma_2(t)$ is $t = \pi/2$, namely, the parameter value at which $\gamma_1$ and $\gamma_2$ reach the equator $S^{n-1}$ of $S^n$ (note that this happens only if $-\gamma_2'(0) = \gamma_1'(0)$). It follows that any unit speed geodesic in $\mathbf{R}P^n$ is minimizing until time $\pi/2$; it is also clear that such a geodesic is not minimizing past time $\pi/2$. It follows that $\mathrm{Cut}(\bar{p})$ is the image of the equator $S^{n-1} \subset S^n$ under $\pi$, and is thus isometric to $\mathbf{R}P^{n-1}$.

### Rectangular flat 2-tori

The next example we consider is a rectangular 2-torus $\mathbf{R}^2/\Gamma$, where $\Gamma$ is spanned by an orthogonal basis $\{v_1, v_2\}$ of $\mathbf{R}^2$. We want to describe $\mathrm{Cut}(\bar{p})$, where $\bar{p} = \pi(p)$ for some $p \in \mathbf{R}^2$ and $\pi : \mathbf{R}^2 \to \mathbf{R}^2/\Gamma$ is the projection. For simplicity, assume $p = \frac{1}{2}(v_1 + v_2)$; this entails no loss of generality as $\mathbf{R}^2/\Gamma$ is homogeneous. Then $p$ is the center of the rectangle $\mathcal{R} = \{\, a_1v_1 + a_2v_2 \in \mathbf{R}^2 \mid 0 \leq a_1, a_2 \leq 1\,\}$. If $\bar{x} = \pi(x)$ for some $x \in \mathbf{R}^2$, then the geodesics joining $\bar{p}$ to $\bar{x}$ are exactly the projections of the line segments in $\mathbf{R}^2$ joining $p$ to a point in $x + \Gamma$. It follows that if $\gamma$ is a line in $\mathbf{R}^2$ starting at $p$ and $\bar{\gamma} = \pi \circ \gamma$ is the corresponding geodesic in $\mathbf{R}^2/\Gamma$ starting at $\bar{p}$, then $\bar{\gamma}$ is minimizing before $\gamma$ goes out of $\mathcal{R}$, and not afterwards. It follows that $\exp_p(\mathrm{D}_{\bar{p}}) = \pi(\mathrm{int}\,\mathcal{R})$ and $\mathrm{Cut}(\bar{p}) = \pi(\partial\mathcal{R})$ is homeomorphic to the bouquet of two circles $S^1 \vee S^1$.

### Riemannian submersions and $\mathbf{C}P^n$

We first describe the behavior of geodesics with regard to Riemannian submersions. Let $\pi : \tilde{M} \to M$ be a Riemannian submersion, and denote by $\mathcal{H}$ the associated horizontal distribution in $\tilde{M}$. A smooth curve in $M$ is called *horizontal* if it is everywhere tangent to $\mathcal{H}$.

**3.5.1 Proposition** *Let $\pi : \tilde{M} \to M$ be a Riemannian submersion.*

*a. We have that $\pi$ is distance-nonincreasing (or* non-expanding*), namely,*

$$d(\pi(\tilde{x}), \pi(\tilde{y})) \leq d(\tilde{x}, \tilde{y})$$

*for every $\tilde{x}$, $\tilde{y} \in \tilde{M}$.*

*b. Let $\gamma$ be a geodesic of $M$. Given $\tilde{p} \in \pi^{-1}(\gamma(0))$, there exists a unique locally defined horizontal lift $\tilde{\gamma}$ of $\gamma$ with $\tilde{\gamma}(0) = \tilde{p}$, and $\tilde{\gamma}$ is a geodesic of $\tilde{M}$.*

*c. Let $\tilde{\gamma}$ be a geodesic of $\tilde{M}$. If $\tilde{\gamma}'(0)$ is a horizontal vector, then $\tilde{\gamma}'(t)$ is horizontal for every $t$ in the domain of $\tilde{\gamma}$ and the curve $\pi \circ \tilde{\gamma}$ is a geodesic of $M$ of the same length as $\tilde{\gamma}$.*

*d. If $\tilde{M}$ is complete, then so is $M$.*

*Proof.* (a) If $\tilde{\gamma}$ is a piecewise smooth curve on $\tilde{M}$ joining $\tilde{x}$ and $\tilde{y}$, then the curve $\pi \circ \tilde{\gamma}$ on $M$ is also piecewise smooth and joins $\pi(\tilde{x})$ and $\pi(\tilde{y})$. Moreover, $L(\pi \circ \tilde{\gamma}) \leq L(\tilde{\gamma})$, because the projection $d\pi : T\tilde{M} \to TM$ kills the vertical components of vectors and preserves the horizontal ones. It follows that $d(\pi(\tilde{x}), \pi(\tilde{y})) \leq d(\tilde{x}, \tilde{y})$.

(b) If $\gamma$ is constant, there is nothing to be proven, so we can assume that $\gamma$ is an immersion. Then there is $\epsilon > 0$ such that $N = \gamma(-\epsilon, \epsilon)$ is an embedded submanifold of $M$. Since $\pi$ is a submersion, the pre-image $\tilde{N} = \pi^{-1}(N)$ is an embedded submanifold of $\tilde{M}$. Now there is a smooth function $\phi : \tilde{N} \to (-\epsilon, \epsilon)$ such that $\pi(\tilde{x}) = \gamma(\phi(\tilde{x}))$ for every $\tilde{x} \in N$. Using this function, we can define a smooth horizontal vector field on $\tilde{N}$ by setting

$$\tilde{X}_{\tilde{x}} = (d\pi_{\tilde{x}}|_{\mathcal{H}_{\tilde{x}}})^{-1}(\gamma'(\phi(\tilde{x}))). \tag{3.5.2}$$

Given $\tilde{p} \in \pi^{-1}(\gamma(0)) \in \tilde{N}$, let $\tilde{\gamma}$ be the integral curve of $\tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{p}$. Then $\tilde{\gamma}$ is a horizontal curve locally defined around 0, and $\pi \circ \tilde{\gamma} = \gamma$ because of (3.5.2). It remains to see that $\tilde{\gamma}$ is a geodesic. Indeed, using Theorem 3.2.6 and (a) we have that for every $t_0$ in the domain of $\tilde{\gamma}$, there exists $\delta > 0$ such that

$$L(\tilde{\gamma}|_{[t_0,t_0+h]}) = L(\gamma|_{[t_0,t_0+h]}) = d(\gamma(t_0), \gamma(t_0+h)) \leq d(\tilde{\gamma}(t_0), \tilde{\gamma}(t_0+h))$$

for $0 < h < \delta$, and there is a similar formula for $-\delta < h < 0$. It follows that $\tilde{\gamma}$ is locally minimizing. Since $||\tilde{\gamma}'|| = ||\gamma||$ is a constant, $\tilde{\gamma}$ is already parametrized proportionally to arc-length, hence it is a geodesic.

(c) Let $\tilde{\gamma}$ be a geodesic of $\tilde{M}$ such that $\tilde{\gamma}'(t_0)$ is horizontal for some $t_0$ in the domain of $\tilde{\gamma}$. Put $\tilde{p} = \tilde{\gamma}(t_0)$ and suppose $\gamma$ is the geodesic of $M$ with initial conditions $\gamma(t_0) = \pi(\tilde{p})$ and $\gamma'(t_0) = d\pi_{\tilde{p}}(\tilde{\gamma}'(t_0))$. Using (b), we have a horizontal lift $\tilde{\eta}$ of $\gamma$ with $\tilde{\eta}(t_0) = \tilde{p}$, locally defined around $t_0$, which is also a geodesic of $\tilde{M}$. Since $\tilde{\gamma}'(t_0)$ and $\tilde{\eta}'(t_0)$ are both horizontal vectors, they coincide and it follows that $\tilde{\gamma}$ and $\tilde{\eta}$ coincide on an open interval around $t_0$; on this interval, $\tilde{\gamma}'$ is horizontal and $\pi \circ \tilde{\gamma}$ is a geodesic. From the fact that the set of points in the domain of $\tilde{\gamma}$ where $\tilde{\gamma}'$ is horizontal is closed, we deduce that $\tilde{\gamma}'$ is horizontal wherever it is defined and $\pi \circ \tilde{\gamma}$ is a geodesic everywhere. The assertion about the lengths of $\tilde{\gamma}$ and $\gamma$ plainly follows from the fact that $d\pi_{\tilde{x}} : \mathcal{H}_{\tilde{x}} \to T_{\pi(\tilde{x})}M$ is a linear isometry for $\tilde{x} \in \tilde{M}$.

(d) Let $\gamma$ be a geodesic of $M$. By (b), $\gamma$ admits a horizontal lift $\tilde{\gamma}$ which turns out to be defined on $\mathbf{R}$ due to the completeness of $\tilde{M}$. It follows from (c) that $\pi \circ \tilde{\gamma}$ is a geodesic of $M$ defined on $\mathbf{R}$, which must clearly extend $\gamma$. Hence $M$ is complete. □

In the preceding proposition, it can happen that $M$ is complete but $\tilde{M}$ is not. This happens for instance if $\pi$ is the inclusion of a proper open subset of $\mathbf{R}^n$ into $\mathbf{R}^n$.

Next we turn to the question of describing the cut-locus of $\mathbf{C}P^n$. Consider the Riemannian submersion $\pi : S^{2n+1} \to \mathbf{C}P^n$ where as usual we view $S^{2n+1}$ as the unit sphere in $\mathbf{C}^{n+1}$. Note that $\mathbf{C}P^n$ is complete by Proposition 3.5.1(d). Let $\tilde{p} \in S^{2n+1}$. Since the fibers of $\pi$ are just the $S^1$-orbits, the vertical space $\mathcal{V}_{\tilde{p}} = \mathbf{R}(i\tilde{p})$. It follows that the horizontal space $\mathcal{H}_{\tilde{p}} \subset T_{\tilde{p}}S^{2n+1}$ is the orthogonal complement of $\mathbf{R}\{\tilde{p}, i\tilde{p}\} = \mathbf{C}\tilde{p}$ in $\mathbf{C}^{2n+1}$. In view of the proposition, the unit speed geodesics of $\mathbf{C}P^n$ starting at $p = \pi(\tilde{p})$ are of the form $\gamma(t) = \pi(\cos t\tilde{p} + \sin t\tilde{v})$, where $\tilde{v}$ is orthogonal to $\tilde{p}$ and $i\tilde{p}$. It follows that geodesics are defined on $\mathbf{R}$ and periodic of period $\pi$.

We agree to retain the above notations and consider another unit speed geodesic starting at $\tilde{p}$, $\eta(t) = \pi(\cos t\tilde{p} + \sin t\tilde{u})$, where $\tilde{u} \in \mathcal{H}_{\tilde{p}}$. Starting at $t = 0$, $\cos t\tilde{p} + \sin t\tilde{v}$ and $\cos t\tilde{p} + \sin t\tilde{u}$ become linearly dependent over $\mathbf{C}$ for the first time at $t = \pi$ (if $\tilde{v}$, $\tilde{u}$ are linearly independent over $\mathbf{C}$) or at $t = \pi/2$ (if $\tilde{v}$, $\tilde{u}$ are linearly dependent over $\mathbf{C}$). This means that $\gamma$ and $\eta$ meet for the first time at $t = \pi$ in the first case and at $t = \pi/2$ in the second one. It follows that $\gamma$ is minimizing on $[0, t_0]$ for $t_0 \leq \pi/2$. By using Lemma 3.4.1, It also follows that $\gamma$ is not minimizing on $[0, t_0]$ for $t_0 > \pi/2$.

It follows from the discussion in the previous paragraph that $\mathrm{D}_p = B(0_p, \frac{\pi}{2})$ and a typical point in $\mathrm{Cut}(p)$ is of the form $\gamma(\frac{\pi}{2}) = \pi(\tilde{v})$, where $\tilde{v}$ is a unit vector in $\mathcal{H}_{\tilde{p}}$. Since the unit sphere of $\mathcal{H}_{\tilde{p}}$ is isometric to $S^{2n-1}$, $\mathrm{Cut}(p) = \pi(S^{2n-1})$ turns out to be isometric to $\mathbf{C}P^{n-1}$.

## 3.6 Additional notes

§1 Let $(X, d)$ be a connected metric space and define the *length* of a continuous curve $\gamma : [a, b] \to X$ to be the supremum of the lengths of all "polygonal paths" inscribed in $\gamma$ that join $\gamma(a)$ to $\gamma(b)$, namely,

$$L(\gamma) = \sup_P \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where $P : a = t_0 < t_1 < \cdots < t_n = b$ runs over all subdivisions of the interval $[a, b]$. A curve is called *rectifiable* if its length is finite. Now $(X, d)$ is called a *length space* if the distance between any two points can be "almost" realized by the length of a continuous curve joining the two points, namely, for every $x, y \in X$,

$$d(x, y) = \inf_\gamma L(\gamma),$$

where $\gamma$ runs over the set of all continuous curves joining $x$ to $y$. Any piecewise smooth curve in a connected Riemannian manifold is rectifiable and its length in this sense coincides with its length in the sense of (1.3.5). It follows that the underlying metric space of a connected Riemannian manifold is a length space, but length spaces of course form a much larger class of metric spaces involving no a priori differentiability properties. Many concepts and results of Riemannian geometry admit generalizations to the class of length spaces. For instance, geodesics in length spaces are defined to be the continuous, locally minimizing curves, and one proves that if $(X, d)$ is a complete locally compact length space, then any two points are joined by a minimizing geodesic. There is a distance in the set of isometry classes of compact metric spaces called the *Gromov-Hausdorff distance* which turns it into a complete metric space itself (for noncompact spaces, a slightly more general notion of distance is used), and length spaces form a closed subset in this topology. In this sense, length spaces appear as limits of Riemannian manifolds. For an introduction to general length spaces, see [BBI01].

§2 Next, we give an interesting class of examples of length spaces. Namely, one starts with a connected Riemannian manifold $(M, g)$ of dimension $n$ equipped with a smooth distribution $\mathcal{D}$ of dimension $k$, where $1 < k < n$, and, for $x, y \in M$, declares $d(x, y) = \inf_\gamma L(\gamma)$ where the infimum is taken over the piecewise smooth curves $\gamma$ joining $x$ to $y$ such that $\gamma'$ *is tangent to* $\mathcal{D}$ *whenever defined*. If $\mathcal{D}$ is sufficiently generic, in the sense that iterated brackets of arbitrary length of locally defined sections of $\mathcal{D}$ span $TM$ at every point, then one shows that $d$ is finite and $(M, d)$ is a length space. Note that in this definition we have only used the restriction of $g$ to the sections of $\mathcal{D}$. A triple $(M, \mathcal{D}, g)$ where $M$ is a smooth manifold, $\mathcal{D}$ is a bracket-generating smooth distribution as above and $g$ is a smoothly varying choice of inner products on the fibers of $\mathcal{D}$ is called a *sub-Riemannian manifold*, and the associated length space $(M, d)$ is called a *Carnot-Carathéodory space*; such spaces appear for instance in mechanics with non-holonomic constraints and geometric control theory. A very interesting feature of a Carnot-Carathéodory space is that its Hausdorff dimension is always stricly bigger than its manifold dimension. For further reading about sub-Riemannian geometry, we recommend [BR96, Mon02].

## 3.7 Exercises

**1** Let $(M, g)$ be a connected Riemannian manifold and consider the underlying metric space structure $(M, d)$. Prove that any isometry $f$ of $(M, g)$ is *distance-preserving*, that is, it satisfies the condition that $d(f(x), f(y)) = d(x, y)$ for every $x, y \in M$.

**2** Describe the isometry group $G$ of $\mathbf{R}^n$:

*a.* Show that $G$ is generated by orthogonal transformations and translations.
*b.* Show that $G$ is isomorphic to the semidirect product $\mathbf{O}(n) \ltimes \mathbf{R}^n$, where

$$(B, w) \cdot (A, v) = (BA, Bv + w)$$

for $A$, $B \in \mathbf{O}(n)$ and $v$, $w \in \mathbf{R}^n$.
(Hint: Use the result of the previous exercise.)

**3** Prove that every isometry of the unit sphere $S^n$ of Euclidean space $\mathbf{R}^{n+1}$ is the restriction of a linear orthogonal transformation of $\mathbf{R}^{n+1}$. Deduce that the isometry group of $S^n$ is isomorphic to $\mathbf{O}(n+1)$. What is the isometry group of $\mathbf{R}P^n$?

**4** Prove that every isometry of the hyperboloid model of $\mathbf{R}H^n$ is the restriction of a linear Lorentzian orthochronous transformation of $\mathbf{R}^{1,n}$. Deduce that the isometry group of $\mathbf{R}H^n$ is isomorphic to $\mathbf{O}^+(1, n)$.

**5** A *ray* in a complete Riemannian manifold $M$ is a unit speed geodesic $\gamma : [0, +\infty) \to \mathbf{R}$ such that $d(\gamma(0), \gamma(t)) = t$ for all $t \geq 0$. We say that the ray $\gamma$ *emanates from* $\gamma(0)$.

Let $M$ be a complete Riemannian manifold and assume that $M$ is noncompact. Prove that, for every $p \in M$, there exists a ray $\gamma$ emanating from $p$.

**6** A *line* in a complete Riemannian manifold $M$ is a unit speed geodesic $\gamma : \mathbf{R} \to M$ such that $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t$, $s \geq 0$. Also, $M$ is called *connected at infinity* if for every compact set $K \subset M$ there is a compact set $C \supset K$ such that any two points in $M \setminus C$ can be joined by a curve in $M \setminus K$. If $M$ is not connected at infinity, we say that $M$ is *disconnected at infinity.*

Let $M$ be a complete Riemannian manifold and assume that $M$ is noncompact and disconnected at infinity. Prove that $M$ contains a line.

**7** Prove that the following assertions for a Riemannian manifold $M$ are equivalent:
*a.* $M$ is complete.
*b.* There exists $p \in M$ such that the function $x \mapsto d(p, x)$ is a proper function on $M$.
*c.* For every $p \in M$, the function $x \mapsto d(p, x)$ is a proper function on $M$.

**8** A smooth curve $\gamma : I \to M$ in a Riemannian manifold $M$ defined on an interval $I \subset \mathbf{R}$ is said to be *divergent* if the image of $\gamma$ does not lie in any compact subset of $M$.

Prove that a Riemannian manifold is complete if and only if every divergent curve in $M$ has infinite length.

**9** Prove that on any smooth manifold a complete Riemannian metric can be defined.

**10** Let $M$ be a smooth manifold with the property that it is complete with respect to any Riemannian metric in it. Prove that $M$ must be compact. (Hint: Use the results of exercises 5 and 8.)

**11** Describe the cut locus of a point in an hexagonal flat 2-torus. Note that its homeomorphism type is different from that of the cut locus of a point in a rectangular flat 2-torus (compare Examples 3.5).

**12** Let $M_i$ be complete Riemannian manifolds, where $i = 1, 2$.

*a*. Show that the product Riemannian manifold $M_1 \times M_2$ is also complete.
*b*. Let $p_i \in M_i$, where $i = 1, 2$. Show that the cut locus of $(p_1, p_2)$ in $M_1 \times M_2$ is given by $\big(\mathrm{Cut}(p_1) \times M_2\big) \cup \big(M_1 \times \mathrm{Cut}(p_2)\big)$.

**13** A Riemannian manifold $M$ is called *homogeneous* if given any two points of $M$ there exists an isometry of $M$ that maps one point to the other.

Prove that a homogeneous Riemannian manifold is complete.

**14** A Riemannian manifold $M$ is called *two point-homogeneous* if given any two equidistant pairs of points of $M$ there exists an isometry of $M$ that maps one pair to the other.

Prove that a Riemannian manifold is two point-homogeneous if and only if it is isotropic.

**15** Let $f,\, g : M \to N$ be local isometries between Riemannian manifolds where $M$ is connected. Assume there exists $p \in M$ such that $f(p) = g(p) = q$ and $df_p = dg_p : T_pM \to T_qN$. Prove that $f = g$. (Hint: Show that the set of points of $M$ where $f$ and $g$ coincide up to first order is closed and open.)

**16** Let $\gamma : (a, b) \to M$ be a smooth curve in a Riemannian manifold $M$. Prove that

$$||\gamma'(t)|| = \lim_{h \to 0} \frac{d\big(\gamma(t+h), \gamma(t)\big)}{h}$$

for $t \in (a, b)$. (Hint: Use a normal neighborhood of $\gamma(t)$.)

**17** Let $(M, g)$ and $(M', g')$ be Riemannian manifolds, and let $d$ and $d'$ be the associated distances, respectively. Show that a distance-preserving map $f : M \to M'$ (cf. exercise 1) is smooth and a local isometry. (Hint: use a normal neighborhood for the smoothness and exercise 16 to prove it is a local isometry.) Conclude that if $f$ is in addition surjective, then it is a global isometry.

**18** Let $M$ be a compact Riemannian manifold of dimension at least two. Prove that the following assertions are equivalent:
*a*. $M$ is simply-connected;
*b*. $Cut(p)$ is simply-connected for all $p \in M$;
*c*. $Cut(p)$ is simply-connected for some $p \in M$.

**19** Let $\pi : M \to N$ be a smooth submersion and fix a complementary subbundle $\mathcal{H}$ to the vertical bundle $\mathcal{V} = \ker d\pi$ in $TM$. Prove that any smooth curve in $N$ locally admits a horizontal smooth lift. (Hint: Work in a coordinate system on which $\pi$ has the standard form of submersions, and express the condition that a smooth curve in $M$ is the horizontal lift of a given smooth curve in $N$ as a system of linear ordinary differential equations.)