CHAPTER 1

Riemannian manifolds

1.1 Introduction

A Riemannian metric is a family of smoothly varying inner products on the tangent spaces of a smooth manifold. Riemannian metrics are thus infinitesimal objects, but they can be used to measure distances on the manifold. They were introduced by Riemann in his seminal work [Rie53] in 1854. At that time, the concept of a manifold was extremely vague and, except for some known global examples, most of the work of the geometers focused on local considerations, so the modern concept of a Riemannian manifold took quite some time to evolve to its present form. We point out the seemingly obvious fact that a given smooth manifold can be equipped with many different Riemannian metrics. This is really one of the great insights of Riemann, namely, the separation between the concepts of space and metric.

This chapter is mainly concerned with examples.

1.2 Riemannian metrics

Let M be a smooth manifold. A *Riemannian metric* g on M is a smoothly varying family of inner products on the tangent spaces of M. Namely, g associates to each $p \in M$ a positive definite symmetric bilinear form on T_pM,

$$g_p : T_pM \times T_pM \to \mathbf{R},$$

and the smoothness condition on g refers to the fact that the function

$$p \in M \mapsto g_p(X_p, Y_p) \in \mathbf{R}$$

must be smooth for every locally defined smooth vector fields X, Y in M. A *Riemannian manifold* is a pair (M, g) where M is a differentiable manifold and g is a Riemannian metric on M. Later on (but not in this chapter), we will often simplify the notation and refer to M as a Riemannian manifold where the Riemannian metric is implicit.

Let (M, g) be a Riemannian manifold. If $(U, \varphi = (x^1, \ldots, x^n))$ is a chart of M, a local expression for g can be given as follows. Let $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ be the coordinate vector fields, and let $\{dx^1, \ldots, dx^n\}$ be the dual 1-forms. For $p \in U$ and u, $v \in T_pM$, we write

$$u = \sum_i u^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{and} \quad v = \sum_j v^j \frac{\partial}{\partial x^j}\Big|_p.$$

Then, by bilinearity,

$$\begin{aligned} g_p(u,v) &= \sum_{i,j} u^i v^j g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= \sum_{i,j} g_{ij}(p) u^i v^j, \end{aligned}$$

where we have set

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Note that $g_{ij} = g_{ji}$. Hence we can write

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j = \sum_{i \leq j} \tilde{g}_{ij}\, dx^i dx^j, \tag{1.2.1}$$

where $\tilde{g}_{ii} = g_{ii}$, $\tilde{g}_{ij} = 2g_{ij}$ if $i < j$, and $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$.

Next, let $(U', \varphi' = (x'^1, \ldots, x'^n))$ be another chart of M such that $U \cap U' \neq \varnothing$. Then

$$\frac{\partial}{\partial x'^i} = \sum_k \frac{\partial x^k}{\partial x'^i} \frac{\partial}{\partial x^k},$$

so the relation between the local expressions of g with respect to (U, φ) and (U', φ') is given by

$$g'_{ij} = g\left(\frac{\partial}{\partial x'^i}, \frac{\partial}{\partial x'^j}\right) = \sum_{k,l} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j}\, g_{kl}.$$

1.2.2 Examples (a) The canonical Euclidean metric is expressed in Cartesian coordinates by $g = dx^2 + dy^2$. Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ yields that

$$dx = \cos\theta dr - r\sin\theta d\theta \quad \text{and} \quad dy = \sin\theta dr + r\cos\theta d\theta,$$

so

$$\begin{aligned} g &= dx^2 + dy^2 \\ &= (\cos^2\theta dr^2 + r^2 \sin^2\theta d\theta^2 - 2r\sin\theta\cos\theta dr d\theta) \\ &\qquad +(\sin^2\theta dr^2 + r^2\cos^2\theta d\theta^2 + 2r\sin\theta\cos\theta dr d\theta) \\ &= dr^2 + r^2 d\theta^2. \end{aligned}$$

(b) A classical example is the surface of revolution parametrized by

$$\mathbf{x}(r,\theta) = (a(r)\cos\theta, a(r)\sin\theta, b(r)),$$

where $a > 0$, b are smooth functions defined on some interval and the generatrix $\gamma(r) = (a(r), 0, b(r))$ has $||\gamma'||^2 = (a')^2 + (b')^2 = 1$, equipped with the metric g induced from $\mathbf{R}^3$. Namely, the tangent spaces to the surface are subspaces of $\mathbf{R}^3$, so we can endow them with inner products just by taking the restrictions of the Euclidean dot product in $\mathbf{R}^3$. The tangent spaces are spanned by the partial derivatives $\mathbf{x}_r = (\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r})$, $\mathbf{x}_\theta = (\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta})$, and then $g = (\mathbf{x}_r \cdot \mathbf{x}_r)\, dr^2 + 2(\mathbf{x}_r \cdot \mathbf{x}_\theta)\, dr d\theta + (\mathbf{x}_\theta \cdot \mathbf{x}_\theta)\, d\theta^2$. Equivalently, from

$$\begin{aligned} dx &= a'(r)\cos\theta\, dr - a(r)\sin\theta\, d\theta \\ dy &= a'(r)\sin\theta\, dr + a(r)\cos\theta\, d\theta \\ dz &= b'(r)\, dr \end{aligned}$$

we obtain

$$\begin{aligned} g &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + a(r)^2\, d\theta^2. \end{aligned}$$

⋆

The functions g_{ij} are smooth on U and, for each $p \in U$, the matrix $(g_{ij}(p))$ is symmetric and positive-definite. Conversely, a Riemannian metric in U can be obviously specified by these data.

1.2.3 Proposition *Every smooth manifold can be endowed with a Riemannian metric.*

Proof. Let $M = \cup_\alpha U_\alpha$ be a covering of M by domains of charts $\{(U_\alpha, \varphi_\alpha)\}$. For each α, consider the Riemannian metric g_α in U_α whose local expression $((g_\alpha)_{ij})$ is the identity matrix. Let $\{\rho_\alpha\}$ be a smooth partition of unity of M subordinate to the covering $\{U_\alpha\}$, and define

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

Since the family of supports of the ρ_α is locally finite, the above sum is locally finite, and hence g is well defined and smooth, and it is bilinear and symmetric at each point. Since $\rho_\alpha \geq 0$ for all α and $\sum_\alpha \rho_\alpha = 1$, it also follows that g is positive definite, and thus is a Riemannian metric in M. □

The proof of the preceding proposition suggests the fact that there exists a vast array of Riemannian metrics on a given smooth manifold. Even taking into account equivalence classes of Riemannian manifolds, the fact is that there many uninteresting examples of Riemannian manifolds, so an important part of the work of the differential geometer is to sort out relevant families of examples.

Let (M, g) and (M', g') be Riemannian manifolds. A *isometry* between (M, g) and (M', g') is diffeomorphism $f : M \to M'$ whose differential is a linear isometry between the corresponding tangent spaces, namely,

$$g_p(u, v) = g'_{f(p)}(df_p(u), df_p(v)),$$

for every $p \in M$ and u, $v \in T_pM$. We say that (M, g) and (M', g') are *isometric Riemannian manifolds* if there exists an isometry between them. This completes the definition of the category of Riemannian manifolds and isometric maps. Note that the set of all isometries of a Riemannian manifold (M, g) forms a group, called the *isometry group* of (M, g), with respect to the operation of composition of mappings, which we will denote by $\mathrm{Isom}(M, g)$. Here we quote without proof the following important theorem [MS39].

1.2.4 Theorem (Myers-Steenrod) *The isometry group* $\mathrm{Isom}(M, g)$ *of a Riemannian manifold* (M, g) *has the structure of a Lie group with respect to the compact-open topology. Its isotropy subgroup at an arbitrary fixed point is compact. Moreover,* $\mathrm{Isom}(M, g)$ *is compact if* M *is compact.*

The isometry group is a Riemannian-geometric invariant in the sense that if $f : (M, g) \to (M', g)$ is an isometry between Riemannian manifolds, then $\alpha \mapsto f \circ \alpha \circ f^{-1}$ defines an isomorphism $\mathrm{Isom}(M, g) \to \mathrm{Isom}(M', g')$.

A *local isometry* from (M, g) into (M', g') is a smooth map $f : M \to M'$ satisfying the condition that every point $p \in M$ admits a neighborhood U such that the restriction of f to U is an isometry onto its image. In particular, f is a local diffeomorphism. Note that a local isometry which is bijective is an isometry.

1.3 Examples

The Euclidean space

The Euclidean space is $\mathbf{R}^n$ equipped with its standard scalar product. The essential feature of $\mathbf{R}^n$ as a smooth manifold is that, since it is *the* model space for finite dimensional smooth manifolds, it admits a global chart given by the identity map. Of course, the identity map establishes canonical isomorphisms of the tangent spaces of $\mathbf{R}^n$ at each of its points with $\mathbf{R}^n$ itself. Therefore an arbitrary Riemannian metric in $\mathbf{R}^n$ can be viewed as a smooth family of inner products in $\mathbf{R}^n$. In particular, by taking the constant family given by the standard scalar product, we get the canonical Riemannian structure in $\mathbf{R}^n$. In this book, unless explicitly stated, we will always use its canonical metric when referring to $\mathbf{R}^n$ as a Riemannian manifold.

If $(x_1, \ldots, x_n)$ denote the standard coordinates on $\mathbf{R}^n$, then it is readily seen that the local expression of the canonical metric is

$$dx_1^2 + \cdots + dx_n^2. \tag{1.3.1}$$

More generally, if a Riemannian manifold (M, g) admits local coordinates such that the local expression of g is as in (1.3.1), then (M, g) is called *flat* and g is called a *flat metric* on M. Note that, if g is a flat metric on M, then the coordinates used to express g as in (1.3.1) immediately define a local isometry between (M, g) and Euclidean space $\mathbf{R}^n$.

Riemannian submanifolds and isometric immersions

Let (M, g) be a Riemannian manifold and consider an immersed submanifold $\iota : N \to M$. This means that N is a smooth manifold and ι is an injective immersion. Then the Riemannian metric g induces a Riemannian metric g_N in N as follows. Let $p \in N$. The tangent space T_pN can be viewed as a subspace of T_pM via the injective map $d\iota_p : T_pN \to T_{\iota(p)}M$. We define $(g_N)_p$ to be simply the restriction of g to this subspace, namely,

$$(g_N)_p(u, v) = g_{\iota(p)}(d\iota_p(u), d\iota_p(v)),$$

where u, $v \in T_pN$. It is clear that g_N is a Riemannian metric. We call g_N the *induced Riemannian metric in* N, and we call (N, g_N) a *Riemannian submanifold* of (M, g).

Note that the definition of g_N makes sense even if ι is an immersion that is not necessarily injective. In general, we call g_N the *pulled-back metric*, write $g_N = \iota^* g$, and say that $\iota : (N, g_N) \to (M, g)$ is an *isometric immersion* (of course, any immersion must be locally injective). On another note, an isometry $f : (M, g) \to (M', g')$ is a diffeomorphism satisfying $f^*(g') = g$.

A very important particular case is that of Riemannian submanifolds of Euclidean space (compare example 1.2.2(b)) Historically speaking, the study of Riemannian manifolds was preceded by the theory of curves and surfaces in $\mathbf{R}^3$. In the classical theory, one uses parametrizations instead of local charts, and these objects are called *parametrized curves* and *parametrized surfaces* since they usually already come with the parametrization. In the most general case, the parametrization is only assumed to be smooth. One talks about a *regular curve* or a *regular surface* if one wants the parametrization to be an immersion. Of course, in this case it follows that the parametrization is locally an embedding. This is good enough for the classical theory, since it is really concerned with local computations.

The sphere S^n

The canonical Riemannian metric in the sphere S^n is the Riemannian metric induced by its embedding in $\mathbf{R}^{n+1}$ as the sphere of unit radius. When one refers to S^n as a Riemannian manifold with its canonical Riemannian metric, sometimes one speaks of "the unit sphere", or "the metric sphere", or the "Euclidean sphere", or "the round sphere". One also uses the notation $S^n(R)$ to specify a sphere of radius R embedded in $\mathbf{R}^{n+1}$ with the induced metric. In this book, unless explicitly stated, we will always use the canonical metric when referring to S^n as a Riemannian manifold.

Product Riemannian manifolds

Let (M_i, g_i), where $i = 1, 2$, denote two Riemannian manifolds. Then the product smooth manifold $M = M_1 \times M_2$ admits a canonical Riemannian metric g, called the *product Riemannian metric*, given as follows. The tangent space of M at a point $p = (p_1, p_2) \in M_1 \times M_2$ splits as $T_pM = T_{p_1}M_1 \oplus T_{p_2}M_2$. Given u, $v \in T_pM$, write accordingly $u = u_1 + u_2$ and $v = v_1 + v_2$, and define

$$g_p(u, v) = g_{p_1}(u_1, v_1) + g_{p_2}(u_2, v_2).$$

It is clear that g is a Riemannian metric. Note that it follows from this definition that $T_{p_1}M_1 \oplus \{0\}$ is orthogonal to $\{0\} \oplus T_{p_2}M_2$. We will sometimes write that $(M, g) = (M_1, g_1) \times (M_2, g_2)$, or that $g = g_1 + g_2$.

It is immediate to see that Euclidean space $\mathbf{R}^n$ is the Riemannian product of n copies of $\mathbf{R}$.

Conformal Riemannian metrics

Let (M, g) be a Riemannian manifold. If f is a nowhere zero smooth function on M, then f^2g defined by

$$(f^2g)_p(u, v) = f^2(p)g_p(u, v),$$

where $p \in M$, u, $v \in T_pM$, is a new Riemannian metric on M which is said to be *conformal* to g. The idea behind this definition is that g and f^2g define the same angles between pairs of tangent vectors. We say that (M, g) is *conformally flat* if M can be covered by open sets on each of which g is conformal to a flat metric.

A particular case happens if f is a nonzero constant in which f^2g is said to be *homothetic* to g.

The real hyperbolic space $\mathbf{R}H^n$

To begin with, consider the Lorentzian inner product in $\mathbf{R}^{n+1}$ given by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n,$$

where $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n) \in \mathbf{R}^{n+1}$. We will write $\mathbf{R}^{1,n}$ to denote $\mathbf{R}^{n+1}$ with such a Lorentzian inner product. Note that if $p \in \mathbf{R}^{1,n}$ is such that $\langle p, p\rangle < 0$, then the restriction of $\langle , \rangle$ to $\langle p\rangle^\perp$ (the the orthogonal complement to p with regard to $\langle , \rangle$) is positive-definite (compare Exercise 15). Note also that the equation $\langle x, x\rangle = -1$ defines a two-sheeted hyperboloid in $\mathbf{R}^{1,n}$.

Now we can define the *real hyperbolic space* as the following submanifold of $\mathbf{R}^{1,n}$,

$$\mathbf{R}H^n = \{\, x \in \mathbf{R}^{1,n} \mid \langle x, x\rangle = -1 \quad \text{and} \quad x_0 > 0 \,\},$$

equipped with a Riemannian metric g given by the restriction of $\langle , \rangle$ to the tangent spaces at its points. Since the tangent space of the hyperboloid at a point p is given by $\langle p\rangle^\perp$, the Riemannian metric g turns out to be well defined. Actually, this submanifold is sometimes called the *hyperboloid model of* $\mathbf{R}H^n$ (compare Exercises 3 and 4). This model brings about the duality between S^n and $\mathbf{R}H^n$ in the sense that one can think of the hyperboloid as the sphere of unit imaginary radius in $R^{1,n}$. Of course, as a smooth manifold, $\mathbf{R}H^n$ is diffeomorphic to $\mathbf{R}^n$.

Flat tori

A *lattice* Γ in $\mathbf{R}^n$ (or, more generally, in a real vector space) is the additive subgroup of $\mathbf{R}^n$ consisting of integral linear combinations of the vectors in a fixed basis. Namely, if $\{v_1, \ldots, v_n\}$ is a basis of $\mathbf{R}^n$, then it defines the lattice $\Gamma = \{\sum_{j=1}^n m_j v_j \mid m_1, \ldots, m_n \in \mathbf{Z}\}$. For a given lattice Γ we consider the quotient group $\mathbf{R}^n/\Gamma$ in which two elements p, $q \in \mathbf{R}^n$ are identified if $q - p \in \Gamma$. We will show that $M = \mathbf{R}^n/\Gamma$ has the structure of a compact smooth manifold of dimension n diffeomorphic to a product of n copies of S^1, which we denote by T^n. Moreover there is a naturally defined flat metric g_Γ on M; the resulting Riemannian manifold is called a *flat torus*. We also denote it by (T^n, g_Γ).

Relevant for the topology of M will be the discreteness of Γ as an additive subgroup of $\mathbf{R}^n$, namely: any bounded subset of $\mathbf{R}^n$ meets Γ in finitely many points only. In fact, if $p = \sum_{j=1}^n m_j v_j$ is a lattice point viewed as a column vector, then

$$p = M \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

where M is the (invertible) matrix having the $v_1, \ldots, v_n$ as columns. We obtain

$$|m_i| \leq \left(\sum_{j=1}^n m_j^2 \right)^{1/2} \leq ||M^{-1}||\,||p||$$

for all $i = 1, \ldots, n$, where $||\cdot||$ denotes the Euclidean norm. Therefore if we require p to lie in a given bounded subset of $\mathbf{R}^n$, then there are only finitely many possibilities for the integers m_j, and thus only finitely many such lattice points. Note that discreteness of Γ implies that Γ, and thus any equivalence class $p + \Gamma$, is a closed subset of $\mathbf{R}^n$.

Equip M with the quotient topology induced by the canonical projection $\pi : \mathbf{R}^n \to M$ that maps each $p \in \mathbf{R}^n$ to its equivalence class $[p] = p + \Gamma$. Then π is continuous. It follows that M is compact since it coincides with the image of $\{\sum_{j=1}^n x_j v_j \mid 0 \leq x_j \leq 1\}$ under the projection π. Moreover, π is an open map, as for an open subset W of $\mathbf{R}^n$ we have that $\pi^{-1}(\pi(W)) = \cup_{\gamma \in \Gamma} (W + \gamma)$ is a union of open sets and thus open. It follows that the projection of a countable basis of open sets of $\mathbf{R}^n$ is a countable basis of open sets of M. We also see that the quotient topology is Hausdorff. In fact, given $[p]$, $[q] \in \mathbf{R}^n/\Gamma$, $[p] \neq [q]$, the minimal distance r_{pq} from p to a point in the closed subset $q + \Gamma$ is positive. Let W_p, W_q be the balls of radius $\frac{r_{pq}}{2}$ centered at p, q, respectively. A point $x \in W_p \cap (W_q + \Gamma)$ satisfies $d(x, p) < \frac{r}{2}$ and $d(x, q + \gamma) < \frac{r}{2}$ for some $\gamma \in \Gamma$, and therefore $d(p, q+\gamma) \leq d(p, x) + d(x, q+\gamma) < r$ leading to a contradiction. It follows that $W_p \cap (W_q + \Gamma) = \varnothing$ and hence $\pi(W_p)$, $\pi(W_q)$ are disjoint open neighborhoods of $[p]$, $[q]$, respectively.

We next check that $\pi : \mathbf{R}^n \to M$ is a covering. In fact, discreteness of Γ implies that the minimal distance s from a non-zero lattice point to the origin is positive. Note that s is also the minimal distance from any given point $p \in \mathbf{R}^n$ to another point in $p + \Gamma$. Let V be the ball of radius $\frac{s}{2}$ centered at p. Then $V \cap (V + \gamma) = \varnothing$ 0for all $\gamma \in \Gamma \setminus \{0\}$. Note also that $\pi : V \to \pi(V)$ is continuous, open and injective, thus a homemorphism. Now $\pi^{-1}(\pi(V)) = \cup_{\gamma \in \Gamma}(V + \gamma)$ is a disjoint union of open sets on each of which π is a homeomorphism onto $\pi(V)$, proving that $\pi(V)$ is an evenly covered neighborhood and hence π is a covering map. Since $\mathbf{R}^n$ is simply-connected, this is the universal covering and the fundamental group of M is isomorphic to Γ.

Now we have natural local charts for M defined on any evenly covered neighborhood $U = \pi(V)$ as above. Indeed, write $\pi^{-1}U = \cup_{\gamma\in\Gamma}(V+\gamma)$ and take as chart $\varphi_V = (\pi|_V)^{-1} : U \to V$. If $U' = \pi(V')$ is another evenly covered neighborhood as above with $U\cap U' \neq \varnothing$, consider a connected component W of $U \cap U'$, take $p \in V$ such that $[p] \in W$ and note that there is a unique $\gamma \in \Gamma$ such that $p+\gamma \in V'$. Now $\tau_\gamma \circ \varphi_V|_W$ and $\varphi_{V'}|_W$, where τ_γ denotes the translation by γ, are both lifts of the identity map of $\pi(W)$ and coincide on $[p]$, hence $\tau_\gamma \circ \varphi_V|_W = \varphi_{V'}|_W$ (Theorem 0.2.12). This proves that the transition map $\varphi_{V'} \circ \varphi_V^{-1}$ coincides with τ_γ on W and is thus smooth. In this way we have defined a smooth atlas for M. The covering map $\pi : \mathbf{R}^n \to M$ is smooth and in fact a local diffeomorphism because $\pi|_V$ composed with φ_V on the left yields as local representation the identity, so we indeed have a smooth covering. The smooth structure on M is the unique one that makes $\pi : \mathbf{R}^n \to M$ into a smooth covering (this is more than a covering whose covering map is smooth, compare page 8!).

The transition maps of the above atlas are restrictions of translations of $\mathbf{R}^n$ and thus isometries. In account of this, M acquires a natural quotient Riemannian metric g_Γ, which is the unique one making the covering map π into a local isometry. In fact this requirement implies uniqueness of g_Γ, as it imposes that on an evenly covered neighborhood $U = \pi(V)$ as above, the local chart $\varphi_V = (\pi|_V)^{-1}$ must be a local isometry and so $g_\Gamma = \varphi_V^* g$ on U, where g denotes the canonical metric in $\mathbf{R}^n$. To have existence of g_Γ, we need to check that it is well defined, namely, for another evenly covered neighborhood $U' = \pi(V')$ as above with $U \cap U' \neq \varnothing$ it holds that $\varphi_V^* g = \varphi_{V'}^* g$ on $U \cap U'$. However, this follows from $\varphi_{V'}^* g = \left((\varphi_{V'}\varphi_V^{-1})\varphi_V\right)^* g = \varphi_V^*(\varphi_{V'}\varphi_V^{-1})^* g = \varphi_V^* g$ as $(\varphi_{V'}\varphi_V^{-1})^* g = g$. Note that g_Γ is a flat metric.

As a smooth manifold, M is diffeomorphic to the n-torus T^n. In fact, define a map $f : \mathbf{R}^n \to T^n$ by setting

$$f\Big(\sum_{j=1}^n x_j v_j\Big) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}),$$

where we view S^1 as the set of unit complex numbers. Then f is constant on Γ, so it induces a bijection $\bar f : M \to T^n$. Suitable restrictions of

$$(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}) \mapsto (x_1, \ldots, x_n)$$

define local charts of T^n whose domains cover it. Now $f = \bar f \circ \pi$ composed on the left with such charts of T^n give $\sum_{j=1}^n x_j v_j \mapsto (x_1, \ldots, x_n)$, the restriction of an invertible linear map. It follows that $\bar f$ is a local diffeomorphism and hence a diffeomorphism.

We remark that different lattices may give rise to nonisometric flat tori, although they will always be locally isometric one to the other since they are all isometrically covered by Euclidean space; in other words, for two given lattices Γ, Γ', suitable restrictions of the identity map id : $\mathbf{R}^n \to \mathbf{R}^n$ induce locally defined isometries $\mathbf{R}^n/\Gamma \to \mathbf{R}^n/\Gamma'$.

One way to globally distinguish the isometry classes of tori obtained from different lattices is to show that they have different isometry groups. To fix ideas, let $n = 2$, and consider in $\mathbf{R}^2$ the lattices Γ, Γ' respectively generated by the bases $\{(1,0),(0,1)\}$ and $\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}$. Then $\mathbf{R}^2/\Gamma$ is called a square flat torus and $\mathbf{R}^2/\Gamma'$ is called an hexagonal flat torus. The isotropy subgroup of the square torus at an arbitrary point is isomorphic to the dihedral group D_4 (of order 8) whereas the isotropy subgroup of the hexagonal torus at an arbitrary point is isomorphic to the dihedral group D_3. Hence $\mathbf{R}^2/\Gamma$ and $\mathbf{R}^2/\Gamma'$ are not isometric. See exercise 9 for a characterization of isometric flat tori.

We finish the discussion of this example by noting that we could have introduced the smooth structure on M and the smooth covering $\pi : \mathbf{R}^n \to M$ by invoking Theorem 0.2.13, which we have avoided only for pedagogical reasons. In fact, the elements of Γ can be identified with the translations of $\mathbf{R}^n$ that they define and, in this way, Γ becomes a discrete group acting on $\mathbf{R}^n$. Plainly, the action is free. It is also proper, as this follows from the existence of $r > 0$ such that $d(p, q+\Gamma) \geq r$ if $p \neq q$ and $d(p, p+\Gamma \setminus \{0\}) \geq r$, which was shown above. In the next subsection, we follow and extend this alternative approach to incorporate the construction of the quotient metric.

Riemannian coverings

A *Riemannian covering* between two Riemannian manifolds is a smooth covering that is also a local isometry. For instance, for a lattice Γ in $\mathbf{R}^n$ the projection $\pi : \mathbf{R}^n \to \mathbf{R}^n/\Gamma$ is a Riemannian covering.

If $\tilde{M}$ is a smooth manifold and Γ is a discrete group acting freely and properly by diffeomorphisms on $\tilde{M}$, then the quotient space $M = \Gamma\backslash\tilde{M}$ endowed with the quotient topology admits a unique structure of smooth manifold such that the projection $\pi : \tilde{M} \to M$ is a smooth covering, owing to Theorem 0.2.13. If we assume, in addition, that $\tilde{M}$ is equipped with a Riemannian metric $\tilde{g}$ and Γ acts on $\tilde{M}$ by isometries, then we can show that there is a unique Riemannian metric g on M, called the *quotient metric*, so that $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ becomes a Riemannian covering, as follows. Around any point $p \in M$, there is an evenly covered neighborhood U such that $\pi^{-1}U = \cup_{i\in I}\tilde{U}_i$. If π is to be a local isometry, we must have

$$g = \left((\pi|_{\tilde{U}_i})^{-1}\right)^* \tilde{g}$$

on U, for any $i \in I$. In more pedestrian terms, we are forced to have

$$g_q(u, v) = \tilde{g}_{\tilde{q}_i}((d\pi_{\tilde{q}_i})^{-1}(u), (d\pi_{\tilde{q}_i})^{-1}(v)), \tag{1.3.2}$$

for all $q \in U$, u, $v \in T_qM$, $i \in I$, where $\tilde{q}_i = (\pi|_{\tilde{U}_i})^{-1}(q)$ is the unique point in the fiber $\pi^{-1}(q)$ that lies in $\tilde{U}_i$. We claim that this definition of g_q does not depend on the choice of point in $\pi^{-1}(q)$. In fact, if $\tilde{q}_j$ is another point in $\pi^{-1}(q)$, there is a unique $\gamma \in \Gamma$ such that $\gamma(\tilde{q}_i) = \tilde{q}_j$. Since $\pi \circ \gamma = \pi$, the chain rule gives that $d\pi_{\tilde{q}_j} \circ d\gamma_{\tilde{q}_i} = d\pi_{\tilde{q}_i}$, so

$$\begin{aligned} \tilde{g}_{\tilde{q}_i}((d\pi_{\tilde{q}_i})^{-1}(u), (d\pi_{\tilde{q}_i})^{-1}(v)) &= \tilde{g}_{\tilde{q}_i}((d\gamma_{\tilde{q}_i})^{-1}(d\pi_{\tilde{q}_j})^{-1}(u), (d\gamma_{\tilde{q}_i})^{-1}(d\pi_{\tilde{q}_j})^{-1}(v)) \\ &= \tilde{g}_{\tilde{q}_j}((d\pi_{\tilde{q}_j})^{-1}(u), (d\pi_{\tilde{q}_j})^{-1}(v)), \end{aligned}$$

since $d\gamma_{\tilde{q}_i} : T_{\tilde{q}_i}\tilde{M} \to T_{\tilde{q}_j}\tilde{M}$ is a linear isometry, checking the claim. Note that g is smooth since it is locally given as a pull-back metric.

On the other hand, if we start with a Riemannian manifold (M, g) and a smooth covering $\pi : \tilde{M} \to M$, then π is in particular an immersion, so we can endow $\tilde{M}$ with the pulled-back metric $\tilde{g}$ and $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ becomes a Riemannian covering. Let Γ denote the group of deck transformations of $\pi : \tilde{M} \to M$. An element $\gamma \in \Gamma$ satisfies $\pi \circ \gamma = \pi$. Since π is a local isometry, we have that γ is a local isometry, and being a bijection, it must be a global isometry. Hence the group Γ consists of isometries of $\tilde{M}$. If we assume, in addition, that $\pi : \tilde{M} \to M$ is a regular covering (meaning that Γ acts transitively on each fiber of π; this is true, for instance, if $\pi : \tilde{M} \to M$ is the universal covering), then M is diffeomorphic to the orbit space $\Gamma\backslash\tilde{M}$, and since we already know that $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ is a Riemannian covering, it follows from the uniqueness result of the previous paragraph that g must be the quotient metric of $\tilde{g}$.

The real projective space $\mathbf{R}P^n$

As a set, $\mathbf{R}P^n$ is the set of all lines through the origin in $\mathbf{R}^{n+1}$. It can also be naturally viewed as a quotient space in two ways. In the first one, we define an equivalence relation among points in $\mathbf{R}^{n+1} \setminus \{0\}$ by declaring x and y to be equivalent if they lie in the same line, namely, if there exists $\lambda \in \mathbf{R} \setminus \{0\}$ such that $y = \lambda x$. In the second one, we simply note that every line meets the unit sphere in $\mathbf{R}^{n+1}$ in two antipodal points, so we can also view $\mathbf{R}P^n$ as a quotient space of S^n and, in this case, $x, y \in S^n$ are equivalent if and only if $y = \pm x$. Of course, in both cases $\mathbf{R}P^n$ acquires the same quotient topology.

Next, we reformulate our point of view slightly by introducing the group Γ consisting of two isometries of S^n, namely the identity map and the antipodal map. Then Γ obviously acts freely and properly (it is a finite group!) on S^n, and the resulting quotient smooth structure makes $\mathbf{R}P^n$ into a smooth manifold. Furthermore, as the action of Γ is also isometric, $\mathbf{R}P^n$ immediately acquires a Riemannian metric such that $\pi : S^n \to \mathbf{R}P^n$ is a Riemannian covering.

The Klein bottle

Let $\tilde{M} = \mathbf{R}^2$, let $\{v_1, v_2\}$ be a basis of $\mathbf{R}^2$, and let Γ be the discrete group of transformations of $\mathbf{R}^2$ generated by the affine linear maps

$$\gamma_1(x_1v_1 + x_2v_2) = \left(x_1 + \frac{1}{2}\right)v_1 - x_2v_2 \quad \text{and} \quad \gamma_2(x_1v_1 + x_2v_2) = x_1v_1 + (x_2 + 1)v_2.$$

It is easy to see that Γ acts freely and properly on $\mathbf{R}^2$, so we get a quotient manifold $\mathbf{R}^2/\Gamma$ which is called the *Klein bottle* K^2. It is a compact non-orientable manifold, since γ_1 reverses the orientation of $\mathbf{R}^2$. It follows that K^2 cannot be embedded in $\mathbf{R}^3$ by the Jordan-Brouwer separation theorem; however, it is easy to see that it can be immersed there.

Consider $\mathbf{R}^2$ equipped with its canonical metric. Note that γ_2 is always an isometry of $\mathbf{R}^2$, but so is γ_1 if and only if the basis $\{v_1, v_2\}$ is orthogonal. In this case, Γ acts by isometries on $\mathbf{R}^2$ and K^2 inherits a flat metric so that the projection $\mathbf{R}^2 \to K^2$ is a Riemannian covering.

Riemannian submersions

Let $\pi : M \to N$ be a smooth submersion between two smooth manifolds. Then $\mathcal{V}_p = \ker d\pi_p$ for $p \in M$ defines a smooth distribution on M which is called the *vertical distribution*. Clearly, $\mathcal{V}$ can also be given by the tangent spaces of the fibers of π. In general, there is no canonical choice of a complementary distribution of $\mathcal{V}$ in TM, but in the case in which M comes equipped with a Riemannian metric, one can naturally construct such a complement $\mathcal{H}$ by setting $\mathcal{H}_p$ to be the orthogonal complement of $\mathcal{V}_p$ in T_pM. Then $\mathcal{H}$ is a smooth distribution which is called the *horizontal distribution*. Note that $d\pi_p$ induces an isomorphism between $\mathcal{H}_p$ and $T_{\pi(p)}N$ for every $p \in M$.

Having these preliminary remarks at hand, we can now define a smooth submersion $\pi : (M, g) \to (N, h)$ between two Riemannian manifolds to be a *Riemannian submersion* if $d\pi_p$ induces an isometry between $\mathcal{H}_p$ and $T_{\pi(p)}N$ for every $p \in M$. Note that Riemannian coverings are particular cases of Riemannian submersions.

Let (M, g) and (N, h) be Riemannian manifolds. A quite trivial example of a Riemannian submersion is the projection $(M \times N, g + h) \to (M, g)$ (or $(M \times N, g + h) \to (N, h)$). More generally, if f is a nowhere zero smooth function on N, the projection from $(M \times N, f^2g + h)$ onto (N, h) is a Riemannian submersion. In this case, the fibers of the submersion are homothetic but not necessarily isometric one to the other. A Riemannian manifold of the form $(M \times N, f^2 g + h)$ is called a *warped product*.

Recall that if $\tilde{M}$ is a smooth manifold and G is a Lie group acting freely and properly on $\tilde{M}$, then the quotient space $M = G\backslash\tilde{M}$ endowed with the quotient topology admits a unique structure of smooth manifold such that the projection $\pi : \tilde{M} \to M$ is a (surjective) submersion (Theorem 0.4.16). If in addition we assume that $\tilde{M}$ is equipped with a Riemannian metric $\tilde{g}$ and G acts on $\tilde{M}$ by isometries, then we can show that there is a unique Riemannian metric g on M, called the *quotient metric*, so that $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ becomes a Riemannian submersion. Indeed, given a point $p \in M$ and tangent vectors u, $v \in T_pM$, we set

$$g_p(u, v) = \tilde{g}_{\tilde{p}}(\tilde{u}, \tilde{v}), \tag{1.3.3}$$

where $\tilde{p}$ is any point in the fiber $\pi^{-1}(p)$ and $\tilde{u}$, $\tilde{v}$ are the unique vectors in $\mathcal{H}_{\tilde{p}}$ satisfying $d\pi_{\tilde{p}}(\tilde{u}) = u$ and $d\pi_{\tilde{p}}(\tilde{v}) = v$. The proof that $\tilde{g}$ is well defined is similar to the proof that the quotient metric is well defined in the case of a Riemannian covering, namely, choosing a different point $\tilde{p}' \in \pi^{-1}(p)$, one has unique vectors $\tilde{u}'$, $\tilde{v}' \in \mathcal{H}_{\tilde{p}'}$ that project to u, v, but $\tilde{g}_{\tilde{p}'}(\tilde{u}', \tilde{v}')$ gives the same result as above because $\tilde{p}' = \Phi(g)\tilde{p}$ for some $g \in G$, $d(\Phi(g))_{\tilde{p}} : \mathcal{H}_{\tilde{p}} \to \mathcal{H}_{\tilde{p}'}$ is an isometry and maps $\tilde{u}$, $\tilde{v}$ to $\tilde{u}'$, $\tilde{v}'$ respectively. The proof that $\tilde{g}$ is smooth is also similar, but needs an extra ingredient. Let $P_{\tilde{p}} : T_{\tilde{p}}\tilde{M} \to \mathcal{H}_{\tilde{p}}$ denote the orthogonal projection. It is known that $\pi : \tilde{M} \to M$ admits local sections, so let $s : U \to \tilde{M}$ be a local section defined on an open set U of M. Now we can rewrite (1.3.3) as

$$g_q(u, v) = \tilde{g}_{s(q)}(P_{s(q)} ds_q(u), P_{s(q)} ds_q(v)),$$

where $q \in U$. Since $\mathcal{V}$ as a distribution is locally defined by smooth vector fields, it is easy to check that P takes locally defined smooth vector fields on TM to locally defined smooth vector fields on TM. It follows that g is smooth. Finally, the requirement that π be a Riemannian submersion forces g to be given by formula (1.3.3), and this shows the uniqueness of g.

The complex projective space $\mathbf{C}P^n$

The definition of $\mathbf{C}P^n$ is similar to that of $\mathbf{R}P^n$ in that we replace real numbers by complex numbers. Namely, as a set, $\mathbf{C}P^n$ is the set of all complex lines through the origin in $\mathbf{C}^{n+1}$, so it can be viewed as the quotient of $\mathbf{C}^{n+1} \setminus \{0\}$ by the multiplicative group $\mathbf{C} \setminus \{0\}$ as well as the quotient of the unit sphere S^{2n+1} of $\mathbf{C}^{n+1}$ (via its canonical identification with $\mathbf{R}^{2n+2}$) by the multiplicative group of unit complex numbers S^1. Here the action of S^1 on S^{2n+1} is given by multiplication of the coordinates (since $\mathbf{C}$ is commutative, it is unimportant whether S^1 multiplies on the left or on the right). This action is clearly free and it is also proper since S^1 is compact. Further, the multiplication $L_z : S^{2n+1} \to S^{2n+1}$ by a unit complex number $z \in S^1$ is an isometry. In fact, S^{2n+1} has the induced metric from $\mathbf{R}^{2n+2}$, the Euclidean scalar product is the real part of the Hermitian inner product $(\cdot, \cdot)$ of $\mathbf{C}^{n+1}$ and $(L_z x, L_z y) = (zx, zy) = ||z||^2(x, y) = (x, y)$ for all x, $y \in \mathbf{C}^{n+1}$. It follows that $\mathbf{C}P^n = S^{2n+1}/S^1$ has the structure of a compact smooth manifold of dimension $2n$. Moreover there is a natural Riemannian metric which makes the projection $\pi : S^{2n+1} \to \mathbf{C}P^n$ into a Riemannian submersion. This quotient metric is classically called the *Fubini-Study metric* on $\mathbf{C}P^n$.

We want to explicitly construct the smooth structure on $\mathbf{C}P^n$ and prove that $\pi : S^{2n+1} \to \mathbf{C}P^n$ is a submersion in order to better familiarize ourselves with such an important example. For each $p \in \mathbf{C}P^n$, we construct a local chart around p. View p as a one-dimensional subspace of $\mathbf{C}^{n+1}$ and denote its Hermitian orthogonal complement by $p^\perp$. The subset of all lines which are not parallel to $p^\perp$ is an open subset of $\mathbf{C}P^n$, which we denote by $\mathbf{C}P^n \setminus p^\perp$. Fix a unit vector $\tilde{p}$ lying in the line p. The local chart is

$$\varphi^p : \mathbf{C}P^n \setminus p^\perp \to p^\perp, \qquad q \mapsto \frac{1}{(\tilde{q},\tilde{p})}\tilde{q} - \tilde{p},$$

where $\tilde{q}$ is any nonzero vector lying in q. In other words, q meets the affine hyperplane $\tilde{p} + p^\perp$ at a unique point $\frac{1}{(\tilde{q},\tilde{p})}\tilde{q}$ which we orthogonally project to $p^\perp$ to get $\varphi^p(q)$. (Note that $p^\perp$ can be identified with $\mathbf{R}^{2n}$ simply by choosing a basis.) The inverse of φ^p is the map that takes $v \in p^\perp$ to the line through $\tilde{p} + v$. Therefore, for $p' \in \mathbf{C}P^n$, we see that the transition map $\varphi^{p'} \circ (\varphi^p)^{-1} : \{ v \in p^\perp \mid v + \tilde{p} \notin p'^\perp \} \to \{ v' \in p'^\perp \mid v' + \tilde{p}' \notin p^\perp \}$ is given by

$$v \mapsto \frac{1}{(v+\tilde{p},\tilde{p}')}(v+\tilde{p}) - \tilde{p}', \tag{1.3.4}$$

and hence smooth.

Next we prove that the projection $\pi : S^{2n+1} \to \mathbf{C}P^n$ is a smooth submersion. Let $\tilde{p} \in S^{2n+1}$. Since the fibers of π are just the S^1-orbits, the vertical space $\mathcal{V}_{\tilde{p}} = \mathbf{R}(i\tilde{p})$. It follows that the horizontal space $\mathcal{H}_{\tilde{p}} \subset T_{\tilde{p}}S^{2n+1}$ is the Euclidean orthogonal complement of $\mathbf{R}\{\tilde{p}, i\tilde{p}\} = \mathbf{C}\tilde{p}$ in $\mathbf{C}^{2n+1}$, namely, $p^\perp$ where $p = \pi(\tilde{p})$. It suffices to check that $d\pi_{\tilde{p}}$ is an isomorphism from $\mathcal{H}_{\tilde{p}}$ onto $T_p\mathbf{C}P^n$, or, $d(\varphi^p \circ \pi)_{\tilde{p}}$ is an isomorphism from $p^\perp$ to itself. Let v be a unit vector in $p^\perp$. Then $t \mapsto \cos t\, \tilde{p} + \sin t\, v$ is a curve in S^{2n+1} with initial point $\tilde{p}$ and initial speed v, so using that $(\cos t\, \tilde{p} + \sin t\, v, \tilde{p}) = \cos t$ we have

$$\begin{aligned} d(\varphi^p \circ \pi)_{\tilde{p}}(v) &= \frac{d}{dt}\Big|_{t=0} (\varphi^p \circ \pi)(\cos t\, \tilde{p} + \sin t\, v) \\ &= \frac{d}{dt}\Big|_{t=0} \frac{1}{\cos t}(\cos t\, \tilde{p} + \sin t\, v) - \tilde{p} \\ &= v, \end{aligned}$$

completing the check.

One-dimensional Riemannian manifolds

Let (M,g) be a Riemannian manifold and let $\gamma : [a,b] \to M$ be a piecewise C^1 curve. Then the *length* of γ is defined to be

$$L(\gamma) = \int_a^b g_{\gamma(t)}(\gamma'(t),\gamma'(t))^{1/2}\, dt. \tag{1.3.5}$$

It is easily seen that the length of a curve does not change under re-parametrization. Moreover, every regular curve (i.e. satisfying $\gamma'(t) \neq 0$ for all t) admits a natural parametrization given by arc-length. Namely, let

$$s(t) = \int_a^t g_{\gamma(\tau)}(\gamma'(\tau),\gamma'(\tau))^{1/2}\, d\tau.$$

Then $\dfrac{ds}{dt} = g_{\gamma(t)}(\gamma'(t),\gamma'(t))^{1/2}(t) > 0$, so s can be taken as a new parameter, and then

$$L(\gamma|_{[0,s]}) = s$$

and

$$(\gamma^* g)_t = g_{\gamma(t)}(\gamma'(t),\gamma'(t))dt^2 = ds^2. \tag{1.3.6}$$

Suppose now that (M, g) is a one-dimensional Riemannian manifold. Then any connected component of M is diffeomorphic either to $\mathbf{R}$ or to S^1. In any case, a neighborhood of any point $p \in M$ can be viewed as a regular smooth curve in M and, in a parametrization by arc-length, the local expression of the metric g is the same, namely, given by (1.3.6). It follows that all the one-dimensional Riemannian manifolds are locally isometric among themselves.

Lie groups ★

The natural class of Riemannian metrics to be considered in Lie groups is the class of Riemannian metrics that possess some kind of invariance, be it left, right or both. Let G be a Lie group. A *left-invariant Riemannian metric* on G is a Riemannian metric with respect to which the left translations of G are isometries. Similarly, a *right-invariant Riemannian metric* is defined. A Riemannian metric on G that is both left- and right-invariant is called a *bi-invariant Riemannian metric.*

Left-invariant Riemannian metrics (henceforth, left-invariant metrics) are easy to construct on any given Lie group G. In fact, given any inner product $\langle, \rangle$ in its Lie algebra $\mathfrak{g}$, which we identify with the tangent space at the identity T_1G, one sets $g_1 = \langle, \rangle$ and uses the left translations to pull back g_1 to the other tangent spaces, namely one sets

$$g_x(u, v) = g_1\big(d(L_{x^{-1}})_x(u)\, ,\, d(L_{x^{-1}})_x(v) \big),$$

where $x \in G$ and u, $v \in T_xG$. This defines a smooth Riemannian metric, since $g(X, Y)$ is constant (and hence smooth) for any pair (X, Y) of left-invariant vector fields, and any smooth vector field on G is a linear combination of left-invariant vector fields with smooth functions as coefficients. By the very construction of g, the $d(L_x)_1$ for $x \in G$ are linear isometries, so the composition of linear isometries $d(L_x)_y = d(L_{xy})_1 \circ d(L_y)_1^{-1}$ is also a linear isometry for x, $y \in G$. This checks that all the left-translations are isometries and hence that g is left-invariant. (Equivalently, one can define g by choosing a global frame of left-invariant vector fields on G and declaring it to be orthonormal at every point of G.) It follows that the set of left-invariant metrics in G is in bijection with the set of inner products on $\mathfrak{g}$. Of course, similar remarks apply to right-invariant metrics.

Bi-invariant metrics are more difficult to come up with. Starting with a fixed left-invariant metric g on G, we want to find conditions for g to be also right-invariant. Reasoning similarly as in the previous paragraph, we see that it is necessary and sufficient that the $d(R_x)_1$ for $x \in G$ be linear isometries. Further, by differentiating the obvious identity $R_x = L_x \circ \mathrm{Inn}(x^{-1})$ at 1, we get that

$$d(R_x)_1 = d(L_x)_1 \circ \mathrm{Ad}(x^{-1})$$

for $x \in G$. From this identity, we get that g is right-invariant if and only if the $\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$ for $x \in G$ are linear isometries with respect to $\langle, \rangle = g_1$. In this case, $\langle, \rangle$ is called an Ad-*invariant inner product* on $\mathfrak{g}$.

In view of the previous discussion, applying the following proposition to the adjoint representation of a compact Lie group on its Lie algebra yields that *any compact Lie group admits a bi-invariant Riemannian metric.*

1.3.7 Proposition *Let $\rho : G \to \mathbf{GL}(V)$ be a representation of a Lie group on a real vector space V such that the closure $\rho(G)$ is relatively compact in $\mathbf{GL}(V)$. Then there exists an inner product $\langle, \rangle$ on V with respect to which the $\rho(x)$ for $x \in G$ are orthogonal transformations.*

Proof. Let $\tilde{G}$ denote the closure of $\rho(G)$ in $\mathbf{GL}(V)$. Then ρ factors through the inclusion $\tilde{\rho} : \tilde{G} \to \mathbf{GL}(V)$ and it suffices to prove the result for $\tilde{\rho}$ instead of ρ. By assumption, $\tilde{G}$ is compact, so without loss of generality we may assume in the following that G is compact.

Let $\langle , \rangle_0$ be any inner product on V and fix a right-invariant Haar measure dx on G. Set

$$\langle u, v\rangle = \int_G \langle \rho(x)u, \rho(x)v\rangle_0 \, dx,$$

where u, $v \in V$. It is easy to see that this defines a positive-definite bilinear symmetric form $\langle , \rangle$ on V. Moreover, if $y \in G$, then

$$\begin{aligned}
\langle \rho(y)u, \rho(y)v\rangle &= \int_G \langle \rho(x)\rho(y)u, \rho(x)\rho(y)v\rangle_0 \, dx \\
&= \int_G \langle \rho(xy)u, \rho(xy)v\rangle_0 \, dx \\
&= \langle u, v\rangle,
\end{aligned}$$

where in the last equality we have used that dx is right-invariant. Note that we have used the compactness of G only to guarantee that the above integrands have compact support. □

In later chapters, we will explain the special properties that bi-invariant metrics on Lie groups have.

Homogeneous spaces ★

It is apparent that for a generic Riemannian manifold (M, g), the isometry group $\mathrm{Isom}(M, g)$ is trivial. Indeed, Riemannian manifolds with large isometry groups have a good deal of symmetries. In particular, in the case in which $\mathrm{Isom}(M, g)$ is transitive on M, (M, g) is called a *Riemannian homogeneous space* or a *homogeneous Riemannian manifold*. Explicitly, this means that given any two points of M there exists an isometry of M that maps one point to the other. In this case, of course it may happen that a subgroup of $\mathrm{Isom}(M, g)$ is already transitive on M.

Let (M, g) be a homogeneous Riemannian manifold, and let G be a subgroup of $\mathrm{Isom}(M, g)$ acting transitively on M. Then the isotropy subgroup H at an arbitrary fixed point $p \in M$ is compact and M is diffeomorphic to the quotient space G/H. In this case, we also say that the Riemannian metric g on M is *G-invariant*.

Recall that if G is a Lie group and H is a closed subgroup of G, then there exists a unique structure of smooth manifold on the quotient G/H such that the projection $G \to G/H$ is a submersion and the action of G on G/H by left translations is smooth. (Theorem 0.4.18). A manifold of the form G/H is called a homogeneous space. In some cases, one can also start with a homogeneous space G/H and construct G-invariant metrics on G/H. For instance, if G is equipped with a left-invariant metric *that is also right-invariant with respect to H*, then it follows that the quotient G/H inherits a quotient Riemannian metric such that the projection $G \to G/H$ is a Riemannian submersion and the action of G on G/H by left translations is isometric. In this way, G/H becomes a Riemannian homogeneous space. A particular, important case of this construction is when the Riemannian metric on G that we start with is bi-invariant; in this case, G/H is called a *normal homogeneous space*. In general, a homogeneous space G/H for arbitrary G, H may admit several distinct G-invariant Riemannian metrics, or may admit no such metrics at all.

Let $M = G/H$ be a homogeneous space, where H is the isotropy subgroup at $p \in M$. Then the *isotropy representation* at p is the homomorphism

$$H \to O(T_pM), \qquad h \mapsto dh_p.$$

1.3.8 Lemma *The isotropy representation of G/H at p is equivalent to the adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$.*

1.3.9 Proposition *a. There exists a G-invariant Riemannian metric on G/H if and only if the image of the adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$ is relatively compact in $GL(\mathfrak{g}/\mathfrak{k})$.*

b. In case the condition in (a) is true, the G-invariant metrics on G/H are in bijective correspondence with the $\mathrm{Ad}_G(H)$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.

1.4 Exercises

1 Show that the Riemannian product of $(0, +\infty)$ and S^{n-1} is isometric to the cylinder

$$C = \{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 = 1 \quad \text{and} \quad x_0 > 0 \}.$$

2 The *catenoid* is the surface of revolution in $\mathbf{R}^3$ with the z-axis as axis of revolution and the catenary $x = \cosh z$ in the xz-plane as generating curve. The *helicoid* is the ruled surface in $\mathbf{R}^3$ consisting of all the lines parallel to the xy plane that intersect the z-axis and the helix $t \mapsto (\cos t, \sin t, t)$.

a. Write natural parametrizations for the catenoid and the helicoid.

b. Consider the catenoid and the helicoid with the metrics induced from $\mathbf{R}^3$, and find the local expressions of these metrics with respect to the parametrizations in item (a).

c. Show that the local expressions in item (b) coincide, possibly up to a change of coordinates, and deduce that the catenoid and the helicoid are locally isometric.

d. Show that the catenoid and the helicoid cannot be isometric because of their topology.

3 Consider the real hyperbolic space $(\mathbf{R}H^n, g)$ as defined in section 1.3. Let $\mathbf{D}^n$ be the open unit disk of $\mathbf{R}^n$ embedded in $\mathbf{R}^{n+1}$ as

$$\mathbf{D}^n = \{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} \mid x_0 = 0 \quad \text{and} \quad x_1^2 + \cdots + x_n^2 < 1 \}.$$

Define a map $f : \mathbf{R}H^n \to \mathbf{D}^n$ by setting $f(x)$ to be the unique point of $\mathbf{D}^n$ lying in the line joining $x \in \mathbf{R}H^n$ and the point $(-1, 0, \ldots, 0) \in \mathbf{R}^{n+1}$. Prove that f is a diffeomorphism and, setting $g_1 = (f^{-1})^* g$, we have that

$$g_1|_x = \frac{4}{(1 - \langle x, x \rangle)^2} \left(dx_1^2 + \cdots + dx_n^2 \right),$$

where $x = (0, x_1, \ldots, x_n) \in \mathbf{D}^n$. Deduce that $\mathbf{R}H^n$ is conformally flat.

$(\mathbf{D}^n, g_1)$ is called the *Poincaré disk model* of $\mathbf{R}H^n$.

4 Consider the open unit disk $\mathbf{D}^n = \{ (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1^2 + \cdots x_n^2 < 1 \}$ equipped with the metric g_1 as in Exercise 3. Prove that the inversion of $\mathbf{R}^n$ on the sphere of center $(-1, 0, \ldots, 0)$ and radius $\sqrt{2}$ defines a diffeomorphism f_1 from $\mathbf{D}^n$ onto the upper half-space

$$\mathbf{R}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 > 0 \},$$

and that the metric $g_2 = (f_1^{-1})^* g_1$ is given by

$$g_2|_x = \frac{1}{x_1^2} \left(dx_1^2 + \cdots + dx_n^2 \right),$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$.

$(\mathbf{R}^n_+, g_2)$ is called the *Poincaré upper half-space model* of $\mathbf{R}H^n$.

5 Consider the Poincaré upper half-plane model $\mathbf{R}^2_+ = \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$ with the metric $g_2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$ (case $n = 2$ in Exercise 4). Check that the following transformations of $\mathbf{R}^2_+$ into itself are isometries:

a. $\tau_a(x,y) = (x + a, y)$ for $a \in \mathbf{R}$;

b. $h_r(x,y) = (rx, ry)$ for $r > 0$;

c. $R(x,y) = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$.

Deduce from (a) and (b) that $\mathbf{R}_2^+$ is homogeneous.

6 Use stereographic projection to prove that S^n is conformally flat.

7 Consider the parametrized curve

$$\begin{cases} x &= t - \tanh t \\ y &= \frac{1}{\cosh t} \end{cases}$$

The surface of revolution in $\mathbf{R}^3$ constructed by revolving it around the x-axis is called the *pseudo-sphere*. Note that the pseudo-sphere is singular along the circle obtained by revolving the point $(0,1)$.

a. Prove that the pseudo-sphere with the singular circle taken away is locally isometric to the upper half plane model of $\mathbf{R}H^2$.

b. Show that the Gaussian curvature of the pseudo-sphere is -1.

8 Let Γ be the lattice in $\mathbf{R}^n$ defined by the basis $\{v_1, \dots, v_n\}$, and denote by g_Γ the Riemannian metric that it defines on T^n. Show that in some product chart of $T^n = S^1 \times \cdots \times S^1$ the local expression

$$g_\Gamma = \sum_{i,j} \langle v_i, v_j \rangle \, dx_i \otimes dx_j$$

holds, where $\langle , \rangle$ denotes the standard scalar product in $\mathbf{R}^n$.

9 Let Γ and Γ' be two lattices in $\mathbf{R}^n$, and denote by g_Γ, $g_{\Gamma'}$ the Riemannian metrics that they define on T^n, respectively.

a. Prove that (T^n, g_Γ) is isometric to $(T^n, g_{\Gamma'})$ if and only if there exists an isometry $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(\Gamma) = \Gamma'$. (Hint: You may use the result of exercise 2 of chapter 3.)

b. Use part (a) to see that (T^n, g_Γ) is isometric to the Riemannian product of n copies of S^1 if and only if Γ is the lattice associated to an orthonormal basis of $\mathbf{R}^n$.

10 Let Γ be the lattice of $\mathbf{R}^2$ spanned by an orthogonal basis $\{v_1, v_2\}$ and consider the associated rectangular flat torus T^2.

a. Prove that the map γ of $\mathbf{R}^2$ defined by $\gamma(x_1 v_1 + x_2 v_2) = (x_1 + \frac{1}{2})v_1 - x_2 v_2$ induces an isometry of T^2 of order two.

b. Prove that T^2 double covers a Klein bottle K^2.

11 Prove that $\mathbf{R}^n \setminus \{0\}$ is isometric to the warped product $((0, +\infty) \times S^{n-1}, dr^2 + r^2 g)$, where r denotes the coordinate on $(0, +\infty)$ and g denotes the standard Riemannian metric on S^{n-1}.

12 Let G be a Lie group equal to one of $\mathbf{O}(n)$, $\mathbf{U}(n)$ or $\mathbf{SU}(n)$, and denote its Lie algebra by $\mathfrak{g}$. Prove that for any $c > 0$

$$\langle X, Y \rangle = -c \, \mathrm{trace}\,(XY),$$

where X, $Y \in \mathfrak{g}$, defines an Ad-invariant inner product on $\mathfrak{g}$.

13 Consider the special unitary group $\mathbf{SU}(2)$ equipped with a bi-invariant metric induced from an Ad-invariant inner product on $\mathfrak{su}(2)$ as in the previous exercise with $c = \frac{1}{2}$. Show that the map

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α, $\beta \in \mathbf{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, defines an isometry from $\mathbf{SU}(2)$ onto S^3. Here $\mathbf{C}^2$ is identified with $\mathbf{R}^4$ and S^3 is viewed as the unit sphere in $\mathbf{R}^4$.

14 Show that $\mathbf{R}P^1$ equipped with the quotient metric from $S^1(1)$ is isometric to $S^1(\frac{1}{2})$. Show that $\mathbf{C}P^1$ equipped with the Fubini-Study metric is isometric to $S^2(\frac{1}{2})$.

15 (Sylvester's law of inertia) Let $B : V \times V \to \mathbf{R}$ be a symmetric bilinear form on a finite-dimensional real vector space V. For each basis $E = (e_1, \ldots, e_n)$ of V, we associate a symmetric matrix $B_E = (B(e_i, e_j))$.

a. Check that $B(u, v) = v_E^t B_E u_E$ for all u, $v \in V$, where u_E (resp. v_E) denotes the column vector representing the vector u (resp. v) in the basis E.

b. Suppose $F = (f_1, \ldots, f_n)$ is another basis of V such that

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

for a real matrix A of order n. Show that $B_E = A B_F A^t$.

c. Prove that there exists a basis E of V such that B_E has the form

$$\begin{pmatrix} I_{n-i-k} & 0 & 0 \\ 0 & -I_i & 0 \\ 0 & 0 & 0_k \end{pmatrix},$$

where I_m denotes an identity block of order m, and 0_m denotes a null block of order m.

d. Prove that there is a B-orthogonal decomposition

$$V = V_+ \oplus V_- \oplus V_0$$

where B is positive definite on V_+ and negative definite on V_-, V_0 is the kernel of B (the set of vectors B-orthogonal to V), $i = \dim V_-$ and $k = \dim V_0$. Prove also that i is the maximal dimension of a subspace of V on which B is negative definite. Deduce that i and k are invariants of B. They are respectively called the *index* and *nullity* of B. Of course, B is nondegenerate if and only if $k = 0$, and it is positive definite if and only if $k = i = 0$.

e. Check that the Lorentzian metric of $\mathbf{R}^{1,n}$ restricts to a positive definite symmetric bilinear form on the tangent spaces to the hyperboloid modeling $\mathbf{R}H^n$.

1.5 Additional notes

§1 Riemannian manifolds were defined as abstract smooth manifolds equipped with Riemannian metrics. One class of examples of Riemannian manifolds is of course furnished by the Riemannian submanifolds of Euclidean space. On the other hand, a very deep theorem of Nash [Nas56] states that every abstract Riemannian manifold admits an isometric embedding into Euclidean space, so that it can be viewed as an embedded Riemannian submanifold of Euclidean space. In view of this, one might be tempted to ask why bother to consider abstract Riemannian manifolds in the first place. The reason is that Nash's theorem is an existence result: for a given Riemannian manifold, it does not supply an explicit embedding of it into Euclidean space. Even if an isometric embedding is known, there may be more than one (up to congruence) or there may be no canonical one. Also, an explicit embedding may be too complicated to describe. Finally, a particular embedding is sometimes distracting because it highlights some specific features of the manifold at the expense of some other features, which may be undesirable.

§2 From the point of view of foundations of the theory of smooth manifolds, the following assertions are equivalent for a smooth manifold M whose underlying topological space is assumed to be Hausdorff but not necessarily second-countable:

a. The topology of M is paracompact.

b. M admits smooth partitions of unity.

c. M admits Riemannian metrics.

In fact, as is standard in the theory of smooth manifolds, second-countability of the topology of M (together with the Hausdorff property) implies its paracompactness and this is used to prove the existence of smooth partitions of unity [War83, chapter 1]. Next, Riemannian metrics are constructed on M by using partitions of unity as we did in Proposition 1.2.3. Finally, the underlying topology of a Riemannian manifold is metrizable according to Proposition 3.2.3, and every metric space is paracompact.

§3 The pseudo-sphere constructed in Exercise 7 was introduced by Beltrami [Bel68] in 1868 as a local model for the Lobachevskyan geometry. This means that the geodesic lines and their segments on the pseudo-sphere play the role of straight lines and their segments on the Lobachevsky plane. In 1900, Hilbert posed the question of whether there exists a surface in three-dimensional Euclidean space whose intrinsic geometry coincides completely with the geometry of the Lobachevsky plane. Using a simple reasoning, it follows that if such a surface does exist, it must have constant negative curvature and be complete (see chapter 3 for the notion of completeness).

As early as 1901, Hilbert solved this problem [Hil01] (see also [Hop89, chapter IX]), and in the negative sense, so that no complete surface of constant negative curvature exists in three-dimensional Euclidean space. This theorem has attracted the attention of geometers over a number of decades, and continues to do so today. The reason for this is that a number of interesting questions are related to it and to its proof. For instance, the occurrence of a singular circle on the pseudo-sphere is not coincidental, but is in line with Hilbert's theorem.