A short course on the differential geometry of curves and surfaces in Euclidean spaces

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Contents

Chapter 1

Curves

1.1 Regular curves

A *regular parameterized curve* in $\mathbb{R}^n$ is a continuously differentiable map $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, such that $\gamma'(t) \neq 0$ for $t \in I$. This condition implies that γ admits a tangent line at every point. A *regular curve* is an equivalence class of regular parameterized curves, where $\gamma \sim \eta$ if and only if $\eta = \gamma \circ \varphi$ for a continuously differentiable $\varphi : J \to I$, $\varphi' > 0$. We shall normally deal with curves satisfying some higher differentiability condition, like class $\mathcal{C}^k$ for $k \in \{1, 2, \dots, \infty\}$.

Examples 1.1 1. A *line* $\gamma(t) = p + tv = (x_0 + at, y_o + bt, z_0 + ct)$, where $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ is a point and $v = (a, b, c) \in \mathbb{R}^n$ is a vector.

2. The *circle* $\gamma(t) = (\cos t, \sin t)$ in the plane, or, more generally, $\gamma(t) = (x_0 + R\cos\omega t, y_0 + R\sin\omega t)$.

3. The *helix* $\gamma(t) = (a\cos t, a\sin t, bt)$, where a, $b \neq 0$.

4. The *semi-cubical parabola* $\gamma(t) = (t^2, t^3)$.

5. The *cathenary* $\gamma(t) = (t, \cosh(at))$, where $a > 0$.

6. The *tractrix* $\gamma(t) = (e^{-t}, \int_0^t \sqrt{1 - e^{-2\xi}}\, d\xi)$.

The *length* of a regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ is

$$L(\gamma) = \int_a^b ||\gamma'(t)||\, dt.$$

It is invariant under reparameterization.

Lemma 1.2 *Every regular curve $\gamma : [a, b] \to \mathbb{R}^n$ admits a reparameterization by arc length, that is, $\eta : [0, \ell] \to \mathbb{R}^n$, where $\ell = L(\gamma)$, such that $L(\eta|_{[0,t]}) = t$; equivalently, $||\eta'|| \equiv 1$, and we say that γ has unit speed.*

Proof. Define

$$\psi(t) = \int_a^t ||\gamma'(\xi)||\, d\xi.$$

Then $\psi : [a, b] \to [0, \ell]$, $\psi' > 0$ and we can take $\varphi = \psi^{-1}$, $\eta = \gamma \circ \varphi$. $\square$

Unless explicit mention to the contrary, we shall generally assume that our curves are parameterized by arc-length.

1.2 Plane curves

Let $\gamma : I \to \mathbb{R}^2$ be a curve parameterized by arc-length of class $\mathcal{C}^2$. Then $||\gamma'(s)|| = 1$ for all s. The curvature of γ is the rate of change of the direction of γ. Namely, let

$$\mathbf{t}(s) = \gamma'(s)$$

be the unit tangent vector at time s, and complete it to a positively oriented orthonormal base $\mathbf{t}(s)$, $\mathbf{n}(s)$ of $\mathbb{R}^2$. Then $\langle \mathbf{t}, \mathbf{t} \rangle = 1$ implies $\langle \mathbf{t}, \mathbf{t}' \rangle = 0$, so $\mathbf{t}' = \kappa \mathbf{n}$ for some continuous function $\kappa : I \to \mathbb{R}$. Similarly, $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ yields $\mathbf{n}' = -\kappa \mathbf{t}$. We can write

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^2$.

Proposition 1.3 *Suppose $\gamma : I \to \mathbb{R}^2$ is a curve parameterized by arc-length. Then κ is constant if and only γ is either part of a circle (if $\kappa \neq 0$) or part of a line (if $\kappa = 0$).*

Proof. If κ is identically zero, then the Frenet-Serret equations give $\mathbf{t}' = 0$ so $\gamma' = \mathbf{t}$ is a constant vector $\mathbf{t}_0$ in the plane and $\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} \gamma'(\xi)\, d\xi = \gamma(s_0) + (s - s_0)v_0$, for $s_0 \in I$.

If κ is a nonzero constant, by changing the orientaion we may assume that $\kappa > 0$. We first show that

$$c(s) := \gamma(s) + \frac{1}{\kappa}\mathbf{n}(s)$$

is a constant curve. In fact,

$$\begin{aligned} c' &= \gamma' + \frac{1}{\kappa}\mathbf{n}' \\ &= \mathbf{t} + \frac{1}{\kappa}(-\kappa \mathbf{t}) \\ &= 0. \end{aligned}$$

Set $c(s) = c_0$ for all $s \in I$. Since

$$||\gamma(s) - c_0|| = \frac{1}{\kappa}$$

for all $s \in I$, we deduce that γ is part of the circle of center c_0 and radius $1/\kappa$, as wished. $\square$

Theorem 1.4 (Fundamental theorem of plane curves) *The curvature is a complete invariant of plane curves, up to rigid motion. More precisely, given a continuous function $\alpha : [a, b] \to \mathbb{R}$ there is a unique curve in the plane defined on $[a, b]$, parametrized by arc-length, whose curvature at time $s \in [a, b]$ is $\alpha(s)$, up to a translation and rotation of the plane.*

Proof. For the existence, set $\gamma(s) = (x(s), y(s))$, where

$$x(s) = \int_a^s \cos\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta,\ y(s) = \int_a^s \sin\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta$$

for $s \in [a, b]$. Then γ has curvature function given by α.

Conversely, suppose $\gamma : [a, b] \to \mathbb{R}$, $\gamma(s) = (x(s), y(s))$ is parameterized by arc-length and has curvature α. The Frenet-Serret frame t, n along γ can be written

$$t(s) = (\cos\theta(s), \sin\theta(s)),\ n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now

$$\alpha(s) = \langle t'(s), n(s)\rangle = \theta'(s),$$

so

$$\theta(s) = \theta(a) + \int_a^s \alpha(\xi)\, d\xi.$$

Also, $t = (x', y')$ yields

$$x(s) = x(a) + \int_a^s \cos(\theta(\tau))\, d\tau,\ y(s) = y(a) + \int_a^s \sin(\theta(\tau))\, d\tau.$$

This determines completely γ up to the values of $x(a)$, $y(a)$, $\theta(a)$, that is, up to translation and rotation. □

Regular curves in $\mathbb{R}^2$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$ and the curvature κ of γ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\kappa}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

for $t \in I$. Denote the velocity of γ by $\nu(t) = ||\gamma'(t)||$ and recall that $\varphi' = \nu$ (Lemma 1.2). The function ν is the appropriate correction term when we want to write Frenet-Serret equations for a curve γ of arbitrary speed, as we show in the sequel.

Since $\tilde{\gamma}$ is of unit speed, we have the Frenet-Serret equations $\tilde{\mathbf{t}}' = \tilde{\kappa}\tilde{\mathbf{n}}$, $\tilde{\mathbf{n}}' = -\tilde{\kappa}\tilde{\mathbf{t}}$. Now

$$\begin{aligned} \mathbf{t}'(u) &= \tilde{\mathbf{t}}'(\varphi(u))\varphi'(u) \\ &= \tilde{\kappa}(\varphi(u))\nu(u)\tilde{\mathbf{n}}(\varphi(u)) \\ &= \kappa(u)\nu(u)\mathbf{n}(u), \end{aligned}$$

and similarly

$$\mathbf{n}'(u) = -\kappa(u)\nu(u)\mathbf{t}(u).$$

So we have the following Frenet-Serret equations for γ:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa\nu \\ -\kappa\nu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}.$$

In particular,

$$\kappa = \frac{1}{\nu}\langle \mathbf{t}', \mathbf{n}\rangle. \tag{1.5}$$

In practice, sometimes can be hard to find the explicit reparameterization by arc-length of a given regular curve, so equation (1.5) comes in handy to compute the curvature in such cases.

Example 1.6 We compute the curvature of the catenary $\gamma(u) = (u, \cosh u)$. We have $\gamma'(u) = (1, \sinh u)$ and $\nu(u) = (1 + \sinh^2 u)^{1/2} = \cosh u$, so γ has variable speed. We first seek to reparameterize γ by arc-length. We have $\varphi = \int \nu$ yields $\varphi(u) = \sinh u$ and $\varphi^{-1}(s) = \operatorname{arcsinh} s = \log(s + \sqrt{s^2+1})$, so

$$\tilde{\gamma}(s) = \gamma(\varphi^{-1}(s)) = (\log(s + \sqrt{s^2+1}), \sqrt{1+s^2})$$

is a reparameteriztion by arc-length. Pursuing this line of reasoning would require us to differentiate $\tilde{\gamma}$ twice (in the end we would still need to change back to the variable u), which is possible but not worth it. Instead, we start again and use (1.5). We have

$$\mathbf{t}(u) = \frac{1}{\nu(u)}\gamma'(u) = (\mathrm{sech} u, \tanh u),$$

so

$$\mathbf{n}(u) = (-\tanh u, \mathrm{sech} u),$$
$$\mathbf{t}'(u) = (-\tanh u \ \mathrm{sech}\, u, \mathrm{sech}^2\, u),$$

and

$$\kappa(u) = \frac{1}{\cosh u}(\tanh^2 u\, \mathrm{sech} u + \mathrm{sech}^3 u) = \mathrm{sech}^2 u.$$

Example 1.7 We can generalize Example 1.6. If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve of arbitrary speed and $\gamma(u) = (x(u), y(u))$, then $\nu = (x'^2 + y'^2)^{1/2}$, so

$$\mathbf{t} = \frac{1}{\nu}\gamma' = \frac{1}{(x'^2+y'^2)^{1/2}}(x', y'),$$

and

$$\mathbf{n} = \frac{1}{(x'^2 + y'^2)^{1/2}}(-y', x').$$

Now

$$\begin{aligned} \mathbf{t}' &= -\frac{x'x'' + y'y''}{(x'^2 + y'^2)^{3/2}}(x', y') + \frac{1}{(x'^2 + y'^2)^{1/2}}(x'' + y'') \\ &= \frac{1}{(x'^2 + y'^2)^{3/2}}(x''y'^2 - x'y'y'', x'^2y'' - x'x''y'). \end{aligned}$$

So

$$\begin{aligned} \langle \mathbf{t}', \mathbf{n} \rangle &= \frac{1}{(x'^2 + y'^2)^2}(-x''y'^3 + x'y'^2y'' + x'^3y'' - x'^2x''y') \\ &= \frac{1}{x'^2 + y'^2}(x'y'' - x''y'), \end{aligned}$$

and (1.5) yields

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

Remark 1.8 (i) If $\gamma_1 : I_1 \to \mathbb{R}^2$ and $\gamma_2 : I_2 \to \mathbb{R}^2$ are two unit speed reparameterizations preserving the orientation of a given regular parameterized curve γ then $\gamma_1(t) = \gamma_2(\varphi(t))$ for some $\varphi : I_1 \to I_2$ with $\varphi' > 0$. Then $\gamma_1' = \varphi'\gamma_2'$ with $||\gamma_1'|| = ||\gamma_2'|| = 1$, so $\varphi' \equiv 1$ implying that $\varphi(t) = t + t_0$ for some $t_0 \in \mathbb{R}$. We deduce that the definition of curvature of a regular parameterized curve γ of arbitrary speed does not depend on the reparameterization by arc-length that we choose.

(ii) The curvature of a regular parameterized curve in the plane thus defined is invariant under reparameterization preserving the orientation.

(iii) The curvature of a regular paparemeterized curve in the plane changes sign under a change of orientation.

1.3 Space curves

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of class $\mathcal{C}^3$ and assume that $\gamma'' \neq 0$ everywhere. Then we can associate an adapted trihedron to $\gamma(s)$ for each $s \in I$. We put:

$$\mathbf{t} = \gamma' \text{ (}\textit{tangent}\text{)}, \mathbf{n} = \frac{\gamma''}{||\gamma''||} \text{ (}\textit{normal}\text{)}, \mathbf{b} = \mathbf{t} \times \mathbf{n} \text{ (}\textit{binormal}\text{)}.$$

The *curvature* is $\kappa = ||\gamma''||$. It follows that $\mathbf{t}' = \kappa n$. Since $\mathbf{n}(s)$ is a unit vector for all s, $\mathbf{n}' \perp \mathbf{n}$ so

$$\begin{aligned} \mathbf{n}' &= \langle \mathbf{n}', \mathbf{t} \rangle \mathbf{t} + \langle \mathbf{n}', \mathbf{b} \rangle \mathbf{b} \\ &= -\langle \mathbf{n}, \mathbf{t}' \rangle \mathbf{t} + \langle \mathbf{n}', \mathbf{b} \rangle \mathbf{b}. \end{aligned}$$

We define the *torsion* $\tau = \langle \mathbf{n}', \mathbf{b} \rangle$. Now

$$\mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b}.$$

Finally,

$$\begin{aligned} \mathbf{b}' &= \mathbf{t}' \times \mathbf{n} + \mathbf{t} \times \mathbf{n}' \\ &= \kappa \mathbf{n} \times \mathbf{n} + \mathbf{t} \times (-\kappa \mathbf{t} + \tau \mathbf{b}) \\ &= -\tau \mathbf{n}. \end{aligned}$$

We summarize this discussion in matrix notation:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^3$.

Remark 1.9 A space curve with nonzero curvature is planar if and only if $\tau \equiv 0$.

Example 1.10 We compute the curvature and torsion of the helix

$$\gamma(s) = (a\cos(s/c), a\sin(s/c), b(s/c)), \ s \in \mathbb{R},$$

for $a > 0$, $b \in \mathbb{R}$ and $c \neq 0$. We have

$$\gamma'(s) = (-(a/c)\sin(s/c), (a/c)\cos(s/c), b/c),$$

so γ is parameterized by arc-length precisely when

$$a^2 + b^2 = c^2, \tag{1.11}$$

and then $t(s) = \gamma'(s)$. Further,

$$\gamma''(s) = (-(a/c^2)\cos(s/c), -(a/c^2)\sin(s/c), 0),$$

so

$$\mathbf{n}(s) = (-\cos(s/c), -\sin(s/c), 0)$$

and

$$\mathbf{b}(s) = ((b/c)\sin(s/c), -(b/c)\cos(s/c), a/c).$$

We compute

$$\mathbf{n}'(s) = ((1/c)\sin(s/c), -(1/c)\cos(s/c), 0)$$

and

$$\mathbf{b}'(s) = ((b/c^2)\cos(s/c), (b/c^2)\sin(s/c), 0).$$

It follows that

$$\kappa(s) = ||\gamma''(s)|| = a/c^2$$

and

$$\tau(s) = \langle n'(s), b(s) \rangle = b/c^2$$

are constant functions. Moreover $\kappa^2 + \tau^2 = 1/c^2$, so

$$a = \frac{\kappa}{\kappa^2 + \tau^2} \quad \text{and} \quad b = \frac{\tau}{\kappa^2 + \tau^2}. \tag{1.12}$$

Therefore, given κ, τ, we can solve equations (1.11), (1.12) for a, b, c and obtain a unique helix with curvature κ and torsion τ.

Theorem 1.13 (Fundamental theorem of space curves) *The curvature and torsion are complete invariants of space curves. More precisely, given continuous functions α, $\beta : [a, b] \to \mathbb{R}$ with $\alpha(s) > 0$ for all s, there exists a unique regular curve in $\mathbb{R}^3$ defined on $[a, b]$, parameterized by arc-length, of class C^3, whose curvature and torsion at time $s \in [a, b]$ are respectively given by $\alpha(s)$ and $\beta(s)$, up to a translation and rotation of $\mathbb{R}^3$.*

Proof. Consider

$$A = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}$$

as a matrix-valued function $[a, b] \to \mathbb{R}^{3\times 3}$. We consider the first order system of linear differential equations

$$F' = AF$$

for a matrix-valued $F : [a, b] \to \mathbb{R}^{3\times 3}$, given by the Frenet-Serret equations. Here the lines of F will yield the Frenet-Serret frame of our curve γ to be constructed, namely, $F(s) = (\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$. For a given initial condition $F(a) = (e_1, e_2, e_3)$, which is a positively oriented orthonormal basis of $\mathbb{R}^3$, the system has a unique solution $F(s)$ of class $\mathcal{C}^3$ defined for $s \in [a, b]$.

We claim that $F(s)$ is an ortogonal matrix of determinant 1 for all $s \in [a, b]$. The crucial fact involved here is that $A(s)$ is a skew-symmetric matrix. In fact, set $G = FF^t$. Then $G(a) = I$ and

$$\begin{aligned} G' &= (FF^t)' \\ &= F'F^t + F(F^t)' \\ &= F'F^t + F(F')^t \\ &= AFF^t + FF^tA^t \\ &= AG + GA^t. \end{aligned}$$

Since the constant function given by the identity matrix also satisfies the differential equation $G' = AG + GA^t$, due to the fact that $A(s) + A^t(s) = 0$ for all s, by the uniqueness theorem of solutions of first order ODE, $G(s) = I$ for all s. This proves that $F(s)$ is an orthogonal matrix and hence $\det F(s) = \pm 1$ for all

s. Since the determinant is a continuous function and $\det F(0) = 1$, we deduce that $\det F(s) = 1$ for all s.

Now $F(s) = (\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ is a trihedron for all s. For a given initial point $\gamma(a) = p \in \mathbb{R}^3$, the curve is completely determined by

$$\gamma(s) = p + \int_a^s \mathbf{t}(\xi)\, d\xi.$$

From the equation $F' = AF$ we see that (t, n, b) is the Frenet-Serret frame along γ and α, β are its curvature and torsion respectively. Note that the ambiguity in the construction of γ precisely amounts to the choices of point p and positive orthonormal basis (e_1, e_2, e_3), so any two choices differ by a translation and a rotation. □

Remark 1.14 (Local form) Let $\gamma : I \to \mathbb{R}^3$ be a regular curve of class $\mathcal{C}^3$ parameterized by arc-length and suppose that $\kappa > 0$ so that the Frenet-Serret frame is well-defined. We may assume that $0 \in I$, $\gamma(0) = 0$ and $(t(0), n(0), b(0))$ is the canonical basis of $\mathbb{R}^3$. Then the Taylor expansion of $\gamma(s) = (x(s), y(s), z(s))$ at $s = 0$ yields:

$$\begin{aligned}
x(s) &= s - \frac{\kappa(0)^2}{6} s^3 + R_x, \\
y(s) &= \frac{\kappa(0)}{2} s^2 + \frac{\kappa'(0)}{6} s^3 + R_y, \\
z(s) &= \frac{\kappa(0)\tau(0)}{6} s^3 + R_z,
\end{aligned}$$

where $\lim_{s\to 0} \frac{1}{s^3}(R_x, R_y, R_z) = 0$. Therefore the projections of γ in the (t, n)-plane (*osculating plane*), (n, b)-plane (*normal plane*, (t, b)-plane (*rectifying plane plane*) has the form of a parabola, semi-cubical parabola (if $\tau(0) \neq 0$), cubical parabola (if $\tau(0) \neq 0$), respectively, up to third order.

Regular curves in $\mathbb{R}^3$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the curvature κ and the torsion τ of γ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\mathbf{b}}$, $\tilde{\kappa}$, $\tilde{\tau}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\mathbf{b}(t) = \tilde{\mathbf{b}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

$$\tau(t) = \tilde{\tau}(\varphi(t))$$

for $t \in I$.

Set $\nu = ||\gamma'||$. As in the case of $\mathbb{R}^2$, we deduce the Frenet-Serret equations for γ:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

In the following, we find formuale for κ and τ in terms of the first three derivatives of γ. We have

$$\gamma' = \nu\mathbf{t},$$

$$\begin{aligned} \gamma'' &= \nu'\mathbf{t} + \nu\mathbf{t}' \\ &= \nu'\mathbf{t} + \kappa\nu^2\mathbf{n}, \end{aligned}$$

and

$$\begin{aligned} \gamma''' &= \nu''\mathbf{t} + \nu'(\kappa\mathbf{n}) + (\kappa'\nu^2 + 2\kappa\nu\nu')\mathbf{n} + \kappa\nu^2(-\kappa\nu\mathbf{t} + \tau\nu\mathbf{b}) \\ &= (\nu'' - \kappa^2\nu^2)\mathbf{t} + (\kappa'\nu^2 + 3\kappa\nu\nu')\mathbf{n} + \kappa\tau\nu^3\mathbf{b}. \end{aligned}$$

We deduce that $\gamma' \times \gamma'' = \kappa\nu^3\mathbf{b}$, so

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}. \tag{1.15}$$

Further $||\gamma' \times \gamma''|| = \kappa\nu^3$ and $(\gamma' \times \gamma'') \cdot \gamma''' = \kappa^2\tau\nu^6$, so

$$\tau = \frac{\gamma' \times \gamma'' \cdot \gamma'''}{||\gamma' \times \gamma''||^2}. \tag{1.16}$$

Example 1.17 It is very easy to compute curvature and torsion of the space curve $\gamma(t) = (t, t^2, t^3)$ using (1.15) and (1.16), as opposed to the moethod of finding a reparameteriztion by arc-length. Indeed

$$\gamma'(t) = (1, 2t, 3t^2),$$

$$\gamma''(t) = (0, 2, 6t)$$

and

$$\gamma'''(t) = (0, 0, 6).$$

Now

$$\gamma' \times \gamma'' = (6t^2, -6t, 2),$$

and

$$\kappa = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{(9t^4 + 4t^2 + 1)^3}},$$

$$\tau = \frac{3}{9t^4 + 9t^2 + 1}.$$

1.4 Global theory

Our discussion so far has been mostly local in nature, that is, we have studied properties of curves that depend on the behavior of the curve in a neighborhood of a point, like for instance the curvature and the torsion. The global differential geometry of curves studies curves as whole objects. For example, length is a global concept, and the property of being a closed curve is a global property.

In this section we survey on some of the most famous and interesting questions pertaining to the global differential geometry of curves. We neither intend to present an exhaustive treatment nor to delve into the details of proofs, but just sketch a few geometric ideas.

1.4.1 The rotation index

A regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ of class $\mathcal{C}^k$ is said to be *closed* if γ and its derivatives up to order k coincide at a and b:

$$\gamma(a) = \gamma(b), \dots, \gamma^{(k)}(a) = \gamma^{(k)}(b).$$

Equivalently, γ extends to a $\mathcal{C}^k$ $(b - a)$-periodic map $\mathbb{R} \to \mathbb{R}^n$.

A closed regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ is called *simple* if it has no self-intersections, that is, $\gamma|_{(a,b)}$ is injective as a map.

For simplicity, hereafter we consider only curves with class C^∞. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed curve parameterized by arc length in the plane. Let $\theta(s)$ be an angle determination of its tangent direction. On the one hand, we have seen that $\theta'(s) = \kappa(s)$, so

$$\int_a^b \kappa(s)\, ds = \theta(b) - \theta(a).$$

On the other hand, since γ is closed,

$$\theta(b) - \theta(a) = 2\pi k$$

for some $k \in \mathbb{Z}$. The integer k is called the *rotation index* of γ. It is clear that the rotation index of a closed curves changes sign if we change the orientation of the curve. For a closed regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^2$ of arbitrary speed, its rotation index is defined as the rotation index of a reparameterization by arc-length, so that it equals

$$\frac{1}{2\pi} \int_a^b \kappa(t) ||\gamma'(t)||\, dt.$$

Hence we can talk of the rotation index of a regular curve in $\mathbb{R}^2$ (without reference to parameterization).

Theorem 1.18 (Hopf's Umlaufsatz 1935) *The rotation index of a simple closed regular curve is* ± 1.

The proof is not very difficult, but we omit it.

Two regular parameterized curves γ_0, $\gamma_1 : [a,b] \to \mathbb{R}^n$ are called *regularly homotopic* if there exists a map $F : [a,b] \times [0,1] \to \mathbb{R}^n$ of class $\mathcal{C}^\infty$ such that:

- $F(s,0) = \gamma_0(s)$ and $F(s,1) = \gamma_1(s)$ for all $s \in [a,b]$;
- if we set $\gamma_t(s) = F(s,t)$, then $\gamma_t : [a,b] \to \mathbb{R}^n$ is a regular parameterized curve for all $t \in [0,1]$.

If, in addition, γ_0 and γ_1 are closed then γ_t is required to be closed for all t.

It is not difficult to see that if γ_0 and γ_1 are two closed regular curves which are regularly homotopic then they have the same rotation index. Conversely, we state:

Theorem 1.19 (Whitney-Graustein 1937) *Two closed regular curves in the plane are homotopic if and only if they have the same rotation index.*

1.4.2 Total absolute curvature

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a closed regular parameterized curve. Recall that

$$\int_a^b \kappa(t)||\gamma'(t)||\,dt = \int_a^b \kappa(s)\,ds$$

equals 2π times the rotation index of γ, where s is arc-length parameter. The *total absolute curvature* of γ is

$$\int_a^b |\kappa(t)|||\gamma'(t)||\,dt = \int_a^b |\kappa(s)|\,ds$$

(note that the absolute value in the integrand is important only in case $n = 2$).

A closed regular curve $\gamma : [a,b] \to \mathbb{R}^2$ is called *convex* if, for all $s \in [a,b]$, $\gamma([a,b])$ is contained in one half-plane determined by the tangent line at s.

Theorem 1.20 (Fenchel 1929) *The total absolute curvature of a regular curve in* $\mathbb{R}^3$ *is bounded below by* 2π*, and equality holds if and only if the curve is planar and convex.*

Proof. We work with a parameterization by arc-length. The total absolute curvature of a curve $\gamma : [a,b] \to \mathbb{R}^3$ parameterized by arc-length equals the length of its spherical image $\alpha : [a,b] \to S^2(1)$, where $\alpha(s) = t(s) = \gamma'(s)$.

If α is contained in a hemisphere then $\alpha(s) \cdot v \geq 0$ for all s and some unit vector v. But

$$0 = (\gamma(b) - \gamma(a)) \cdot v = \int_a^b \alpha(s) \cdot v\,ds \geq 0,$$

so γ must be planar.

If α is not contained in a hemisphere, let $s_0 \in [a, b]$ divide α into two curves of the same length, $\alpha_1 = \alpha|_{[a,s_0]}$ and $\alpha_2 = \alpha|_{[s_0,b]}$. Up to a rotation, we may assume $\alpha(0)$ and $\alpha(s_0)$ are symmetric with respect to the north pole. One of α_0, α_1 crosses the equator, say α_0 crosses the equator at p. Reflect t_0 on the plane through $\alpha(0)$, the north pole and $\alpha(s_0)$ to obtain a closed curve α_2 in $S^2(1)$ passing through p and $-p$.

Since α_2 is closed and passes through two antipodal points, clearly $L(\alpha_2) \geq 2\pi$, with equality holding only in case α is contained in the equator. On the other hand,

$$L(\alpha_2) = 2L(\alpha_0) = L(\alpha_0) + L(\alpha_1) = L(\alpha),$$

as desired.

To finish, note that a simple closed curve in $\mathbb{R}^2$ has curvature κ not changing sign if and only if it is convex. $\square$

Theorem 1.21 (Fary-Milnor) *The total absolute curvature of a non-trivially knotted regular curve in $\mathbb{R}^3$ is strictly bounded below by 4π.*

Chapter 2

Surfaces: basic definitions

A *regular parameterized surface* is a smooth mapping $\varphi : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$, of maximal rank. This is equivalent to saying that the rank of φ is 2 or that φ is an immersion. Such a φ is called a *parameterization*.

Let (u, v) be coordinates in $\mathbb{R}^2$, (x, y, z) be coordinates in $\mathbb{R}^3$. Then

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$x(u, v)$, $y(u, v)$, $z(u, v)$ admit partial derivatives of all orders and the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank two. This is equivalent to

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(y, z)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(z, x)}{\partial(u, v)} \neq 0,$$

or to the columns of the Jacobian matrix, denoted φ_u and φ_v, to be linearly independent.

A *surface* is a subset S of $\mathbb{R}^3$ satisfying:

(1) $S = \cup_{i \in I}$, where V_i is an open subset of S and $\varphi_i : U_i \subset \mathbb{R}^2 \to \varphi_i(U_i) = V_i$ is a parameterization for all $i \in I$. In other words, every point $p \in S$ lies in an open subset $W \subset \mathbb{R}^3$ such that $W \cap S$ is the image of a smooth immersion of an open subset of $\mathbb{R}^2$ into $\mathbb{R}^3$.

(2) Each $\varphi_i : U_i \to V_i$ is a homeomorphism. The continuity of φ_i^{-1} means that for given $i \in I$, $q \in V_i$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi_i^{-1}(\underbrace{B(q, \delta)}_{\text{ball in } \mathbf{R}^3} \cap V_i) \subset \underbrace{B(\varphi_i^{-1}(q), \epsilon)}_{\text{ball in } \mathbf{R}^2}.$$

2.1 Examples

1. The graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open, is a regular parameterized surface, where the parameterization is given by $\varphi(u, v) = (u, v, f(u, v))$. Note that

$$(d\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

has rank two.

2. If $S \subset \mathbb{R}^3$ is a subset such that any one of its points lies in a open subset of S which is a graph as in (1) (with respect to any one of the three coordinate planes), then S is a surface. It only remains to check that the parameterizations constructed in (1) are homeomorphisms. But this follows from the fact that $\varphi^{-1} = \pi|_{\varphi(U)}$ is continuous, where $\pi(x, y, z) = (x, y)$ is continuous.

3. The unit sphere is defined as

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Any point of S^2 lies in one of the following six open subsets, which are graphs, given by $z = \pm\sqrt{1 - x^2 - y^2}$, $y = \pm\sqrt{1 - x^2 - z^2}$, $x = \pm\sqrt{1 - y^2 - z^2}$.

2.2 Inverse images of regular values

Let $F : W \to \mathbb{R}$ be a smooth map, where $W \subset \mathbb{R}^3$ is open. A point $p \in W$ is called a *critical point* of F is $dF_p = 0$; otherwise, it is called a *regular point*. A point $q \in \mathbb{R}$ is called a *critical value* of F if there exists a critical point of F in $F^{-1}(q)$; otherwise, it is called a *regular value*. Note that a point $q \in \mathbb{R}$ lying outside the image of F is automatically a regular value of F.

Theorem 2.1 *If q is a regular value of F and $F^{-1}(q) \neq \varnothing$, then $S = F^{-1}(q)$ is a surface.*

Proof. It suffices to show that every point of S lies in an open subset of S which is a graph. Let $p = (x_0, y_0, z_0) \in S$. Then $dF_p = \begin{pmatrix} \frac{\partial F}{\partial x}(p) \\ \frac{\partial F}{\partial y}(p) \\ \frac{\partial F}{\partial z}(p) \end{pmatrix} \neq 0$ by the assumption. Without loss of generality, assume that $\frac{\partial F}{\partial z}(p) \neq 0$. By the implicit function theorem, there exist open neighborhoods $\tilde{V}$ of (x_0, y_0, z_0) in $\mathbb{R}^3$ and U of (x_0, y_0) in $\mathbb{R}^2$ and a smooth function $f : U \to \mathbb{R}$ such that $F(x, y, z) = q$, $(x, y, z) \in \tilde{V}$ if and only if $z = f(x, y)$, $(x, y) \in U$. Hence $V = \tilde{V} \cap S$ is the graph of f and an open neighborhood of p in S. $\square$

2.3 More examples

4. Spheres can also be seen as inverse images of regular values. Let $F(x,y,z) = x^2+y^2+z^2$. Then $(dF_{(x,y,z)})^t = (2x\ 2y\ 2z) = (0\ 0\ 0)$ if and only if $(x,y,z) = (0,0,0)$. Since $(0,0,0) \notin F^{-1}(r^2)$ for $r>0$, we have that the sphere $F^{-1}(r^2)$ of radius $r>0$ is a surface. Similarly, the ellipsoids $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (a, b, $c>0$) are surfaces.

5. The hyperboloids $x^2+y^2-z^2=r^2$ (one sheet) and $x^2+y^2-z^2=-r^2$ (two sheets) are surfaces, $r>0$. The cone $x^2+y^2-z^2=0$ is not a surface in a neighborhood of its vertex $(0,0,0)$.

6. The tori of revolution are surfaces given by the equation $z^2+(\sqrt{x^2+y^2}-a)^2=r^2$, where $a, r>0$.

7. More generally, one can consider surfaces of revolution. Let $\gamma(t) = (f(t),0,g(t))$ be a regular parameterized curve, $t\in(a,b)$. Define

$$\varphi(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u,v)\in(a,b)\times(v_0,v_0+2\pi)$. One can cover the surface by varying v_0 in $\mathbb{R}$. But there are conditions on γ for φ to be an immersion. One has

$$\frac{\partial(x,y)}{\partial(u,v)} = ff', \quad \frac{\partial(y,z)}{\partial(u,v)} = -fg'\cos v, \quad \frac{\partial(z,x)}{\partial(u,v)} = -fg'\sin v,$$

so

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 = f^2||\dot{\gamma}||^2,$$

and φ is an immersion if and only if $f>0$. Note also that φ is injective if and only if γ is injective. One also checks that φ^{-1} is continuous by writing its explicit expression.

8. Let $\xi:[0,2\pi]\to S^2$ be the smooth curve given in spherical coordinates as $\varphi=\theta/2$ (θ: longitude, φ; co-latitude) Then

$$\xi(t) = (\cos t\sin(t/2), \sin t\sin(t/s), \cos(t/2)),$$

and we can parameterize the Möbius band as

$$\varphi(u,v) = \alpha(u) + v\xi(u),$$

where $\alpha(t) = (\cos t, \sin t, 0)$ is the unit circle in $\mathbb{R}^2$. Since $\varphi_u\cdot\varphi_v = 0$, φ is an immersion.

2.3.1 Graphs

According to Example 2.1(2), every subset $S\subset\mathbb{R}^3$ which is locally a graph is a surface. Conversely:

Proposition 2.2 *If $S \subset \mathbb{R}^3$ is a surface then it is locally a graph, in the sense that, for every $p \in S$, there is an open subset W of $\mathbb{R}^3$ containing p such that $W \cap S$ is the graph of one of the forms $z = f(x, y)$, $y = f(x, z)$ or $x = f(y, z)$, where f is a mooth function on a open subset V of $\mathbb{R}^2$.*

Proof. Let $\varphi : U \to \mathbb{R}^3$ be a parameterization of S around $p = \varphi(u_0, v_0)$, where $(u_0, v_0) \in U$. Since φ has rank 2 at (u_0, v_0), there is a non-zero 2×2 minor of its Jacobian matrix at (u_0, v_0). Without loss of generality, let us say that

$$\frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) \neq 0.$$

Denote the orthogonal projection from $\mathbb{R}^3$ onto the plane $z = 0$ by $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Then

$$J(\pi \circ \varphi)_{(u_0,v_0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \circ (J\varphi)_{(u_o,v_0)} = \frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) \neq 0.$$

Therefore we can apply the Inverse Mapping Theorem to $\pi \circ \varphi : U \to \mathbb{R}2$ at (u_0, v_0). It says there exist an open neighborhood $\tilde{U}$ of (u_0, v_0) in $\mathbb{R}^2$ and an open neighborhoof V of $\pi \circ \varphi(u_0, v_0)$ in $\mathbb{R}^2$ such that $\pi \circ \varphi : \tilde{U} \to V$ is a bijection and its inverse is smooth.

We claim that $\varphi(\tilde{U})$ is the graph of the map

$$f = \pi' \circ \varphi \circ (\pi \circ \varphi|_{\tilde{U}})^{-1} : V \to \mathbb{R},$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}$ is the orthogonal projection onto the z-axis. Indeed: (x, y, z) lies in the graph of f if and only if $(x, y) \in V$ and $z = f(x, y)$. The latter is equivalent to

$$\begin{aligned} \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y) &= \left(\pi \circ \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y), \pi' \circ \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y)\right) \\ &= (x, y, z). \end{aligned}$$

But this says that (x, y, z) is the image of $(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y)$ (an arbitrary point of $\tilde{U}$) under φ. This proves the claim and finishes the proof. □

Example 2.3 The cone C given by $z = \sqrt{x^2 + y^2}$ is not a surface near $(0, 0, 0)$. In fact, if C were a surface near the origin, it would be locally graph there. It cannot be a graph of the form $x = f(y, z)$ or $y = f(x, z)$, since the projection of C to the planes $x = 0$ and $y = 0$ are not injective. If it were a graph of the form $z = f(x, y)$, then we would need to have $f(x, y) = \sqrt{x^2 + y^2}$. But this function f is not even differentiable at 0.

2.4 Change of parameters

Theorem 2.4 *Let $S \subset \mathbb{R}^3$ be a surface and let $\varphi : U \to \varphi(U)$, $\psi : V \to \psi(V)$ be two parameterizations of S, where $U, V \subset \mathbb{R}^2$ are open. Then the* change of parameters *$h = \varphi^{-1} \circ \psi : \psi^{-1}(\varphi(U)) \to \varphi^{-1}(\psi(V))$ is a diffeomorphism between open sets of $\mathbb{R}^2$.*

Proof. h is a homeomorphism because it is the composite map of two homeomorphisms. Note that a similar argument cannot be used to say that h is smooth, because it does not make sense (yet) to say that φ^{-1} is smooth.

Let $p = (u_0, v_0) \in U$, $q \in V$, $\varphi(p) = \psi(q)$. Since φ is an immersion, $d\varphi$ has rank two and we may assume WLOG that $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$. Write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in U$ and define

$$\Phi(u,v,w) = (x(u,v), y(u,v), z(u,v) + w),$$

where $(u,v,w) \in U \times \mathbb{R}$. Then $\Phi : U \times \mathbb{R} \to \mathbb{R}^3$ is smooth and

$$\det(d\Phi_{(u_0,v_0,0)}) = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{array} \right|_{(u_0,v_0,0)} = \frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0.$$

Since $d\Phi_{(u_0,v_0,0)}$ is non-singular, by the inverse function theorem, Φ^{-1} is defined and is smooth on some open neighborhood W of $\varphi(p)$ in $\mathbb{R}^3$. Since $\Phi|_{U\times\{0\}} = \varphi$, we have that $\Phi^{-1}_{\varphi(U)\cap W} = \varphi^{-1}|_{\varphi(U)\cap W}$. Since $W \cap \varphi(U)$ is open in S and ψ is a homeomorphism, $\psi^{-1}(W \cap \varphi(U)) \subset V$ is open. Now

$$h|_{\psi^{-1}(W)} = \varphi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))} = \Phi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))}$$

is smooth, because it is the composite map of smooth maps.

Similarly, one sees that h^{-1} is smooth by reversing the rôles of φ and ψ in the argument above. Hence h is a diffeomorphism. $\square$

Corollary 2.5 *Let $S \subset \mathbb{R}^3$ be a surface and suppose $f : W \to \mathbb{R}^3$ is a smooth map defined on the open subset $W \subset \mathbb{R}^m$ such that $f(W) \subset S$. Then $\varphi^{-1} \circ f : W \to \mathbb{R}^2$ is smooth for every parameterization $\varphi : U \to \varphi(U)$ of S.*

Proof. If Φ is as in the proof of the theorem, we have that $\varphi^{-1} \circ f = \Phi^{-1} \circ f$ is the composite of smooth maps between Euclidean spaces. $\square$

As an application of the smoothness of change of parameters, we can make the following definition. Let S be a surface. An application $f : S \to \mathbb{R}^n$ is smooth at a point $p \in S$ if $f \circ \varphi : U \to \mathbb{R}^n$ is smooth at $\varphi^{-1}(p) \in U$, for some parameterization $\varphi : U \to \varphi(U)$ of S with $p \in \varphi(U)$. Note that if $\psi : V \to \psi(V)$ is another parameterization of S with $p \in \psi(V)$, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is smooth at $\varphi^{-1}(p)$, because

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi^{-1} \circ \psi)$$

and the change of parameters $\varphi^{-1} \circ \psi$ is smooth.

Example 2.6 If S is a surface and $F : \mathbb{R}^3 \to \mathbb{R}$ is smooth, then the restriction $f = F|_S : S \to \mathbb{R}$ is smooth. In fact, $f \circ \varphi = F \circ \varphi$ is smooth for any parameterization φ of S. As special cases, we can take the *height function* relative to a, $F(x) = \langle x, a \rangle$, where $a \in \mathbb{R}^3$ is a fixed vector; or the *distance function* from q, $F(x) = ||x - q||^2$, where $q \in \mathbb{R}^3$ is a fixed point.

In particular, if $f : S \to \mathbb{R}^3$ is smooth at $p \in S$ and $\tilde{S} \subset \mathbb{R}^3$ is a surface such that $f(S) \subset \tilde{S}$, then we say that $f : S \to \tilde{S}$ is smooth at p. A bijective smooth map $f : S \to \bar{S}$ whose inverse is also smooth is called a *diffeomorphism*.

2.5 Tangent plane

Let $S \subset \mathbb{R}^3$ be a surface. Recall that a smooth curve $\gamma : I \subset \mathbb{R} \to S$ is simply a smooth curve $\gamma : I \to \mathbb{R}^3$ such that $\gamma(I) \subset S$. Fix a point $p \in S$. A *tangent vector* to S at p is the tangent vector $\dot{\gamma}(0)$ to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. The *tangent plane* to S at p is the collection of all tangent vectors to S at p.

Proposition 2.7 *The tangent space T_pS is the image of the differential*

$$d\varphi_a : \mathbb{R}^2 \to \mathbb{R}^3, \tag{2.8}$$

where $\varphi : U \to \varphi(U)$ is any parameterization of S with $p = \varphi(a)$ and $a \in U$.

Proof. Any vector in the image of (2.8) is of the form $d\varphi_a(w_0)$ for some $w_0 \in \mathbb{R}^2$ and therefore is the tangent vector at 0 of the smooth curve $t \mapsto \varphi(a + tw_0)$.

Conversely, suppose $w = \dot{\gamma}(0)$ is tangent to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. By Corollary 2.5, $\eta := \varphi^{-1} \circ \gamma : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$ is a smooth curve in $\mathbb{R}^2$ with $\eta(0) = a$. Note that $\gamma = \varphi \circ \eta$. By the chain rule

$$v = d\varphi_a(\dot{\eta}(0)) \tag{2.9}$$

lies in the image of (2.8). □

Corollary 2.10 *The tangent plane T_pS is a 2-dimensional vector subspace of $\mathbb{R}^3$. For any parameterization $\varphi : U \to \varphi(U)$ of S with $p = \varphi(a)$, $a \in U$,*

$$\{\tfrac{\partial \varphi}{\partial u}(a), \tfrac{\partial \varphi}{\partial v}(a)\} \tag{2.11}$$

is a basis of T_pS.

It is also convenient to write $\varphi_u := \frac{\partial \varphi}{\partial u}$ and $\varphi_v := \frac{\partial \varphi}{\partial v}$.

Consider a tangent vector $w \in T_pS$, say $w = \dot{\gamma}(0)$ where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, as in the proof of Proposition 2.7. Then $\eta = \varphi^{-1} \circ \gamma$ is a smooth curve in $\mathbb{R}^2$ which we may write as $\eta(t) = (u(t), v(t))$. Since $\dot{\eta}(0) = (u'(0), v'(0))$, eqn. (2.9) yields that

$$w = u'(0)\varphi_u(a) + v'(0)\varphi_v(a),$$

namely, $u'(0), v'(0)$ are the coordinates of w in the basis (2.11). This remark also shows that another smooth curve $\bar{\gamma} : (-\epsilon, \epsilon) \to S$ represents the same w if and only if $\bar{\eta}(t) = \varphi^{-1} \circ \bar{\gamma}(t) = (\bar{u}(t), \bar{v}(t))$ satisfies $(\bar{u}'(0), \bar{v}'(0)) = (u'(0), v'(0))$.

With the same notation as above, suppose now that $f : S \to \tilde{S}$ is a smooth map at $p \in S$. Note that $f \circ \gamma$ is a smooth curve in $\tilde{S}$. The *differential* of f at p is the map

$$df_p : T_pS \to T_{f(p)}\tilde{S}$$

that maps $w = \dot{\gamma}(0) \in T_pS$ to the tangent vector $\dot{\tilde{\gamma}}(0)$, where $\tilde{\gamma} = f \circ \gamma$. We check that $df_p(w)$ does not depend on the choice of curve γ. Let $\varphi : U \to \varphi(U) = V$, $\tilde{\varphi} : \tilde{U} \to \tilde{\varphi}(\tilde{U}) = \tilde{V}$ be parameterizations of S, $\tilde{S}$, resp., with $p = \varphi(a)$, $a \in U$, $f(p) = \tilde{\varphi}(\tilde{a})$, $\tilde{a} \in \tilde{U}$, and such that $f(V) \subset \tilde{V}$. Consider the local representation of f,

$$g = \tilde{\varphi}^{-1} \circ f \circ \varphi : U \to \tilde{U},$$

and write

$$g(u, v) = (g_1(u, v), g_2(u, v))$$

for $(u, v) \in U \subset \mathbb{R}^2$. Then

$$\tilde{\gamma}(t) = \tilde{\varphi}\left(g_1(u(t), v(t)), g_2(u(t), v(t)) \right),$$

so

$$\dot{\tilde{\gamma}}(0) = \left(\frac{\partial g_1}{\partial u}\, u'(0) + \frac{\partial g_1}{\partial v}\, v'(0)\right) \tilde{\varphi}_{\tilde{u}} + \left(\frac{\partial g_2}{\partial u}\, u'(0) + \frac{\partial g_2}{\partial v}\, v'(0)\right) \tilde{\varphi}_{\tilde{v}}.$$

This relation shows that $\dot{\tilde{\gamma}}(0)$ depends only on $u'(0)$, $v'(0)$ and hence has the same value for any smooth curve representing w. This relation can also be rewritten as

$$df_p(w) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix},$$

which shows that df_p is a linear map whose matrix with respect to the bases $\{\varphi_u, \varphi_v\}$, $\{\tilde{\varphi}_{\tilde{u}}, \varphi_{\tilde{v}}\}$ is the 2 by 2 matrix above.

Example 2.12 If S is a surface given as the inverse image under $F : \mathbb{R}^3 \to \mathbb{R}$ of a regular value, then $T_pS = \ker(dF_p)$ for every $p \in S$. In fact, if $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, then $F(\gamma(t))$ is constant for $t \in (-\epsilon, \epsilon)$. By the chain rule, $dF_p(\dot{\gamma}(0)) = 0$. This proves the inclusion $T_pS \subset \ker(dF_p)$ and hence the equality by dimensional reasons.

Example 2.13 Let S be a surface and consider the height function $h : S \to \mathbb{R}$ for a fixed unit vector $\xi \in S^2$, given by $h(x) = x \cdot \xi$ for $x \in S$. We compute the differential $dh_p : T_pS \to \mathbb{R}$ at $p \in S$. Given $w \in T_pS$, there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = w$. Now

$$dh_p(w) = \frac{d}{dt}\Big|_{t=0} h(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \gamma(t) \cdot \xi = \dot{\gamma}(0) \cdot \xi = w \cdot \xi.$$

In particular, $p \in S$ is a critical point of h if and only if ξ is normal to S at p.

Chapter 3

Surfaces: local theory

3.1 The first fundamental form

Let $S \subset \mathbb{R}^3$ be a surface. The *first fundamental form* of S is just the restriction I of the dot product of $\mathbb{R}^3$ to the tangent spaces of S:

$$I_p = \langle \cdot, \cdot \rangle|_{T_pS \times T_pS}.$$

If $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) = V \subset S$ is a parametrization, we already know that $\{\varphi_u(u,v), \varphi_v(u,v)\}$ is a basis of $T_{\varphi(u,v)}S$. Then any tangent vector to S at a point in V can be written $w = a\varphi_u + b\varphi_v$ and thus

$$I(w,w) = a^2 \underbrace{I(\varphi_u, \varphi_u)}_{E(u,v)} + 2ab \underbrace{I(\varphi_u, \varphi_v)}_{F(u,v)} + b^2 \underbrace{I(\varphi_v, \varphi_v)}_{G(u,v)}.$$

E, F, G are smooth functions on U, the so called coefficients of the first fundamental form. If $\{du, dv\}$ denotes the dual basis of $\{\varphi_u, \varphi_v\}$, then $a = du(w)$, $b = dv(w)$ and we can write

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

This is the local expression of I with respect to φ. The matrix associated to this bilinear form is

$$(I) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Examples 3.1 1. A plane can be parametrized by $\varphi(u,v) = p + uw_1 + vw_2$, where p is a point in $\mathbb{R}^3$, $\{w_1, w_2\}$ is an orthonormal set of vectors in $\mathbb{R}^3$, and $(u,v) \in \mathbb{R}^2$. We have $\varphi_u = w_1$, $\varphi_v = w_2$, so $E = G = 1$, $F = 0$ and $I = du^2 + dv^2$.

2. The (right circular) cilinder can be parametrized by $\varphi(u,v) = (\cos u, \sin u, v)$, where $u_0 < u < u_0 + 2\pi$ and $v \in \mathbb{R}$. In this case, $\varphi_u = (-\sin u, \cos u, 0)$, $\varphi_v = (0,0,1)$, so $E = G = 1$, $F = 0$ and $I = du^2 + dv^2$.

3. The helicoid is the union of horizontal lines joining the z-axis to the points of an helix, namely, it is the image of the parametrization $\varphi(u,v) = (v\cos u, v\sin u, au)$ ($a>0$). We see that $I = (a^2+v^2)du^2 + dv^2$.

4. Spherical coordinates $\varphi(u,v) = (\cos v \sin u, \sin v \sin u, \cos u)$, where $0 < u < \pi/2$, $v_0 < v < v_0 + 2\pi$, yield that $I = du^2 + \sin^2 u dv^2$ on the sphere.

Remark 3.2 Two regular parameterized surfaces with the same fundamental forms (like the cilinder minus a vertical line a strip of width 2π of the plane in Example 3.1(2)) are called *isometric*.

Using the first fundamental form, we can define:
Length of a smooth curve $\gamma : (a,b) \to S$ by

$$L(\gamma) = \int_a^b I(\gamma'(t), \gamma'(t))^{1/2}\, dt.$$

Angle between two vectors w_1, $w_2 \in T_pS$ by

$$\cos\angle(w_1,w_2) = \frac{I_p(w_1,w_2)}{I_p(w_1,w_1)^{1/2} I_p(w_2,w_2)^{1/2}}.$$

Surface integral of a compactly supported continuous function $f : S \to \mathbb{R}$. If the support D of f is contained in the image of a parametrization $\varphi : U \to S$, then

$$\int\int_D f\, dS = \int\int_{\varphi^{-1}(D)} f(\varphi(u,v))\, ||\varphi_u \times \varphi_v||\, dudv,$$

where the left hand side is a double integral. Taking another parametrization $\bar\varphi : \bar U \to S$, the change of parameters $(u,v) = (\varphi^{-1}\circ\bar\varphi)(\bar u, \bar v)$ is smooth with Jacobian determinant $\frac{\partial(u,v)}{\partial(\bar u,\bar v)}$. Note that

$$||\bar\varphi_{\bar u} \times \bar\varphi_{\bar v}|| = \left|\frac{\partial(u,v)}{\partial(\bar u,\bar v)}\right| \, ||\varphi_u \times \varphi_v||.$$

The formula of change of variables in the double integral yields that

$$\begin{aligned}
&\int\int_{\bar\varphi^{-1}(D)} f(\bar\varphi(\bar u,\bar v))\, ||\bar\varphi_{\bar u} \times \bar\varphi_{\bar v}||\, d\bar u d\bar v \\
&= \int\int_{\bar\varphi^{-1}(D)} f(\bar\varphi(\bar u,\bar v))\, ||\varphi_u \times \varphi_v|| \left|\frac{\partial(u,v)}{\partial(\bar u,\bar v)}\right| d\bar u d\bar v \\
&= \int\int_{\varphi^{-1}(D)} f(\varphi(u,v))\, ||\varphi_u \times \varphi_v||\, dudv,
\end{aligned}$$

so the definition is independent of the choice of parametrization. In general, onde needs to cover the support of f by finitely many parametrizations and define the surface integral as a sum of double integrals. The proof of independence of the choices involved is more complicated in this case, and we will not go into details. The relation to the first fundamental form is that

$$||\varphi_u \times \varphi_v||^2 = ||\varphi_u||^2 ||\varphi_v||^2 - \langle \varphi_u, \varphi_v\rangle^2,$$

so

$$||\varphi_u \times \varphi_v|| = \sqrt{EG - F^2}.$$

Area of a compact domain $D \subset S$ (say, with piecewise smooth boundary) is

$$\text{area}(D) = \int\int_D 1\, dS.$$

In particular, if D is contained in the image of φ,

$$\text{area}(D) = \int\int_{\varphi^{-1}(D)} \sqrt{EG - F^2}\, dudv.$$

3.2 The Gauss map and the second fundamental form

Let $S \subset \mathbb{R}^3$ be a surface. For each $p \in S$, we want to assign a unit vector $\nu(p)$ which is normal to T_pS; note that there are exactly two possible choices. If it is possible to make such an assigment continuously along the whole of S, we say that S is *orientable*. The resulting map

$$\nu : S \to S^2$$

into the unit sphere is called the *Gauss map*. We will always assume that the Gauss map is continuous.

Examples 3.3 1. If $\varphi : U \to S$ is a parametrization, then we can take

$$\nu = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}.$$

This construction shows that every surface is locally orientable. It also shows that the Gauss map is smooth.

2. If S is given as the inverse image under F of regular value, then we can take

$$\nu = \frac{\nabla F}{||\nabla F||}.$$

Next, we note that the differential

$$d\nu_p : T_pS \to T_{\nu(p)}S^2 = T_pS$$

is an operator on T_pS, since T_qS^2 is always normal to q, for $q \in S^2$. The operator

$$A_p := -d\nu_p : T_pS \to T_pS$$

is called the *Weingarten operator*.

Proposition 3.4 *The Weingarten operator is symmetric:*

$$\langle A_p(w_1), w_2\rangle = \langle w_1, A_p(w_2)\rangle,$$

where w_1, $w_2 \in T_pS$.

Proof. By linearity, it suffices to check the relation for a basis of T_pS. Let $\varphi : U \to S$ be a parametrization; then $\{\varphi_u, \varphi_v\}$ is a tangent frame. Set $N = \nu\circ\varphi$. Then

$$d\nu(\varphi_u) = d\nu\left(\frac{\partial\varphi}{\partial u}\right) = \frac{\partial}{\partial u}(\nu\circ\varphi) = \frac{\partial N}{\partial u}.$$

Similarly $d\nu(\varphi_v) = \frac{\partial N}{\partial v}$. Since $\langle\frac{\partial\varphi}{\partial v}, N\rangle = 0$ on U, differentiating with respect to u,

$$\langle\frac{\partial^2\varphi}{\partial u\partial v}, N\rangle + \langle\frac{\partial\varphi}{\partial v}, \frac{\partial N}{\partial u}\rangle = 0,$$

and similarly

$$\langle\frac{\partial^2\varphi}{\partial v\partial v}, N\rangle + \langle\frac{\partial\varphi}{\partial u}, \frac{\partial N}{\partial v}\rangle = 0,$$

Taking the difference of these equations,

$$\langle\frac{\partial\varphi}{\partial v}, \frac{\partial N}{\partial u}\rangle - \langle\frac{\partial\varphi}{\partial u}, \frac{\partial N}{\partial v}\rangle = 0,$$

which says that

$$\langle\varphi_v, d\nu(\varphi_v)\rangle - \langle\varphi_u, d\nu(\varphi_v)\rangle = 0,$$

as we wished. □

The associated symmetric bilinear form

$$II_p(w_1, w_2) = \langle A_p(w_1), w_2\rangle$$

is called the *second fundamental form*.

Proposition 3.5 *Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve parametrized by arc-length, $\gamma(0) = p$, $\gamma'(0) = w \in T_pS$. Then*

$$\langle\gamma''(0), \nu(p)\rangle = II_p(w, w).$$

Proof. Start with the equation $\langle\gamma'(s), \nu(\gamma(s))\rangle = 0$ and differentiate it at $s = 0$ to obtain

$$\langle\gamma''(0), \nu(p)\rangle + \langle w, \frac{d}{ds}|_{s=0}\nu(\gamma(s))\rangle = 0.$$

Then $\langle\gamma''(0), \nu(p)\rangle = -\langle w, d\nu_p(w)\rangle = II_p(w, w)$, as desired. □

There is a geometric interpretation of last proposition (Meusnier). Given a unit vector $w \in T_pS$, the affine plane through p spanned by w and $\nu(p)$ meets S transversally along a curve which is called the *normal section of* S along w. If γ

is a parametrization by arc-length of this normal section as in the proposition, then the curvature of γ at $p = \gamma(0)$ is

$$\kappa_w = \langle \gamma''(0), \nu(p) \rangle = II_p(w, w),$$

where we view γ as a plane curve and we view its supporting plane as oriented by $\{w, \nu(p)\}$.

Since the Weingarten operator A is symmetric, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pS such that

$$A_p(e_1) = \kappa_1 e_1, \qquad A_p(e_2) = \kappa_2 e_2.$$

The eigenvalues κ_1, κ_2 are called *principal curvatures* at p, and the eigenvectors e_1, e_2 are called *principal directions* at p. Of course, $II(e_1, e_1) = \kappa_1$, $II(e_2, e_2) = \kappa_2$, $II(e_1, e_2) = 0$. It follows that for a unit vector $w = \cos\theta e_1 + \sin\theta e_2 \in T_pS$ we have *Euler's formula*:

$$\kappa_w = I_p(w, w) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Since this a convex linear combination ($\sin^2\theta + \cos^2\theta = 1$), Euler's formula also shows that κ_1, κ_2 are the extrema of the curvatures of the normal sections through p.

In general, a smooth curve γ in S is called a *line of curvature* if $\gamma'(t)$ is a principal direction at $\gamma(t)$ for all t. Note that if $\kappa_1(p) \neq \kappa_2(p)$, then the principal directions at p are uniquely defined, but not otherwise. If $\kappa_1(p) = \kappa_2(p)$, we say that p is an *umbilic point* of S.

Proposition 3.6 *If all the points of a connected surface S are umbilic, then S is contained in a plane or a sphere.*

Proof. We first prove the result in case S is the image V of a parametrization $\varphi : U \to V$. Set $N = \nu \circ \varphi$. By assumption, $d\nu = \lambda \cdot I$, where $\lambda : V \to \mathbb{R}$ is a smooth function. It follows that

$$\begin{aligned} N_u = d\nu(\varphi_u) &= (\lambda \circ \varphi)\varphi_u, \\ N_v = d\nu(\varphi_v) &= (\lambda \circ \varphi)\varphi_v. \end{aligned}$$

We differentiate the first (resp. second) of these equations with respect to v (resp. u) to get

$$\begin{aligned} N_{uv} &= (\lambda \circ \varphi)_v \varphi_u + (\lambda \circ \varphi)\varphi_{uv}, \\ N_{vu} &= (\lambda \circ \varphi)_u \varphi_v + (\lambda \circ \varphi)\varphi_{vu}. \end{aligned}$$

Taking the difference,

$$0 = (\lambda \circ \varphi)_v \varphi_u - (\lambda \circ \varphi)_u \varphi_v.$$

Since $\{\varphi_u, \varphi_v\}$ is linearly independent, the partial derivatives of $\lambda \circ \varphi$ on U are zero. Since U is connected, $\lambda \circ \varphi$ is constant.

Next, we consider two cases. If $\lambda \circ \varphi = 0$, then $N_u = N_v = 0$ so N is constant. This implies $\frac{\partial}{\partial u}\langle \varphi, N\rangle = \frac{\partial}{\partial v}\langle \varphi, N\rangle = 0$ and hence V is contained in an affine plane parallel to $\langle N\rangle^{\perp}$. On the other hand, if $\lambda \circ \varphi = R \neq 0$, then

$$N_u = R\varphi_u, \; N_v = R\varphi_v,$$

so $\varphi - \frac{1}{R} N$ is a constant $q \in \mathbb{R}^3$ and hence V is contained in the sphere of center q and radius $1/|R|$.

In the case of arbitrary S, fix a point $p \in S$ and a parametrized connected open neighborhood V_0 of p. By the previous case, V_0 is contained in a plane or a sphere. Given $x \in S$, by connectedness of S, there exists a continuous curve $\gamma : [0,1] \to S$ joining p to x (S is locally arcwise connected, so it is arcwise connected). For any $t \in [0,1]$, there exists a parametrized neighborhood of $\gamma(t)$ which is contained in a plane or a sphere. By compactness of $\gamma([0,1])$, it is possible to cover it by connected open sets $V_0, V_1, \ldots, V_n$ such that each V_i is contained in a plane or a sphere and $V_i \cap V_{i+1} \neq \varnothing$ (check this!); the latter condition implies that V_{i+1} is contained in the same plane or sphere that contains V_i. The result follows. □

3.3 Curvature of surfaces

Let $S \subset \mathbb{R}^3$ and consider its Weingarten operator $A = -d\nu_p : T_pS \to T_pS$. Recall that A_{ν_p} is symmetric and its eigenvalues $\kappa_1(p)$, $\kappa_2(p)$ are the principal curvatures of S at p. We define:

$$\begin{aligned} \textit{Gaussian curvature} : K(p) &= \det(A_p) = \kappa_1(p) \cdot \kappa_2(p), \\ \textit{Mean curvature} : H(p) &= \frac{1}{2}\mathrm{trace}\,(A_p) = \frac{1}{2}\left(\kappa_1(p) + \kappa_2(p)\right). \end{aligned}$$

Note that κ_1, $\kappa_2 = H \pm \sqrt{H^2 - K}$; we will soon see that H and K are smooth functions on S, and so it follows from this equation that κ_1, κ_2 are continuous functions on S which are smooth away from umbilic points (points characterized by $H^2 = K$).

If we change ν to $-\nu$, then H is changed to $-H$ but K is unchanged. The next example analyses the meaning of the sign of K.

Examples 3.7 1. Let us compute the Gaussian curvature of the graph S of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open. In general, for a parametrization φ and $N = \nu \circ \varphi$,

$$II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u\rangle = -\langle N_u, \varphi_u\rangle = \langle N, \varphi_{uu}\rangle.$$

Similarly,

$$II(\varphi_u, \varphi_v) = -\langle N_u, \varphi_v\rangle = \langle N, \varphi_{uv}\rangle$$

and

$$II(\varphi_v, \varphi_v) = -\langle N_v, \varphi_v \rangle = \langle N, \varphi_{vv} \rangle.$$

In our case,

$$\begin{aligned}
\varphi_u &= (1, 0, f_u), \\
\varphi_v &= (0, 1, f_v), \\
\varphi_{uu} &= (0, 0, f_{uu}), \\
\varphi_{uv} &= (0, 0, f_{uv}), \\
\varphi_{vv} &= (0, 0, f_{vv}), \\
N &= \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
(II) &= \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \\
&= \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \left(\mathrm{Hess}(f) \right).
\end{aligned}$$

We specialize to the case $p = (0, 0, 0) = f(0, 0)$ and T_pS is the xy-plane. Then $f_u(0, 0) = f_v(0, 0) = 0$ and $II_p = \mathrm{Hess}_{(0,0)}(f)$. In particular, if $f(u, v) = au^2 + bv^2$, then

$$(II) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

It follows that $K(p) = 4ab$ is positive (resp. negative) if a and b have the same sign (resp. opposite signs), and it is zero if one of a, b is zero.

Another interesting case is $f(u, v) = u^4 + v^4$. We get $II = 0$.

2. Consider the sphere $S^2(R)$ of radius $R > 0$. We can take $\nu(p) = -\frac{1}{R}\, p$, so $-d\nu_p = \frac{1}{R}\, \mathrm{id}_{T_pS}$ and $K(p) = \frac{1}{R^2} > 0$, $H(p) = \frac{1}{R}$.

A point p in a surface S is called *elliptic* (resp. *hyperbolic*, *parabolic*) if $K(p) > 0$ (resp. $K(p) < 0$, $K(p) = 0$).

3.4 Local expressions for K, H

Fix a parametrization φ of S. Then $\{\varphi_u, \varphi_v\}$ is a tangent frame with respect to which we consider the matrices of the fundamental forms and the Weingarten operator and introduce a new (index) notation for the coefficients.

$$(I) = \begin{pmatrix} \langle \varphi_u, \varphi_u \rangle & \langle \varphi_u, \varphi_v \rangle \\ \langle \varphi_v, \varphi_u \rangle & \langle \varphi_v, \varphi_v \rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

$$(II) = \begin{pmatrix} \langle N, \varphi_{uu} \rangle & \langle N, \varphi_{uv} \rangle \\ \langle N, \varphi_{vu} \rangle & \langle N, \varphi_{vv} \rangle \end{pmatrix} = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$(-d\nu) = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}.$$

We have that

$$h_{11} = II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u \rangle = \langle h_1^1 \varphi_u + h_1^2 \varphi_v, \varphi_u \rangle,$$

so

$$h_{11} = h_1^1 g_{11} + h_1^2 g_{21}.$$

Similarly,

$$h_{12} = h_1^1 g_{12} + h_1^2 g_{22},$$

$$h_{21} = h_2^1 g_{11} + h_2^2 g_{21},$$

$$h_{22} = h_2^1 g_{12} + h_2^2 g_{22}.$$

In matrix form.

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Recall that (I) is invertible since it is positive definite. Thus

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.$$

Back to the classical notation,

$$\begin{aligned} (-d\nu) &= \frac{1}{EG - F^2} \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} \ell G - mF & -\ell F + mE \\ mG - nF & -mF + nE \end{pmatrix}. \end{aligned}$$

Hence

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{\det(II)}{\det(I)}$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)}.$$

It follows from these expressions that K, H are smooth on S.

3.5 Surfaces of revolution

Consider the parametrized surface of revolution

$$\varphi(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u, v) \in (a, b) \times (v_0, v_0 + 2\pi)$ and the geratrix $\gamma(s) = (f(s), 0, g(s))$ is parametrized by arc length. Then

$$\varphi_u = (f' \cos v, f' \sin v, g'), \qquad \varphi_v = (-f \sin v, f \cos v, 0),$$

so

$$E = f'^2 + g'^2 = 1, \quad F = 0, \quad G = f^2.$$

Moreover,

$$\begin{aligned} \varphi_{uu} &= (f'' \cos v, f'' \sin v, g''), \\ \varphi_{uv} &= (-f' \sin v, f' \cos v, 0), \\ \varphi_{vv} &= (-f \cos v, -f \sin v, 0). \end{aligned}$$

We compute the coefficients of II.

$$\begin{aligned} \ell &= \langle N, \varphi_{uu} \rangle = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uu} \rangle}{||\varphi_u \times \varphi_v||} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} f' \cos v & f' \sin v & g' \\ -f \sin v & f \cos v & 0 \\ f'' \cos v & f'' \sin v & g'' \end{vmatrix} \\ &= f'g'' - f''g'. \end{aligned}$$

Similarly,

$$m = \langle N, \varphi_{uv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f' \sin v & f' \cos v & 0 \end{vmatrix} = 0$$

and

$$n = \langle N, \varphi_{vv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f \cos v & -f \sin v & 0 \end{vmatrix} = fg'.$$

Note that $f'g'' - f''g'$ is the signed curvature κ_γ of γ, for $\kappa_\gamma = \langle \gamma'', (-g', 0, f') \rangle$. Now

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{(f'g'' - f''g')g'}{f} = \kappa_\gamma \frac{g'}{f} \tag{3.8}$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{1}{2} \left(\kappa_\gamma + \frac{g'}{f} \right). \tag{3.9}$$

It follows that the principal curvatures

$$\kappa_1 = \kappa_\gamma, \qquad \kappa_2 = \frac{g'}{f}.$$

The identities $F = m = 0$ mean that the fundamental forms are diagonalized in the frame $\{\varphi_u, \varphi_v\}$. In particular, φ_u, φ_v are always principal directions and thus the curves u-constant (parallels) and v-constant (meridians) are lines of curvature.

We can also derive some other useful formulas for K, H. Differentiation of $f'^2 + g'^2 = 1$ gives $f'f'' + g'g'' = 0$. Substituting this identity into (3.8) yields

$$K = \frac{-f''}{f}.$$

The identity also gives $\kappa_1 = \kappa_\gamma = \frac{g''}{f'}$ if $f' \neq 0$, so

$$H = \frac{1}{2}\left(\frac{g''}{f'} + \frac{g'}{f}\right) = \frac{(fg')'}{(f^2)'}. \tag{3.10}$$

We use this formula to prove the following theorem. A surface satisfying $H \equiv 0$ is called *minimal*. This terminology will be explained in section 3.7.

Theorem 3.11 *The only minimal surfaces of revolution are the plane and the catenoid (the surface of revolution generated by a catenary, which is the graph of hyperbolic cosine).*

Proof. We use the above notation. If $f' = 0$ on an interval, then $\kappa_\gamma = f'g'' - f''g'$ also vanishes on that interval, and eqn. (3.9) gives $g' = 0$, which is a contradiction to the fact that γ is regular. Therefore we can work on a neighborhood where f' is never zero. By formula (3.10), we need to solve the equation $fg' = k$, where k is a constant. Using $g' = \pm\sqrt{1 - f'^2}$, we get

$$f' = \pm\sqrt{1 - (k/f)^2}.$$

Note that $|f| \geq |k|$ is a necessary condition. This equation can be easily integrated by rewriting it as

$$\frac{f\,df}{\sqrt{f^2 - k^2}} = \pm ds.$$

We get

$$f(s) = \pm\sqrt{k^2 + (s + c_1)^2}.$$

The constant c_1 can be chosen to be zero by redefining the instant $s = 0$, and we recall that $f > 0$, so we have

$$f(s) = \sqrt{k^2 + s^2}.$$

If $k = 0$ then $f(s) = \pm s$ and g is constant, which corresponds to the case of the plane. Suppose $k \neq 0$ and integrate $g' = k/f$ to get

$$g(s) = k\log(s + \sqrt{k^2 + s^2}) + c_2.$$

We choose the constant $c_2 = -k\log|k|$ so that $\gamma(0) = (|k|, 0, 0)$. Changing the sign of k is equivalent to changing the sign of g, which corresponds to a reflection on the plane $z = 0$, so we may assume $k > 0$. Finally, we make the change of variable

$$t = g(s) = k\log(\sqrt{1 + (s/k)^2} + s/k)$$

to get

$$\gamma(t) = (k\cosh(t/k), 0, t)$$

which is a catenary. $\square$

3.6 Ruled surfaces

A ruled surface is a surface generated by a smooth one-parameter family of lines. More precisely, a (nonnecessarily regular) parametrized surface $\varphi : U \subset \mathbb{R}^2 \to S$ is called a *ruled surface* if there exist a smooth curves $\gamma : I \to \mathbb{R}^3$ and $w : I \to S^2$ such that

$$\varphi(u,v) = \gamma(u) + v\, w(u)$$

where $(u,v) \in I \times \mathbb{R} = U$. The curve γ is called a *directrix* and the lines $\mathbf{R}w(u)$ are called the *rulings*.

Obvious examples of ruled surfaces are planes, cilinders and cones. Other examples are the helicoid, the one-sheeted hyperboloid and the hyperbolic paraboloid (given by the equation $z = xy$ in $\mathbb{R}^3$).

We make some local considerations. Assume that $w'(u) \neq 0$ for all u, in other words, w is regular; this condition is sometimes expressed by saying that the ruled surface is *noncylindrical*. Then it is possible to introduce the so called *standard parameters* on S.

Proposition 3.12 *There exists a unique reparametrization*

$$\tilde\varphi(\tilde u, \tilde v) = \tilde\gamma(\tilde u) + \tilde v\, \tilde w(\tilde u)$$

such that $||\tilde w'|| = 1$ *and* $\langle \tilde\gamma', \tilde w' \rangle = 0$.

Proof. Since w is regular, we can introduce arc-length parameter $\tilde u$ so that $\tilde w(\tilde u) = w(u(\tilde u))$, and then $||\tilde w'|| = 1$. Next, we write $\tilde\gamma(\tilde u) = \gamma(\tilde u) - \tilde v(\tilde u)\tilde w(\tilde u)$ and impose the condition $\langle \tilde\gamma', \tilde w' \rangle = 0$ to get $\tilde v(\tilde u) = -\langle \frac{d}{d\tilde u}\gamma, \tilde w'(u) \rangle$. $\square$

The curve $\tilde\gamma$ is called the *striction line* of the surface and its regular points are called *central points* of the surface; note that $\tilde\gamma$ is not necessarily regular.

Using the standard parametrization, we can compute the curvature of a ruled surface

$$\varphi(u,v) = \gamma(u) + v\, w(u)$$

where $||w|| = ||w'|| = 1$, $\langle \gamma', w' \rangle = 0$. We have

$$\varphi_u = \gamma' + v\, w', \qquad \varphi_v = w,$$

so

$$E = ||\gamma'||^2 + v^2, \quad F = \langle \gamma', w \rangle, \quad G = 1.$$

Since w' is orthogonal to w and γ', there exists a smooth function $\lambda = \lambda(u)$, called the *distribution parameter*, such that

$$\gamma' \times w = \lambda\, w'. \tag{3.13}$$

It follows that

$$||\varphi_u \times \varphi_v|| = \sqrt{EG - F^2} = ||\gamma'||^2 - \langle \gamma', w \rangle^2 + v^2 = \lambda^2 + v^2.$$

In particular, the singular points of φ occur along the striction line ($v = 0$) precisely when $\lambda(u) = 0$.

Next, we compute the coefficients of II. We have

$$\varphi_{uu} = \gamma'' + v\, w', \quad \varphi_{uv} = w', \quad \varphi_{vv} = 0.$$

This implies

$$m = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uv}\rangle}{||\varphi_u \times \varphi_v||} = \frac{\langle \lambda w' + v\, w \times w', w'\rangle}{\sqrt{\lambda^2 + v^2}} = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}$$

and

$$n = 0,$$

what is sufficient to get the formula for the Gaussian curvature:

$$K = \frac{\ell n - m^2}{EG - F^2} = -\frac{\lambda(u)^2}{(\lambda(u)^2 + v^2)^2}.$$

Note that $K \leq 0$, and $K = 0$ precisely along the rulings that meet the striction line at a singular point ($\lambda(u) = 0$), except of course the singular point itself ($v \neq 0$). If $\lambda(u) \neq 0$, this formula also shows the striction line is characterized by the property that the maximum of K along each ruling occurs exactly at the central point.

The computation of ℓ is more involved. We have

$$\begin{aligned} &\langle \varphi_u \times \varphi_v, \varphi_{uu}\rangle \\ &= \lambda\langle w', \gamma''\rangle + \lambda v \langle w', w''\rangle + v\langle w' \times w, \gamma''\rangle + v^2\langle w' \times w, w''\rangle. \end{aligned} \tag{3.14}$$

We analyse separately the four terms on the right-hand side. Introduce the parameter

$$J = \langle w \times w', w''\rangle.$$

Since $\{w, w', w \times w'\}$ is an orthonormal frame,

$$\gamma' = \langle \gamma', w\rangle\, w + \langle \gamma', w \times w'\rangle\, w \times w' = Fw + \lambda\, w \times w'$$

and

$$\langle w', \gamma''\rangle = -\langle w'', \gamma'\rangle = -F\langle w'', w\rangle - \lambda\langle w'', w \times w'\rangle = F - \lambda J,$$

where we have used $\langle w'', w\rangle = -\langle w', w'\rangle = -1$. Equation $||w'|| = 1$ also implies $\langle w', w''\rangle = 0$. In order to analyse the third term in eqn. (3.14), differentiate eqn. (3.13) to get

$$\gamma'' \times w + \gamma' \times w' = \lambda' w' + \lambda w'',$$

and multiply through by w' to write

$$\langle \gamma'' \times w, w'\rangle = \lambda'.$$

Now eqn. (3.14) is

$$\langle \varphi_u \times \varphi_v, \varphi_{uu}\rangle = -Jv^2 - \lambda' v + \lambda(F - \lambda J)$$

and

$$\ell = \frac{-\lambda^2 J + \lambda' v + \lambda(F - \lambda J)}{\sqrt{\lambda^2 + v^2}}.$$

Hence

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = -\frac{Jv^2 + \lambda' v + \lambda(\lambda J + F)}{2(\lambda^2 + v^2)^{3/2}}.$$

Example 3.15 The standard parametrization of the helicoid has $\gamma(u) = (0, 0, bu)$ and $w(u) = (\cos u, \sin u, 0)$. Since $\gamma' \times w = bw'$, the distribution parameter $\lambda = b$ is constant and $K = -b^2/(b^2 + v^2)^2$. Note that $F = 0$ and, since $w'' = -w$, we also have $J = 0$. Hence $H = 0$.

Proposition 3.16 *The only minimal ruled surfaces are the plane and the helicoid.*

Proof. We have just seen that $H = 0$ says that

$$Jv^2 + \lambda' v + \lambda(\lambda J + F) = 0.$$

This is a quadratic polynomial in v whose coefficients, being functions depending only on u, must vanish. It follows that λ is constant and $J = \lambda F = 0$.

Since $J = 0$, w'' is a linear combination of w, w'. But $\langle w'', w' \rangle = 0$ and $\langle w'', w \rangle = -1$, so $w'' = -w$ and w is a circle.

If $\lambda = 0$ then $II = 0$, which corresponds to the case of the plane. Suppose $\lambda \neq 0$. Then $F = 0$ implying that $\gamma' = \lambda\, w \times w'$. Differentiation of this equation yields $\gamma'' = 0$. It follows that γ is a line perpendicular to the circle defined by w. Hence the surface is the helicoid. $\square$

3.7 Minimal surfaces

Let $S \subset \mathbb{R}^3$ be a surface. We say that S is a *minimal surface* if the mean curvature $H \equiv 0$. Historically speaking, this concept is related to the problem of characterizing the surface with smallest area spanned by a given boundary, a problem raised by Lagrange in 1760. The question of showing the existence of such a surface is called the *Plateau problem*, in honor of the Belgian physicist who performed experiments with soap films around 1850, and it was solved completely only in 1930, independently by Jesse Douglas and Tibor Radó, for which they were awarded the first two Fields Medals in 1936.

In order to explain the relation between mean curvature and minimization of surface area, consider a parametrization $\varphi : U \subset \mathbb{R}^2 \to S$, $N = \nu \circ \varphi$ the induced unit normal, and a smooth function $f : U \to \mathbb{R}$. Then we can introduce the *normal variation* of φ along f:

$$\varphi^\epsilon = \varphi + \epsilon f N.$$

Let us compute the first fundamental form of φ^ϵ:

$$\varphi^\epsilon_u = \varphi_u + \epsilon(f_u N + f N_u), \quad \varphi^\epsilon_v = \varphi_v + \epsilon(f_v N + f N_v).$$

Since $\langle \varphi_u, N\rangle = \langle \varphi_v, N\rangle = 0$ and $\langle \varphi_u, N_u\rangle = -\ell$, $\langle \varphi_u, N_v\rangle = \langle \varphi_v, N_u\rangle = -m$, $\langle \varphi_v, N_v\rangle = -n$, we obtain

$$\begin{aligned} E^\epsilon &= E - 2\epsilon f\ell + O(\epsilon^2), \\ F^\epsilon &= F - 2\epsilon f m + O(\epsilon^2), \\ G^\epsilon &= G - 2\epsilon f n + O(\epsilon^2), \end{aligned}$$

where $O(\epsilon^2)$ denotes a continuous function satisfying $\lim_{\epsilon\to 0} O(\epsilon^2)/\epsilon = 0$. It follows that

$$\begin{aligned} E^\epsilon G^\epsilon - (F^\epsilon)^2 &= EG - F^2 - 2\epsilon f(\ell G + nE - 2mF) + O(\epsilon^2) \\ &= (EG - F^2)(1 - 4\epsilon f H) + O(\epsilon^2). \end{aligned}$$

Let now $D \subset U$ be a compact domain and introduce

$$A(\epsilon) = \operatorname{area}(\varphi^\epsilon(D)) = \int\int_D \sqrt{E^\epsilon G^\epsilon - (F^\epsilon)^2}\, dudv.$$

We have

$$\begin{aligned} A'(0) &= \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \int\int_D \left(E^\epsilon G^\epsilon - (F^\epsilon)^2\right)^{1/2}\, dudv \\ &= \int\int_D \frac{\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0} E^\epsilon G^\epsilon - (F^\epsilon)^2}{2(EG - F^2)^{1/2}}\, dudv. \end{aligned}$$

Hence

$$A'(0) = -2\int\int_D fH\sqrt{EG - F^2}\, dudv.$$

This formula is called *first variation of surface area*. As a corollary, we obtain the following characterization of minimal surfaces as critical points of the area functional.

Proposition 3.17 *A surface S is minimal if and only if $A'(0) = 0$ for every parametrization $\varphi : U \to S$, every normal variation of φ, and every compact domain $D \subset U$.*

Proof. If $H(p) \neq 0$ for some $p \in S$, we choose a compact neighborhood $\tilde{D}$ of p in S such that H does not vanish on $\tilde{D}$ and $\tilde{D}$ is contained in the image of a parametrization $\varphi : U \to S$, set $D = \varphi^{-1}(\tilde{D})$, and take $f = H|_U$. We get

$$A'(0) = -2\int\int_D H^2\sqrt{EG - F^2}\, dudv < 0,$$

so the given condition is suficient for the minimality of S. That it is also necessary is obvious. □

3.7.1 Isothermal parameters

When studying minimal surfaces, it is useful to introduce special parameters. A parametrized surface $\varphi : U \to S$ is called *isothermal* if

$$E = G = \lambda^2, \qquad F = 0,$$

where $\lambda \geq 0$ is a smooth function on U; in this case, the parameters $(u, v) \in U$ are also called *isothermal*. Note that φ is regular if and only if $\lambda > 0$. An isothermal parametrization φ is also called *conformal* or *angle preserving* because angles between curves in the surface are equal to the angles between the corresponding curves in the parameter plane.

Note that the mean curvature expressed in terms of isothermal parameters becomes

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{\ell + n}{2\lambda^2}. \tag{3.18}$$

Proposition 3.19 *If φ is isothermal, then $\Delta\varphi = 2\lambda^2 HN$ (here $N = \nu \circ \varphi$ is the unit normal along φ).*

Proof. Here Δ denotes the Laplacian operator and $\Delta\varphi = \varphi_u u + \varphi_v v$. Consider the equations $\langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle$ and $\langle \varphi_u, \varphi_v \rangle = 0$; differentiating the first one with respect to u and the second one with resptec to v, we obtain

$$\langle \varphi_{uu}, \varphi_u \rangle = \langle \varphi_{vu}, \varphi_u \rangle \quad \text{and} \quad \langle \varphi_{uv}, \varphi_v \rangle + \langle \varphi_u, \varphi_{vv} \rangle = 0.$$

Putting these together yields $\langle \Delta\varphi, \varphi_u \rangle = 0$. Similarly, differentiating the first equation with respect to v and the second one with resptec to u, we get that $\langle \Delta\varphi, \varphi_v \rangle = 0$. This shows that $\Delta\varphi$ is a normal vector. Finally,

$$\langle \Delta\varphi, N \rangle = \langle \varphi_{uu}, N \rangle + \langle \varphi_{vv}, N \rangle = \ell + n = 2\lambda^2 H$$

by eqn. (3.18). □

Corollary 3.20 *An isothermal regular parametrized surface $\varphi : U \to S$ is minimal if and only if the coordinate functions of φ are harmonic functions on U.*

Isothermal parameters exist around any point in a surface. In the next section, we present a proof of their existence in the case of minimal surfaces.

Theorem 3.21 *There exist no compact minimal surfaces in $\mathbb{R}^3$.*

Proof. Suppose, to the contrary, that S is a compact minimal surface. Without loss of generality, we may assume S is connected. Consider the coordinate function $x : \mathbb{R}^3 \to \mathbb{R}$. There exists a point $p \in S$ where the restriction $x|_S$ attains its maximum. Let $\varphi : U \to \varphi(U)$ be an isothermal parametrization around p with U connected. Then $x \circ \varphi$ is a harmonic function on U which attains its maximum at an interior point $p \in U$. By the maximum principle, $x \circ \varphi$ is a constant function on U, or, $x \equiv x(p)$ on $\varphi(U)$.

Next, let $q \in S$ be arbitrary and choose a continuous curve $\gamma : [0,1] \to S$ joining p to q. Cover $\gamma([0,1])$ by finitely many connected open sets $V_0 = \varphi(U)$, $V_1, \ldots, V_n$, each V_i equal to the image of an isothermal parametrization φ_i, such that $V_i \cap V_{i+1} \neq \varnothing$ for all $i = 0, 1, \ldots, n$ and $q \in V_n$. Since $x|_S$ attains its maximum value along $V_0 \cap V_1 \neq \varnothing$, the maximum principle applied to $x \circ \varphi_1$ yields that this function is constant along V_1, namely, $x \equiv x(p)$ on V_1. Proceeding by induction, we get that $x \equiv x(p)$ on V_n and hence $x(q) = x(p)$. Since q is arbitrary, this argument proves that $x|S$ is a constant function. The same argument applied to the other coordinate functions y, $z : \mathbb{R}^3 \to \mathbb{R}$ finally shows that S must be a point, a contradiction. $\square$

3.7.2 The Enneper-Weierstrass representation

We discuss now an unexpected connection beween minimal surfaces and theory of functions of one complex variable. Let $\varphi : U \to S$ be a parametrized surface. Denote by x_1, x_2, $x_3 : U \to \mathbb{R}$ the coordinate functions of φ. We introduce the complex functions ($j = 1, 2, 3$):

$$f_j(\zeta) = \frac{\partial x_j}{\partial u} - \frac{\partial x_j}{\partial v}, \qquad \text{where } \zeta = u + iv. \tag{3.22}$$

The function f_j is smooth as a real function $U \subset \mathbb{R}^2 \to \mathbb{R}^2$, so a necessary and suffcient condition for f_j to be holomorphic is given by the Cauchy-Riemann equations $\frac{\partial}{\partial u}\Re f_j = \frac{\partial}{\partial v}\Im f_j$, $\frac{\partial}{\partial v}\Re f_j = -\frac{\partial}{\partial u}\Im f_j$. We deduce that

(a) f_j is holomorphic in ζ if and only if x_j is harmonic in u, v.

Note also the identities:

$$\begin{aligned} f_1^2 + f_2^2 + f_3^2 &= \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2 - \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2 - 2i\sum_{j=1}^{3}\frac{\partial x_j}{\partial u}\cdot\frac{\partial x_j}{\partial v} \\ &= E - G - 2iF \end{aligned}$$

and

$$|f_1|^2 + |f_2|^2 + |f_3|^2 = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2 + \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2 = E + G. \tag{3.23}$$

It follows from these identities that

(b) φ is isothermal if and only if

$$f_1^2 + f_2^2 + f_3^2 = 0. \tag{3.24}$$

(c) If φ is isothermal, then φ is regular if and only if

$$|f_1|^2 + |f_2|^2 + |f_3|^2 \neq 0. \tag{3.25}$$

Proposition 3.26 *Let $\varphi : U \to S$ be an isothermal regular parametrized minimal surface. Then the functions f_j defined by (3.22) are holomorphic and satisfy (3.24) and (3.25). Conversely, if f_1, f_2, f_3 are holomorphic functions defined on an simply-connected domain U which satisfy (3.24) and (3.25), then ($j = 1, 2, 3$)*

$$x_j(\zeta) = \Re \int_{\zeta_0}^{\zeta} f_j(z)\, dz, \qquad \zeta \in U,$$

(for fixed $\zeta_0 \in U$) are the coordinates of an isothermal regular parametrized minimal surface $\varphi : U \to S$ such that eqns. (3.22) are valid.

Proof. One direction follows from assertions (a), (b), (c) above and Corollary 3.20. For the converse, note that $\zeta \mapsto \int_{\zeta_0}^{\zeta} f_j(z)\, dz$ is well defined because U is simply connected and f_j is holomorphic, and yields a holomorphic function on U for which we can apply the Cauchy-Riemann equations:

$$\begin{aligned} \frac{d}{d\zeta} \int^{\zeta} f_j &= \frac{\partial}{\partial u} \Re \int^{\zeta} f_j + i \frac{\partial}{\partial u} \Im \int^{\zeta} f_j \\ &= \frac{\partial}{\partial u} \Re \int^{\zeta} f_j - i \frac{\partial}{\partial v} \Im \int^{\zeta} f_j, \end{aligned}$$

so eqns. (3.22) are valid; the rest now follows from (a), (b), (c) and Corollary 3.20 applied in the opposite direction. □

Note that the functions x_j in the preceding proposition are defined up to an additive constant so that the surface is defined up to a translation.

Thus we see that the local study of minimal surfaces in $\mathbb{R}^3$ is reduced to solving equations (3.24) and (3.25) for triples of holomorphic functions. We next explain how this can be done. Rewrite (3.24) as

$$(f_1 + i f_2)(f_1 - i f_2) = -f_3^2. \tag{3.27}$$

Except in case $f_1 \equiv i f_2$ and $f_3 \equiv 0$ (which is easily seen to correspond to the case of a plane), the functions

$$f = f_1 - i f_2, \qquad g = \frac{f_3}{f_1 - i f_2}$$

are such that f is holomorphic and g is meromorphic. Clearly, $f_3 = fg$, and it follows from eqn. (3.27) that

$$f_1 + i f_2 = \frac{-f_3^2}{f_1 - i f_2} = -fg^2. \tag{3.28}$$

Hence

$$f_1 = \frac{1}{2} f(1 - g^2), \quad \text{and} \quad f_2 = \frac{i}{2} f(1 + g^2).$$

By (3.28), fg^2 is homolorphic and this says that at every pole of g, f has a zero of order at least twice the order of the pole. Further, eqn. (3.25) says that f_1, f_2,

f_3 cannot vanish simultanelously, and this means that f can only have a zero at a pole of g, and then the order of its zero must be exactly twice the order of the pole of g. We summarize this dicussion as follows.

Theorem 3.29 (The Enneper-Weierstrass representation) *Every minimal surface which is not a plane can be locally represented as*

$$\begin{aligned} x_1 &= \Re \int \frac{1}{2} f(\zeta)(1 - g^2(\zeta)\, d\zeta \\ x_2 &= \Re \int \frac{i}{2} f(\zeta)(1 + g^2(\zeta))\, d\zeta \\ x_3 &= \Re \int f(\zeta) g(\zeta)\, d\zeta, \end{aligned}$$

where: f is a holomorphic function on a simply-connected domain U, g is meromorphic on U, f vanishes only at the poles of g, and the order of its zero at such a point is exactly twice the order of the pole of g.

Conversely, every pair functions f, g satisfying these conditions define an isothermal regular parametrized minimal surface via the above equations.

Examples 3.30 1. The catenoid is given by $f(z) = -e^{-z}$, $g(z) = -e^z$.

2. The helicoid is given by $f(z) = -ie^{-z}$, $g(z) = -e^z$.

3. The minimal surface of Enneper (discovered in 1863) is given by $f(z) = 1$, $g(z) = z$. Solving for the parametrization, we obtain $x_1 = u - \frac{1}{3}u^3 + uv^2$, $x_2 = -v - u^2 v + \frac{1}{3}v^3$, $x_3 = u^2 - v^2$.

4. The minimal surface of Scherk (discovered in 1834) is given by $f(z) = 4/(1 - z^4)$, $g(z) = iz$. It can also be parametrized as the graph of $(x, y) \mapsto \log \frac{\cos x}{\cos y}$.

The Enneper-Weierstrass representation not only allows us to construct a great variety of minimal surfaces having interesting properties, but also serve to prove general theorems about minimal surfaces by translating the statements into corresponding statements about holomorphic functions. Unfortunately, developing this philosophy would take us beyond the scope of these notes, so we content ourselves with a small remark. Let us express the basic geometric quantities of an isothermal regular parametrized minimal surface $\varphi : U \to S$ in terms of f, g. We have

$$E = G = \lambda^2, \qquad F = 0,$$

where

$$\begin{aligned} \lambda^2 &= \frac{1}{2} \sum_{j=1}^{3} |f_j|^2 \qquad \text{by (3.23)} \\ &= \frac{1}{4} |f|^2 |1 + g|^2 + \frac{1}{4} |f|^2 |1 + g|^2 + |fg|^2 \\ &= \left(\frac{|f|(1 + |g|^2)}{2} \right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned}\varphi_u \times \varphi_v &= (\,\Im\{f_2\bar{f}_3\}, \Im\{f_3\bar{f}_1\}, \Im\{f_1\bar{f}_2\}\,) \\ &= \frac{|f|^2(1+|g|^2)}{4}\,(\,2\Re g, 2\Im g, |g|^2-1\,),\end{aligned}$$

and

$$||\varphi_u \times \varphi_v|| = \sqrt{EG-F^2} = \lambda^2,$$

so

$$N = \left(\frac{2\Re g}{|g|^2+1}, \frac{2\Im g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right).$$

Recall that *stereographic projection* $\pi : S^2 \setminus \{(0,0,1)\} \to \mathbf{C}$ is the map

$$\pi(x_1,x_2,x_3) = \frac{x_1+ix_2}{1-x_3},$$

and its inverse is

$$\pi^{-1}(z) = \left(\frac{2\Re z}{|z|^2+1}, \frac{2\Im z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Hence

$$N = \pi^{-1} \circ g. \tag{3.31}$$

Proposition 3.32 *Let $\varphi : U \to S$ be an isothermal regular parametrized minimal surface, where U is the entire ζ-plane. Then either S lies in a plane, or the image of the Gauss map takes on all values with at most two exceptions.*

Proof. If S does not lie in a plane, we can construct the function $g(\zeta)$ which is meromorphic on the entire ζ-plane; by Picard's theorem, it either takes all values with at most two exceptions, or else it is constant. Eqn. (3.31) shows that the same alternative applies to N, and in the latter case S lies in a plane. □

3.7.3 Local existence of isothermal parameters for minimal surfaces

Lemma 3.33 *Let S be a minimal surface. Then every point of S lies in the image of an isothermal parametrization of S.*

Proof. Let $p \in S$. First all, we can find a neighborhood of p in which S is the graph of a smooth function which, by relabeling coordinates, can be assumed in the form $z = h(x,y)$ for $(x,y) \in U$ (Check!). The minimal equation for graphs is easily computed to be

$$(1+h_y^2)h_{xx} - 2h_xh_yh_{xy} + (1+h_x^2)h_{yy} = 0.$$

We then have equation

$$\frac{\partial}{\partial x}\frac{1+h_y^2}{W} = \frac{\partial}{\partial y}\frac{h_xh_y}{W}.$$

satisfied on U, where $W = \sqrt{1 + h_x^2 + h_y^2}$. Taking U simply-connected, this implies that we can find a smooth function Φ on U with

$$\frac{\partial \Phi}{\partial x} = \frac{h_x h_y}{W}, \qquad \frac{\partial \Phi}{\partial y} = \frac{1 + h_y^2}{W}.$$

Introduce new coordinates

$$\bar{x} = x, \qquad \bar{y} = \Phi(x, y).$$

One checks

$$\tfrac{\partial x}{\partial \bar{x}} = 1, \quad \tfrac{\partial x}{\partial \bar{y}} = 0, \quad \tfrac{\partial y}{\partial \bar{x}} = -\frac{h_x h_y}{1 + h_y^2}, \quad \tfrac{\partial y}{\partial \bar{y}} = \frac{W}{1 + h_y^2},$$

and the coeffcients of the second fundamental form with respect to $\bar{x}$, $\bar{y}$ are

$$\bar{E} = \bar{G} = \frac{W^2}{1 + h_y^2}, \qquad \bar{F} = 0,$$

as desired. $\square$

Chapter 4

Surfaces: intrinsic geometry

It is the geometry of objects associated to the surface which depend only on the first fundamental form. Obvious examples are lengths of curves, angles between tangent vectors and areas of regions in the the surface. Less obvious examples are geodesics (yet to be defined) and the Gaussian curvature (Theorema Egregium). We will first discuss local questions.

4.1 Isometries and local isometries

A diffeomorphism $f : S \to \bar{S}$ between two surfaces S and $\bar{S}$ is called an *isometry* if it preserves the first fundamental forms, namely,

$$I_p(w_1, w_2) = I_{f(p)}(df_p(w_1), df_p(w_2))$$

for all $p \in S$ and w_1, $w_2 \in T_pS$. Equivalently, the coefficients of the first fundamental forms at corresponding points are equal:

$$E(u, v) = \bar{E}(u, v),\ F(u, v) = \bar{F}(u, v),\ G(u, v) = \bar{G}(u, v)$$

for any parameterization $\varphi : U \to S$ and all $(u, v) \in U$, noting that $\bar{\varphi} = f \circ \varphi : U \to \bar{S}$ will be a parameterization of $\bar{S}$, where E, F, G are computed with respect to φ, and $\bar{E}$, $\bar{F}$, $\bar{G}$ are computed with respect to $\bar{\varphi}$.

A map $f : V \to \bar{S}$ of a neighborhood of $p \in S$ is a *local isometry* at p if there exists a neighborhood $\bar{V}$ of $f(p)$ such that $f : V \to \bar{V}$ is an isometry. If there exists a local isometry into $\bar{S}$ at every $p \in S$, we say S is *locally isometric* to $\bar{S}$. Finally, S and $\bar{S}$ are said *locally isometric* if S is locally isometric to $\bar{S}$ and $\bar{S}$ is locally isometric to S.

It is clear that if $\varphi : U \to S$ and $\bar{\varphi} : U \to \bar{S}$ are two parameterizations with $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ on U, then $\bar{\varphi} \circ \varphi^{-1} : \varphi(U) \to \bar{S}$ and $\varphi \circ \bar{\varphi}^{-1} : \bar{\varphi}(U) \to S$ are local isometries.

Examples 4.1 1. If S and $\bar{S}$ are isometric surfaces, then they are locally isometric. However the converse is not true. In fact, the plane and cylinder are locally isometric, since they have the same first fundamental forms in suitable parameterizations. However, they cannot be isometric because they are not even homeomorphic (any closed curve in the plane is continuously deformable into a point, but there are curves in the cylinder which cannot be deformed continuously into a point).

2. The catenoid and helicoid are locally isometric.

4.2 Directional derivative

We start by recalling the concept of directional derivative of vector fields on $\mathbb{R}^n$. Let $Y : W \to \mathbb{R}^n$ be a smooth vector field defined on an open subset W of $\mathbb{R}^n$, let $v \in \mathbb{R}^n$ be a fixed vector and $p \in W$. Then the *directional derivative* of Y along v at p is

$$D_v Y|_p = dY_p(v) = \lim_{t\to 0} \frac{Y(p+tv) - Y(p)}{t}.$$

Note that if $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$\begin{aligned}
\lim_{t\to 0} \frac{Y(\gamma(t)) - Y(\gamma(0))}{t} &= \frac{d}{dt}\Big|_{t=0} Y(\gamma(t)) \\
&= dY_{\gamma(0)}(\gamma'(0)) \qquad \text{(by the chain rule)} \\
&= dY_p(v) \\
&= D_v Y|p,
\end{aligned}$$

in other words, $D_v Y|_p$ depends only on the values of Y along a smooth curve through p with velocity v.

As a particular case, consider the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and denote by $(x^1, \ldots, x^n)$ the standard coordinates in $\mathbb{R}^n$. Then

$$D_{e_i} Y|_p = \tfrac{\partial Y}{\partial x^i}(p)$$

and, if $v = \sum_i v^i e_i$, then

$$D_v Y = \sum_i v^i \tfrac{\partial Y}{\partial x_i}.$$

A word about notation: If $X : W \to \mathbb{R}^n$ is another smooth vector field, then $X(p)$ is a vector in $\mathbb{R}^n$ and we write

$$D_X Y|_p = D_{X(p)} Y|_p.$$

4.3 Vector fields on surfaces

Let S be a surface in $\mathbb{R}^3$, and let $V \subset S$ be an open subset. A (smooth) *vector field* on V is a (smooth) map $X : V \to \mathbb{R}^3$; recall that this means that $X \circ \varphi : U \to \mathbb{R}^3$ is smooth for any parameterization $\varphi : U \to S$ of S with $\varphi(U) \subset V$. In addition, We say that X is *tangent to* S (resp. *normal to* S) if $X(p) \in T_pS$ (resp. $X(p) \perp T_pS$) for $p \in V$.

Let $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) = V \subset S$ be a parametrization. In the following, we use (u^1, u^2) to denote coordinates on the parameter plane $\mathbb{R}^2$ and (x^1, x^2, x^3) to denote coordinates in the ambient $\mathbb{R}^3$. According to the above definition,

$$\tfrac{\partial \varphi}{\partial u^1} \circ \varphi^{-1}, \tfrac{\partial \varphi}{\partial u^2} \circ \varphi^{-1} : V \to \mathbb{R}^3$$

are vector fields on V tangent to S. For the sake of convenience, henceforth we will abuse terminology and say that $\frac{\partial \varphi}{\partial u^1}$, $\frac{\partial \varphi}{\partial u^2}$ are vector fields tangent to S. Note then that any vector field X on V tangent to S can be written as a linear combination

$$X = a^1 \tfrac{\partial \varphi}{\partial u^1} + a^2 \tfrac{\partial \varphi}{\partial u^2}$$

where a^1, $a^2 : V \to \mathbb{R}$. It is an easy exercise to check that X is smooth if and only if a^1, a^2 are smooth functions (do it!). Similarly, any smooth vector field on V normal to S is of the form

$$X = b \tfrac{\partial \varphi}{\partial u^1} \times \tfrac{\partial \varphi}{\partial u^2}$$

for a smooth function $b : V \to \mathbb{R}$.

4.4 Covariant derivative

In this section we explain how to differentiate a vector field tangent to a surface along another tangent vector field to obtain a third tangent vector field. Let S be a surface in $\mathbb{R}^3$, $V \subset S$ an open subset, $p \in V$. Consider vector fields X, Y on V such that X is tangent to S. Our previous discussion about the directional derivative shows that $D_X Y|_p$ is well defined, namely, it equals $\frac{d}{dt}|_{t=0} Y(\gamma(t))$ where $\gamma : (-\epsilon, \epsilon) \to S$ is smooth and $\gamma(0) = p$, $\gamma'(0) = v$. The association

$$p \mapsto D_X Y|_p$$

is a vector field on V. As a special case, if $\varphi : U \to V$ is a parametrization of S and $\varphi(u) = p$ for some $u \in U$,

$$D_{\frac{\partial \varphi}{\partial u^i}} Y|_p = \tfrac{\partial (Y \circ \varphi)}{\partial u^i}(u).$$

Further, if $Y = \frac{\partial \varphi}{\partial u^j}$,

$$D_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^i}|_p = \frac{\partial^2 \varphi}{\partial u^i u^j}(u).$$

Back to the general case, we now assume that both X and Y are tangent to S. The *covariant derivative* of Y along X at p is

$$\nabla_X Y|_p = (D_X Y|_p)^\top$$

where $(\cdot)^\top$ denotes the orthogonal projection onto T_pS. The symbol "∇" is read "nabla". In this way, the association

$$p \mapsto \nabla_X Y|_p$$

is a vector field on V tangent to S.

Lemma 4.2 (Properties of D and ∇) *Let X, $\tilde{X}$, Y, $\tilde{Y}$ be vector fields tangent to S and let f be a smooth function on S. Then:*

1. $\nabla_{fX+\tilde{X}} Y = f\nabla_X Y + \nabla_{\tilde{X}} Y$;
2. $\nabla_X (Y + \tilde{Y}) = \nabla_X Y + \nabla_X \tilde{Y}$;
3. $\nabla_X (fY) = df(X)Y + f\nabla_X Y$;
4. $X\langle Y, \tilde{Y}\rangle = \langle \nabla_X Y, \tilde{Y}\rangle + \langle Y, \nabla_X \tilde{Y}\rangle$;

and the same identities hold for ∇ replaced by D.

In this statement, fX denotes the tangent vector field $p \mapsto f(p)X(p)$, and $\langle Y, \tilde{Y}\rangle$ denotes the scalar function $p \mapsto \langle Y(p), \tilde{Y}(p)\rangle$ so that $X\langle Y, \tilde{Y}\rangle(p)$ denotes the directional derivative in the direction of $X(p)$.

Proof. We prove (3) and (4) and leave the rest as an exercise. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve with $\gamma(0) = p$, $\gamma'(0) = X(p)$. Then

$$\begin{aligned}
D_X(fY)|_p &= \frac{d}{dt}|_{t=0}(fY)(\gamma(t)) \\
&= \frac{d}{dt}|_{t=0} f(\gamma(t))Y(\gamma(t)) \\
&= \frac{d}{dt}|_{t=0} f(\gamma(t)) \cdot Y(\gamma(0)) + f(\gamma(0))\frac{d}{dt}|_{t=0} Y(\gamma(t)) \\
&= df_p(X(p))Y(p) + f(p)D_X Y|_p.
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_X(fY)|_p &= (D_X(fY))^\top \\
&= (df(X)Y + fD_X Y)^\top \\
&= df(X)Y^\top + f\,(D_X Y)^\top \qquad \text{(orthogonal projection is linear)} \\
&= df(X)Y + f\nabla_X Y \qquad (Y \text{ is tangent}).
\end{aligned}$$

Next,

$$\begin{aligned}
X\langle Y, \tilde{Y}\rangle(p) &= \frac{d}{dt}|_{t=0}\langle Y(\gamma(t)), \tilde{Y}(\gamma(t))\rangle \\
&= \langle \frac{d}{dt}|_{t=0} Y(\gamma(t)), \tilde{Y}(\gamma(0))\rangle + \langle Y(\gamma(0)), \frac{d}{dt}|_{t=0}\tilde{Y}(\gamma(t))\rangle \\
&= \langle D_X Y|_p, \tilde{Y}(p)\rangle + \langle Y(p), D_X \tilde{Y}|_p\rangle \\
&= \langle (D_X Y|_p)^\top, \tilde{Y}(p)\rangle + \langle Y(p), \left(D_X \tilde{Y}|_p\right)^\top\rangle \qquad (Y, \tilde{Y} \text{ are tangent}) \\
&= \langle \nabla_X Y|_p, \tilde{Y}(p)\rangle + \langle Y(p), \nabla_X \tilde{Y}|_p\rangle,
\end{aligned}$$

as we wished. $\square$

Let X, Y be tangent vector fields. Of course, ν is a normal vector field, so $\langle Y, \nu \rangle = 0$ and Lemma 4.2(4) says that

$$\langle D_X Y, \nu \rangle + \langle Y, D_X \nu \rangle = 0.$$

Le $A = -d\nu$ be the Weingarten operator. Then $A(X) = -d\nu(X) = -D_X \nu$ and we have

$$\begin{aligned} \nabla_X Y &= (D_X Y)^\top \\ &= D_X Y - \underbrace{\langle D_X Y, \nu \rangle \nu}_{\text{normal component}} \\ &= D_X Y + \langle Y, D_X \nu \rangle \nu \\ &= D_X Y - \langle Y, AX \rangle \nu \\ &= D_X Y - \Pi(X, Y) \nu. \end{aligned}$$

Hence we arrive at the *Gauss formula*

$$D_X Y = \nabla_X Y + \Pi(X, Y) \nu.$$

In the remaining of this section, we show that the covariant derivative ∇ of a surface S is an intrinsic object, namely, it is completely determined by the first fundamental form I; in particular, locally isometric surfaces have the same covariant derivative. This is not an obvious assertion in view of the fact that $\nabla_X Y$ was defined as the orthogonal projection onto the surface of the directional derivative $D_X Y$, and so the ambient space $\mathbb{R}^3$ was used in this definition.

Let X, Y be vector fields on S. Since we are dealing with a local assertion, we can work in the image of a parametrization φ of S and write

$$X = \sum_{i=1}^{2} a^i \tfrac{\partial \varphi}{\partial u^i}, \qquad Y = \sum_{i=1}^{2} b^j \tfrac{\partial \varphi}{\partial u^j}.$$

By using Lemma 4.2, we first obtain a local expression for $\nabla_X Y$:

$$\begin{aligned} \nabla_X Y &= \sum_{i,j} a^i \nabla_{\frac{\partial \varphi}{\partial u^i}} \left(b^j \frac{\partial \varphi}{\partial u^j} \right) \\ &= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial u^i} \frac{\partial \varphi}{\partial u^j} + b^j \nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} \right). \end{aligned} \tag{4.3}$$

Since $\nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j}$ is tangent to S, we can write

$$\nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} = \sum_k \Gamma^k_{ij} \frac{\partial \varphi}{\partial u^k}, \tag{4.4}$$

for some smooth functions Γ_{ij}^k, the so called *Christofell symbols (of the second kind)*. Substituting into (4.3),

$$\nabla_X Y = \sum_{i,k} a^i \left(\frac{\partial b^k}{\partial u^i} + \sum_j b^j \Gamma_{ij}^k \right) \frac{\partial \varphi}{\partial u^k}, \tag{4.5}$$

which shows that ∇ depends only on $\{\Gamma_{ij}^k\}$.

We can also define the *Christofell symbols of the first kind* by putting

$$\Gamma_{ij,k} = I\left(\nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j}, \frac{\partial \varphi}{\partial u^k} \right).$$

By using (4.4), we have

$$\begin{aligned} \Gamma_{ij,k} &= \sum_\ell \Gamma_{ij}^\ell I\left(\frac{\partial \varphi}{\partial u^\ell}, \frac{\partial \varphi}{\partial u^k} \right) \\ &= \sum_\ell \Gamma_{ij}^\ell g_{\ell k}. \end{aligned}$$

Multiplying through by g^{km}, where (g^{km}) denotes the inverse matrix of $I = (g_{ij})$, we get

$$\Gamma_{ij}^m = \sum_k \Gamma_{ij,k} g^{km}.$$

Hence $\{\Gamma_{ij}^k\}$ depends only on $\{\Gamma_{ij,k}\}$ and I.

In order to complete our argument, we need to show that $\{\Gamma_{ij,k}\}$ depends only on I. It is important to notice that Γ_{ij}^k (and $\Gamma_{ij,k}$) is symmetric with respect to the indices i, j, namely,

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

This is immediate from

$$\nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} = \left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j} \right)^\top.$$

Next, by using Lemma 4.2(4), we write

$$\frac{\partial g_{ij}}{\partial u^k} = \left\langle \nabla_{\frac{\partial \varphi}{\partial u^k}} \frac{\partial \varphi}{\partial u^i}, \frac{\partial \varphi}{\partial u^j} \right\rangle + \left\langle \frac{\partial \varphi}{\partial u^i} \nabla_{\frac{\partial \varphi}{\partial u^k}} \frac{\partial \varphi}{\partial u^j} \right\rangle,$$

so

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ki,j} + \Gamma_{kj,i}.$$

Doing cyclic permutations on (i, j, k), we also obtain

$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{ij,k} + \Gamma_{ik,j}$$

and
$$\frac{\partial g_{ki}}{\partial u^j} = \Gamma_{jk,i} + \Gamma_{ji,k}.$$
Summing the last two equations and subtracting the first one finally yields that
$$2\Gamma_{ij,k} = \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j}, \tag{4.6}$$
which completes the proof that ∇ depends only on $I = (g_{ij})$.

Example 4.7 Consider a surface of revolution parametrized as in section 3.5. Since $F = 0$, we can write
$$\nabla_{\varphi_u}\varphi_u = (D_{\varphi_u}\varphi_u)^\top = (\varphi_{uu})^\top = \langle\varphi_{uu}, \varphi_u\rangle\frac{\varphi_u}{E} + \langle\varphi_{uu}, \varphi_v\rangle\frac{\varphi_v}{G}.$$
Using the formulas from section 3.5,
$$\langle\varphi_{uu}, \varphi_u\rangle = f'f'' + g'g'' = \frac{1}{2}(f'^2 + g'^2)' = 0$$
and
$$\langle\varphi_{uu}, \varphi_v\rangle = 0.$$
Hence
$$\nabla_{\varphi_u}\varphi_u = 0.$$
Similarly, one computes that
$$\nabla_{\varphi_u}\varphi_v = \nabla_{\varphi_v}\varphi_u = \frac{f'}{f}\varphi_v$$
and
$$\nabla_{\varphi_v}\varphi_v = -ff'\varphi_u.$$
We thus obtain
$$\Gamma^1_{11} = \Gamma^2_{11} = 0, \quad \Gamma^1_{12} = 0, \quad \Gamma^2_{12} = \frac{f'}{f}, \quad \Gamma^1_{22} = -ff', \quad \Gamma^2_{22} = 0.$$
In particular, all Christofell symbols vanish along the parallels $u =$ constant corresponding to critical points of f.

4.5 The Lie bracket

If X, Y are vector fields on $\mathbb{R}^n$ or on a surface, then, in general, $D_XY \neq D_YX$. For instance, take $X = x^2 \cdot e_1$, $Y = e_2$. Then
$$D_XY = D_{x^2e_1}e_2 = x^2D_{e_1}e_2 = 0,$$

where $D_{e_1}e_2 = 0$ because e_2 is constant, and

$$D_Y X = D_{e_2}(x^2 \cdot e_1) = dx^2(e_2)e_1 + x^2 D_{e_2}e_1 = e_1,$$

where $dx_2(e_2) = \frac{\partial x_2}{\partial x_2} = 1$. In general, the lack of comutativity is measured by the *Lie bracket*

$$[X, Y] := D_X Y - D_Y X.$$

If X, Y are tangent to S, then

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= (D_X Y - II(X,Y)\nu) - (D_Y X - II(Y,X)\nu) \\ &= D_X Y - D_Y X \\ &= [X,Y]. \end{aligned}$$

As another example, for a parametrization φ of S we have

$$[\tfrac{\partial\varphi}{\partial u^i}, \tfrac{\partial\varphi}{\partial u^j}] = D_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial\varphi}{\partial u^j} - D_{\frac{\partial\varphi}{\partial u^j}} \frac{\partial\varphi}{\partial u^i} = \frac{\partial^2\varphi}{\partial u^i \partial u^j} - \frac{\partial^2\varphi}{\partial u^j \partial u^i} = 0.$$

This example shows that given a tangent frame $\{X_1, X_2\}$ to a surface, a necessary condition for it to be of the form $\{\frac{\partial\varphi}{\partial u^1}, \frac{\partial\varphi}{\partial u^2}\}$ for some parametrization φ is that $[X_1, X_2] = 0$.

Let X, Y be vector fields on S. We obtain a local expression for their Lie bracket. In the image of a parametrization, we can write

$$X = \sum_{i=1}^{2} a^i \tfrac{\partial\varphi}{\partial u^i}, \qquad Y = \sum_{i=1}^{2} b^j \tfrac{\partial\varphi}{\partial u^j}.$$

Using (4.5) and the symmetry of Γ_{ij}^k with respect to i, j, we have, on the image of φ:

$$\begin{aligned} [X,Y] &= \nabla_X Y - \nabla_Y X \\ &= \sum_{i,j} \left(a^i \frac{\partial b^j}{\partial u^i} - b^i \frac{\partial a^j}{\partial u^i} \right) \frac{\partial\varphi}{\partial u^j}. \end{aligned}$$

4.6 Parallel transport

Let $S \subset \mathbb{R}^3$ be a surface and let $\gamma : I \to S$ be a smooth parametrized curve (nonnecessarily regular). A *(smooth) tangent vector field on S along γ* is a (smooth) map $X : I \to \mathbb{R}^3$ such that $X(t) \in T_{\gamma(t)}S$ for all $t \in I$.

Examples 4.8 1. If Y is a tangent vector field on S and $\gamma : I \to S$ is a parametrized curve, then $X(t) = Y(\gamma(t))$ defines a tangent vector field on S along γ.

2. If $\gamma : I \to S$ is a smooth parametrized curve, then $X = \gamma'$ defines a tangent vector field on S along γ.

The *covariant derivative* of a tangent vector field X on S along γ is defined to be

$$\frac{\nabla X}{dt}\Big|_t = \left(\frac{d}{dt}X(t)\right)^{\top}.$$

In example (1) above,

$$\begin{aligned}\frac{\nabla X}{dt}\Big|_t &= \left(\frac{d}{dt}Y(\gamma(t))\right)^{\top}\\ &= \nabla_{\gamma'(t)}Y|_{\gamma(t)}.\end{aligned}$$

In example (2) above,

$$\frac{\nabla X}{dt}\Big|_t = (\gamma''(t))^{\top}.$$

A tangent vector field X on S along γ is said to be *parallel* if $\frac{\nabla X}{dt} \equiv 0$. A smooth parametrized curve $\gamma : I \to S$ is said to be a *geodesic* if γ' is parallel along γ.

In the sequel, we write a local expression of the equation $\frac{\nabla X}{dt} \equiv 0$. In the image of a parametrization φ of S, we can write

$$\gamma(t) = \varphi(u^1(t), u^2(t)), \qquad X(t) = \sum_{i=1}^{2} a^i(t)\frac{\partial\varphi}{\partial u^i}\Big|_{(u^1(t),u^2(t))}$$

where $t \in I$. Then

$$X'(t) = \sum_{i=1}^{2}(a^i)'\frac{\partial\varphi}{\partial u^i} + \sum_{i,j=1}^{2} a^i\frac{\partial^2\varphi}{\partial u^i\partial u^j}(u^j)'$$

and

$$\begin{aligned}\frac{\nabla X}{dt} &= (X'(t))^{\top}\\ &= \sum_{i=k}^{2}(a^i)'\frac{\partial\varphi}{\partial u^i} + \sum_{i,j=1}^{2} a^i(u^j)'\nabla_{\frac{\partial\varphi}{\partial u^i}}\frac{\partial\varphi}{\partial u^j}\\ &= \sum_{i=k}^{2}(a^k)'\frac{\partial\varphi}{\partial u^k} + \sum_{i,j,k=1}^{2} a^i(u^j)'\Gamma^k_{ij}\frac{\partial\varphi}{\partial u^k}\\ &= \sum_{k=1}^{2}\left[(a^k)' + \sum_{i,j=1}^{2}\Gamma^k_{ij}(u^i)'a^j\right]\frac{\partial\varphi}{\partial u^k}.\end{aligned}$$

Hence $\frac{\nabla X}{dt} \equiv 0$ is the following system of first order linear ordinary differential equations in $a^1(t)$, $a^2(t)$:

$$(a^k)'(t) + \sum_{i,j=1}^{2}\Gamma^k_{ij}(u^1(t), u^2(t))\,(u^i)'(t)\,a^j(t) = 0 \qquad (k = 1, 2). \tag{4.9}$$

The theorem on existence and uniqueness of solution of linear ordinary differential equations says that, given the initial values $a^1(t_0) = a_0^1$, $a^2(t_0) = a_0^2$, for some $t_0 \in I$, there exists a unique solution $(a^1(t), a^2(t))$ defined on I and satisfying the given initial values. Geometrically this means that, given $v \in T_{\gamma(t_0)}S$ and taking a parametrization $\varphi : U \to S$ around $\gamma(t_0)$ with $v = a_0^1 \frac{\partial \varphi}{\partial u^1} + a_0^2 \frac{\partial \varphi}{\partial u^2}$, the vector field $X(t) = a^1(t)\frac{\partial \varphi}{\partial u^1} + a^2(t)\frac{\partial \varphi}{\partial u^2}$ is the only parallel vector field along γ such that $X(t_0) = v$, defined on an interval centered at t_0 and lying in $\gamma^{-1}(\varphi(U))$. By covering the image of γ by finitely many opens sets $V_1, \ldots, V_n$, each of which the image of a parametrization, such that $V_i \cap V_{i+1} \neq \varnothing$, and applying this result to each one of $V_1, \ldots, V_n$ in order, we deduce

Proposition 4.10 *Given a smooth parametrized curve $\gamma : [a, b] \to S$ and a tangent vector $v \in T_{\gamma(a)}S$, there exists a unique tangent vector field $X : [a, b] \to \mathbb{R}^3$ on S along γ which is parallel and satisfies $X(a) = v$.*

The vector $X(b) \in T_{\gamma(b)}S$ is called the *parallel transport* of $v = X(a)$ along γ. The parallel transport along γ defines a map $P^\gamma : T_{\gamma(a)}S \to T_{\gamma(b)}S$ which is obviously linear, since the solutions to a linear ODE depend linearly on the initial values.

Proposition 4.11 *If X, Y are parallel vector fields along γ, then $\langle X(t), Y(t)\rangle$, $||X(t)||$ and the angle between $X(t)$ and $Y(t)$ are constant functions.*

Proof. We compute

$$\begin{aligned}
\frac{d}{dt}\langle X(t), Y(t)\rangle &= \langle X'(t), Y(t)\rangle + \langle X(t), Y'(t)\rangle \\
&= \langle \frac{\nabla X}{dt}, Y(t)\rangle + \langle X(t), \frac{\nabla Y}{dt}\rangle \qquad \text{(since } X, Y \text{ are tangent)} \\
&= 0.
\end{aligned}$$

Hence $\langle X(t), Y(t)\rangle$ is constant, and the other assertions follow. □

Corollary 4.12 *$P^\gamma : T_{\gamma(a)}S \to T_{\gamma(b)}S$ is a linear isometry.*

Examples 4.13 1. For the plane, $\varphi(u, v) = (u, v, 0)$ is a parametrization and $I = du^2 + dv^2$. Since the coefficients of I are constant, eqn. (4.6) yields that $\Gamma_{ij}^k = 0$ for all i, j, k. The equations of parallel transport are thus $(a^k)' = 0$, $k = 1, 2$. Hence the parallel vector fields along γ are the constant vector fields. In particular, the parallel transport along γ depends on the endpoints of γ, but not on the curve itself.

2. We consider the cone C of equation $z = k\sqrt{x^2 + y^2}$ for $k > 0$ and $(x, y) \neq 0$. It is obviously a graph of a smooth function, so it is a surface. We can also parametrize it as a surface of revolution by taking the generating curve to be $\gamma(s) = (f(s), 0, g(s))$, where $f(s) = \frac{1}{\sqrt{k^2+1}}s$, $g(s) = \frac{k}{\sqrt{k^2+1}}s$. Then the Gaussian curvature $K = -f''/f = 0$.

In the sequel, we show that the cone is locally isometric to the plane. Note that the angle at the vertex of the cone is $\psi = \operatorname{arccot} k \in (0, \pi/2)$. Consider the open sector of the plane V given in polar coordinates by $r > 0, 0 < \theta < 2\pi \sin\psi$. We define a map

$$\Phi : V \to C, \qquad F(r,\theta) = \left(r\cos\left(\frac{\theta}{\sin\psi}\right)\sin\psi, r\sin\left(\frac{\theta}{\sin\psi}\right)\sin\psi, r\cos\psi \right).$$

Then Φ is smooth and its image $\Phi(V)$ is the cone minus the geratrix $y = 0$, $x > 0$, $z = kx$. The inverse map

$$\Phi^{-1} : \Phi(V) \to V, \qquad \Phi(x,y,z) = \left(\frac{1}{\sin\psi}\sqrt{x^2+y^2}, \sin\psi \operatorname{arccot}\left(\frac{x}{y}\right)\right)$$

is also smooth, since it is smooth as a function of (x, y) for $y \neq 0$. Hence $\Phi : V \to \Phi(V)$ is a diffeomorphism. We finally show that Φ is an isometry, namely, the first fundamental forms of the plane and the cone coincide on points corresponding under Φ.

The open set V is a regular surface parametrized by $\varphi(u,v) = (u\cos v, u\sin v, 0)$, where $(u,v) \in U = (0,+\infty) \times (0, 2\pi\sin\psi)$, and the corresponding coefficients of the first fundamental form are then $E = 1$, $F = 0$, $G = u^2$. Since Φ is a diffeomorphism, $\tilde\varphi = \Phi \circ \varphi$ can be taken as a parametrization of $\Phi(V)$ and then the corresponding coeffcients of the first fundamental form are $\tilde E = 1$, $\tilde F = 0$, $\tilde G = u^2$. Since $E = \tilde E$, $F = \tilde F$, $G = \tilde G$, Φ is an isometry.

From the local expression (4.9), we see that parallel transport is an intrinsic object. Hence the parallel transport along a curve in the cone can be read off the parallel transport along the corresponding curve in the plane. Consider a parallel curve γ in the cone given by $z = z_0$; we compute the parallel transport of its initial tangent vector v after one turn around the cone. The corresponding curve $\tilde\gamma$ in the plane is an arc of a circle of angle $2\pi\sin\psi$. The tangent vector to $\tilde\gamma$ rotates by an angle of measure $2\pi\sin\psi$ after one turn, whereas the parallel transport is the identity. It follows that the parallel transport of v along γ is rotation by $-2\pi\sin\psi$.

3. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. We describe parallel transport around a small circle γ of colatitude φ. There exists a cone which is tangent to the sphere along γ; the angle at the vertex of this cone is $\psi = \frac{\pi}{2} - \varphi$. Since the tangent spaces of the sphere and the cone coincide along γ, also parallel transport along γ is the same whether we view it as a curve in the sphere or in the cone. Therefore parallel transport along the small circle of colatitude φ is rotation of angle $-2\pi\cos\varphi$ (with respect to a suitable orientation). Taking $\varphi \to 0$, by continuity we see that parallel transport along the equator (after one complete turn) is the identity.

4.7 Geodesics

As we have already mentioned, a smooth parametrized curve $\gamma : I \to S$ is a *geodesic* if γ' is parallel along γ. This means that $0 = \frac{\nabla(\gamma')}{dt} = (\gamma'')^\top$, so the acceleration γ'' in $\mathbb{R}^3$ is everywhere normal to S. In other words, γ does not accelerate viewed from S, so that "geodsics are the straightest curves in S".

If γ is geodesic, then $||\gamma'||$ is constant by Proposition 4.11. There are two cases: either $||\gamma'|| = 0$ and γ is a constant curve; or $||\gamma'||$ is a nonzero constant and γ is regular and parametrized proportionally to arc-length.

Equations for geodesics contained in the image of a parametrization φ are immediately deduced from the equations for parallel vector fields (4.9); if $\gamma(t) = \varphi(u^1(t), u^2(t))$, we take $(a^i)' = (u^i)'$ and then

$$(u^k)'' + \sum_{i,j=1}^{2} \Gamma^k_{ij}(u^i)'(u^j)' = 0, \qquad (k = 1, 2). \tag{4.14}$$

This a system of second order non-linear ordinary differential equations. Those equations show that geodesics are intrinsic objects. On the other hand, the nonlinearity implies that in general geodesics are only defined locally. Namely, the theorem of existence and uniqueness for such equations is local in nature, so we have

Proposition 4.15 *Given $p \in S$ and $v \in T_pS$, there exist $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Examples 4.16 1. For the plane, $\Gamma^k_{ij} \equiv 0$, so the geodesic equations are $(u^1)'' = (u^2)'' = 0$. Hence the geodesics are the straight lines.

2. In the sphere S^2, let $p \in S^2$ and $v \in T_pS^2 = (\mathbb{R}v)^\top$, $v \neq 0$, and consider the great circle

$$\gamma(t) = \cos(t||v||)p + \sin(t||v||)\frac{v}{||v||}.$$

Then $\gamma(0) = p$, $\gamma'(0) = v$ and $\gamma''(t) = -||v||^2\gamma(t) \perp T_{\gamma(t)}S^2$, so great circles are geodesics. Since there exist great circles thorugh any point with any speed, by the uniqueness part of Proposition 4.15, the great circles are all the geodesics.

3. The cilinder $x^2 + y^2 = 1$ is locally isometric to the plane. In fact, $\varphi(u, v) = (\cos v, \sin v, u)$ is a local isometry, since by restricting to $(u, v) \in (u_0, u_0 + 2\pi) \times \mathbb{R}$ it becomes a parametrization with $E = 1$, $F = 0$, $G = 1$. It follows that the geodesics of the cilinder are images of the geodesics of the plane under φ. In particular, the geodesics through $\varphi(0, 0)$ are of the form

$$t \mapsto (\cos(at), \sin(at), bt),$$

where a, $b \in \mathbb{R}$. Note that we get horizontal circles ($a \neq 0$, $b = 0$), vertical lines ($a = 0$, $b \neq 0$) and helices (otherwise).

4. Consider a surface of revolution parametrized as in section 3.5. For a curve v-constant, $\eta(t) = \varphi(t, v_0)$, we have $\eta' = \frac{\partial\varphi}{\partial u}$, so by the computations in example 4.7,

$$\frac{\nabla\eta'}{dt} = \nabla_{\eta'}\eta' = \nabla_{\varphi_u}\varphi_u = 0.$$

Hence meridians are always geodesics. Similarly, for the curves u-constant, $u = u_0$, we have $\nabla_{\varphi_v}\varphi_v|_{u=u_0} = -f(u_0)f'(u_0)\varphi_u$ and $f > 0$, so precisely the parallels corresponding to critical points of f are geodesics.

More generally, using the Christofell symbols computed in the quoted example, we can write the geodesic equations as

$$\begin{aligned} u'' - f(u)f'(u)(v')^2 &= 0 \\ v'' + 2\frac{f'(u)}{f(u)}u'v' &= 0 \end{aligned}$$

Note that

$$\begin{aligned} \left(f^2(u)v'\right)' &= 2f(u)f'(u)u'v' + f^2(u)v'' \\ &= f^2(u)\left(v'' + 2\frac{f'(u)}{f(u)}u'v'\right) \\ &= 0 \end{aligned}$$

along a solution curve, so $f^2(u)v'$ is constant along a geodesic (this function is a first integral of the system).

As an application, we consider a geodesic $\gamma(t) = (u(t), v(t))$ and, for each t, the parallel $\zeta(r) = \varphi(u(t), r)$ which crosses $\gamma(t)$ at $r = v(t)$, and compute the inner product

$$\langle\gamma', \zeta'\rangle = \langle u'\tfrac{\partial\varphi}{\partial u} + v'\tfrac{\partial\varphi}{\partial v}, \tfrac{\partial\varphi}{\partial v}\rangle = u'F + v'G = f^2(u)v'$$

to be constant with respect to t. On the other hand,

$$\langle\gamma', \zeta'\rangle = ||\gamma'|| ||\zeta'|| \cos\theta = ||\gamma'|| f(u)\cos\theta,$$

where $||\gamma'||$ is constant and $\theta(t)$ is the angle between $\gamma'(t)$ and $\zeta'(v(t)) = \frac{\partial\varphi}{\partial v}(u(t), v(t))$, and $f(u(t))$ is the radius of the parallel ζ. Hence the cosine of the angle at which γ meets a parallel multiplied by the radius of that parallel is constant; this is known as *Clairaut's relation*.

In fact, the first integral allows to explicitly integrate the geodesic equations. For the sake of simplicity, assume the geodesic γ is parametrized by arc-length; together with Clairaut's relation, this gives the system

$$\begin{aligned} (u')^2 + f^2(v')^2 &= 1 \\ f^2v' &= c \end{aligned}$$

where c is a constant; by changing the orientation of γ if necessary we may assume that $c > 0$. If γ is not a meridian, v' is never zero and we can use v as a parameter along γ, so that $u = u(v)$. We have

$$\begin{aligned}
\left(\frac{du}{dv}\right)^2 &= \frac{(u')^2}{(v')^2} \\
&= \frac{1 - f^2(v')^2}{(v')^2} \\
&= \frac{1}{(v'^2)} - f^2 \\
&= \frac{f^4}{c^2} - f^2 \\
&= \frac{f^2}{c^2}(f^2 - c^2)
\end{aligned}$$

It follows that

$$f \geq c$$

and, if γ is not a parallel, $\frac{du}{dv} \neq 0$ and $f > c$ so that

$$v = c \int \frac{1}{f(u)\sqrt{f(u)^2 - c^2}}\, du.$$

4.8 The integrability equations and the Theorema Egregium

Our next goal is to investigate how true is the fact that the first and second fundamental forms locally determine a surface. We will start by looking at necessary conditions for I, II to correspond to a surface.

Suppose $\varphi : U \to S$ is a regular parametrized surface defined on an open set $U \subset \mathbb{R}^2$, and let $I = (g_{ij})$, $II = (h_{ij})$ be its fundamental forms. Recall the Gauss formula

$$D_X Y = \nabla_X Y + II(X, Y)\nu \tag{4.17}$$

and the Weingarten equation

$$D_X \nu = -A(X), \tag{4.18}$$

where

$$I(AX, Y) = II(X, Y).$$

Writing the Gauss formula in terms of the parametrization, we get

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial u^i \partial u^j} &= D_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} \\
&= \nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} + II\left(\frac{\partial \varphi}{\partial u^i}, \frac{\partial \varphi}{\partial u^j}\right)\nu \\
&= \sum_k \Gamma^k_{ij} \frac{\partial \varphi}{\partial u^k} + h_{ij}\nu.
\end{aligned} \tag{4.19}$$

Doing the same with the Weingarten equation,

$$\begin{aligned}
\frac{\partial \nu}{\partial u^i} &= D_{\frac{\partial \varphi}{\partial u^i}} \nu \\
&= -A\left(\frac{\partial \varphi}{\partial u^i}\right) \\
&= -\sum_j h_i^j \frac{\partial \varphi}{\partial u^j} \\
&= -\sum_{j,k} h_{ik} g^{kj} \frac{\partial \varphi}{\partial u^j}. \qquad (4.20)
\end{aligned}$$

We would like to integrate the equations (4.19) and (4.20) in terms of φ. There are obvious neccesaary conditions for that, namely

$$\begin{aligned}
0 &= \frac{\partial^3 \varphi}{\partial u^k \partial u^i \partial u^j} - \frac{\partial^3 \varphi}{\partial u^j \partial u^i \partial u^k} \\
&= \frac{\partial}{\partial u^k}\left(\sum_s \Gamma_{ij}^s \frac{\partial \varphi}{\partial u^s} + h_{ij}\nu\right) - \frac{\partial}{\partial u^j}\left(\sum_r \Gamma_{ik}^r \frac{\partial \varphi}{\partial u^r} + h_{ik}\nu\right) \\
&= \sum_s \left(\frac{\partial \Gamma_{ij}^s}{\partial u^k}\frac{\partial \varphi}{\partial u^s} + \Gamma_{ij}^s \frac{\partial^2 \varphi}{\partial u^k \partial u^s}\right) + \frac{\partial h_{ij}}{\partial u^k}\nu + h_{ij}\frac{\partial \nu}{\partial u^k} \\
&\quad -\sum_r \left(\frac{\partial \Gamma_{ik}^r}{\partial u^j}\frac{\partial \varphi}{\partial u^r} + \Gamma_{ik}^r \frac{\partial^2 \varphi}{\partial u^j \partial u^r}\right) - \frac{\partial h_{ik}}{\partial u^j}\nu + h_{ik}\frac{\partial \nu}{\partial u^j} \\
&= \sum_s \left(\frac{\partial \Gamma_{ij}^s}{\partial u^k} - \frac{\partial \Gamma_{ik}^s}{\partial u^j}\right)\frac{\partial \varphi}{\partial u^s} \\
&\quad + \sum_r \Gamma_{ij}^r \left(\sum_s \Gamma_{kr}^s \frac{\partial \varphi}{\partial u^s} + h_{kr}\nu\right) - \sum_r \Gamma_{ik}^r \left(\sum_s \Gamma_{jr}^s \frac{\partial \varphi}{\partial u^s} + h_{jr}\nu\right) \\
&\quad + \left(\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j}\right)\nu - h_{ij}\sum_{m,s} h_{km} g^{ms}\frac{\partial \varphi}{\partial u^s} + h_{ik}\sum_{ms} h_{jm} g^{ms}\frac{\partial \varphi}{\partial u^s}.
\end{aligned}$$

Considering separately the tangential and normal components, we respectively get the *Gauss equation*

$$\begin{aligned}
&\frac{\partial \Gamma_{ij}^s}{\partial u^k} - \frac{\partial \Gamma_{ik}^s}{\partial u^j} + \sum_r \left(\Gamma_{ij}^r \Gamma_{rk}^s - \Gamma_{ik}^r \Gamma_{rj}^s\right) \\
&\qquad = \sum_m (h_{ij}h_{km} - h_{ik}h_{jm}) g^{ms} \quad \text{for all } i, j, k, s \qquad (4.21)
\end{aligned}$$

and the *Codazzi-Mainard equation*

$$\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j} + \sum_r \left(\Gamma_{ij}^r h_{rk} - \Gamma_{ik}^r h_{rj}\right) = 0 \quad \text{for all } i, j, k. \qquad (4.22)$$

Taken together, they are known as the *integrability* (or *compatibility*) equations.

As a by-product of our computation, we derive the following consequence of the integrability equations. Put $i = j = 1$, $k = 2$ in the Gauss eqn. (4.21), multiply through by g_{s2}, and sum over s to get

$$\begin{aligned}
\sum_s \left(\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^s_{12}}{\partial u^1} \right) g_{s2} &+ \sum_{r,s} \left(\Gamma^r_{11}\Gamma^s_{r2} - \Gamma^r_{12}\Gamma^s_{r1} \right) g_{s2} \\
&= \sum_{m,s} (h_{11}h_{2m} - h_{12}h_{1m}) g^{ms} g_{s2} \\
&= h_{11}h_{22} - (h_{12})^2 \\
&= \det(II).
\end{aligned}$$

The left-hand side of this equation involves only the g_{ij}'s and the Γ^k_{ij}'s, and we already know that the Christofell symbols are completely determined by the g_{ij}'s. Hence $\det(II)$ depends only on I. Recalling that $K = \det(II)/\det(I)$, we finally get the **Theorema Egregium**.

Theorem 4.23 (Gauss, 1826) *The Gaussian curvature of a surface is an intrinsic invariant of the surface.*

In other words, locally isometric surfaces have the same Gaussian curvature at corresponding points.

Remark 4.24 The expression

$$\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^s_{12}}{\partial u^1} + \sum_r \left(\Gamma^r_{11}\Gamma^s_{r2} - \Gamma^r_{12}\Gamma^s_{r1} \right) =: R^s_{121}$$

is a component of the so called Riemann curvature tensor, and

$$\sum_s R^s_{121} g_{s2} = R_{1212},$$

so the Theorema Egregium can be restated in the form

$$K = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}.$$

4.9 A very quick digression on systems of first order partial differential equations

In constrast to the case of *ordinary* differential equations, systems of PDE's do not always have solutions. We start with a simple example. Consider a vector field $\vec{X} = P\vec{i} + Q\vec{j}$ defined on an open set $U \subset \mathbb{R}^2$, where P, $Q : U \to \mathbb{R}^2$ are smooth functions. Finding a potencial for $\vec{X}$, i.e. a smooth scalar function f on U such that $\operatorname{grad} f = \vec{X}$, is equivalent to solving the system

$$\begin{aligned} \frac{\partial f}{\partial x} &= P(x,y) \\ \frac{\partial f}{\partial y} &= Q(x,y). \end{aligned}$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, a necessary condition for the existence of solutions is that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

namely, the rotational $\operatorname{rot} \vec{X} = \vec{0}$. It is also known that if U is a rectangle, or star-shaped, or even simply-connected, then this condition is also sufficient.

A more general system has the form

$$\begin{aligned} \frac{\partial z}{\partial x} &= P(x,y,z) \\ \frac{\partial z}{\partial y} &= Q(x,y,z). \end{aligned}$$

In this case, the necessary condition is easily seen to be

$$\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} Q = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} P.$$

The general case is dealt with the following theorem, for whose proof the reader is referred to app. B in J. J. Stoker, *Differential Geometry*, Wiley Interscience, 1969.

Theorem 4.25 (Frobenius, 1877) *Consider the first-order system of partial differential equations*

$$\frac{\partial y^i}{\partial x^j} = P^i_j(x^1, \ldots, x^m; y^1, \ldots, y^n),$$

where $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *and* P^i_j *admit continuous derivatives of second order in all arguments. Suppose that the* P^j_i *satisfy the* integrability conditions

$$\frac{\partial P^i_j}{\partial x^k} + \sum_\ell \frac{\partial P^i_j}{\partial y^\ell} P^k_\ell = \frac{\partial P^i_k}{\partial x^j} + \sum_\ell \frac{\partial P^i_k}{\partial y^\ell} P^j_\ell.$$

Then there exist a unique solution satisfying the initial conditions

$$y^i(x^1_0, \ldots, x^m_0) = y^i_0$$

for $1 \leq i \leq n$.

4.10 The fundamental theorem of the local theory of surfaces

The first result asserts the invariance of the fundamental forms of a surface under orientation-preserving rigid motions of Euclidean space, and contains the uniqueness part of Theorem 4.29.

Lemma 4.26 *Let $\varphi : U \to S$ be a regular parametrized surface, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation-preserving rigid motion, i.e. $T(x) = A(x) + b$ where $A \in SO(3)$ and $b \in \mathbb{R}^3$. Put $\tilde{\varphi} = T \circ \varphi$. Then $\tilde{\varphi} : U \to \tilde{S}$ is a regular parametrized surface and the coefficients of the fundamental forms $\tilde{g}_{ij} = g_{ij}$, $\tilde{h}_{ij} = h_{ij}$. Conversely, given two regular parametrized surfaces $\varphi : U \to S$ and $\tilde{\varphi} : U \to \tilde{S}$, where U is connected, satisfying $g_{ij} = \tilde{g}_{ij}$, $h_{ij} = \tilde{h}_{ij}$, there exists an orientation-preserving rigid motion T of $\mathbb{R}^3$ such that $\tilde{\varphi} = T \circ \varphi$.*

Proof. Differentiate $\tilde{\varphi} = A\varphi + b$; owing to the constancy of A, b, we get $\frac{\partial \tilde{\varphi}}{\partial u^i} = A\left(\frac{\partial \varphi}{\partial u^i}\right)$. Using this and the fact that A is orientation-preserving,

$$\tilde{\nu} = \frac{\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}}{||\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}||} = \frac{A\left(\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}\right)}{||A\left(\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}\right)||} = A(\nu).$$

Since A is ortogonal, we immediately see that $\tilde{g}_{ij} = g_{ij}$, $\tilde{h}_{ij} = h_{ij}$.

For the second part, define $A(u) : \mathbb{R}^3 \to \mathbb{R}^3$ for $u \in U$ by setting

$$A(u)\left(\frac{\partial \varphi}{\partial u^i}\Big|_u\right) = \frac{\partial \tilde{\varphi}}{\partial u^i}\Big|_u \quad (i = 1, 2), \qquad A(u)(\nu(u)) = \tilde{\nu}(u).$$

Plainly, $A(u) \in SO(3)$; we next show that A is constant. On one hand,

$$\frac{\partial^2 \tilde{\varphi}}{\partial u^i \partial u^j} = \frac{\partial}{\partial u^i}\left(A\frac{\partial \varphi}{\partial u^j}\right) = \frac{\partial A}{\partial u^i}\frac{\partial \varphi}{\partial u^j} + A\left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j}\right).$$

On the other hand, by the Gauss formula (4.19),

$$\frac{\partial^2 \tilde{\varphi}}{\partial u^i \partial u^j} = \sum_k \tilde{\Gamma}^k_{ij}\frac{\partial \tilde{\varphi}}{\partial u^k} + \tilde{h}_{ij}\tilde{\nu} = \sum_k \Gamma^k_{ij} A\left(\frac{\partial \varphi}{\partial u^k}\right) + h_{ij}A(\nu) = A\left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j}\right).$$

Putting this together yields

$$\frac{\partial A}{\partial u^i}\frac{\partial \varphi}{\partial u^j} = 0. \tag{4.27}$$

Similarly,

$$\frac{\partial \tilde{\nu}}{\partial u^i} = \frac{\partial A}{\partial u^i}\nu + A\left(\frac{\partial \nu}{\partial u^i}\right),$$

and the Weingarten equation

$$\frac{\partial \tilde{\nu}}{\partial u^i} = -\sum_{j,k} \tilde{h}_{ik} \tilde{g}^{kj} \frac{\partial \tilde{\varphi}}{\partial u^j} = -\sum_{j,k} h_{ik} g^{kj} A \left(\frac{\partial \varphi}{\partial u^j} \right) = A \left(\frac{\partial \nu}{\partial u^i} \right)$$

together imply that

$$\frac{\partial A}{\partial u^i} \nu = 0. \tag{4.28}$$

From (4.27) and (4.28), we get that $\frac{\partial A}{\partial u^i} = 0$ on U and hence A is constant. Finally, also $\tilde{\varphi} - A \circ \varphi$ is a constant b, for $\frac{\partial}{\partial u^i}(\tilde{\varphi} - A \circ \varphi) = \frac{\partial \tilde{\varphi}}{\partial u^i} - A(\frac{\partial \varphi}{\partial u^i}) = 0$, which finishes the proof. □

Theorem 4.29 (Bonnet, 1867) *Let be given smooth functions g_{ij}, h_{ij} defined on an open set $U \subset \mathbb{R}^2$ ($1 \leq i, j \leq 2$) such that $g_{ij} = g_{ji}$, $h_{ij} = h_{ji}$ and the matrix (g_{ij}) is positive-definite. Suppose that g_{ij}, h_{ij} satisfy the equations of Gauss (4.21) and Codazzi-Mainardi (4.22). Then, for given initial conditions*

$$u_0 \in U,\ p_0 \in \mathbb{R}^3,\ X_{1,0},\ X_{2,0},\ \nu_0 \in \mathbb{R}^3$$

with ν_0 a unit vector and $\langle X_{i,0}, X_{j,0} \rangle = g_{ij}(u_0)$, there exists an open neighborhood V of u_0 contained in U and a unique regular parametrized surface $\varphi : V \to \mathbb{R}^3$ whose Gauss map is ν with the following properties:

1. *$\varphi(u_0) = p_0$;*
2. *$\frac{\partial \varphi}{\partial u^i}(u_0) = X_{i,0}$;*
3. *$\nu(u_0) = \nu_0$;*
4. *(g_{ij}) and (h_{ij}) are the fundamental forms of φ.*

Proof. We introduce new vector-valued variables X_1, X_2

$$\frac{\partial \varphi}{\partial u^j} = X_j \tag{4.30}$$

and write the Gauss formula and the Weingarten equation in the form of a first-order system of PDE's in X_1, X_2, ν:

$$\begin{aligned} \frac{\partial X_j}{\partial u^i} &= \sum_k \Gamma_{ij}^k X_k + h_{ij} \nu \\ \frac{\partial \nu}{\partial u^i} &= -\sum_j h_{ij} g^{jk} X_k. \end{aligned} \tag{4.31}$$

We first solve (4.31): the integrability conditions of Theorem 4.25 are exactly the equations of Gauss and Codazzi-Mainardi, which are satisfied by assumption, so there exists a unique solution (X_1, X_2, ν) defined on a neighborhood of u_0

and satisfying the initial conditions $(X_{1,0}, X_{2,0}, \nu_0)$ at u_0. Let us check that the solutions satisfy

$$\langle \nu, \nu \rangle = 1, \ \langle \nu, X_i \rangle = 0, \ \langle X_i, X_j \rangle = g_{ij} \tag{4.32}$$

on a neighborhood of u_0. We differentiate the left-hand sides of these equations to get

$$\begin{aligned}
\frac{\partial}{\partial u^i}\langle \nu, \nu \rangle &= 2\langle \frac{\partial \nu}{\partial u^i}, \nu \rangle = -2\sum_{k,l} h_{ik} g^{kl} \langle \nu, X_l \rangle \\
\frac{\partial}{\partial u^i}\langle \nu, X_j \rangle &= \langle \frac{\partial \nu}{\partial u^i}, X_j \rangle + \langle \nu, \frac{\partial X_j}{\partial u^i} \rangle \\
&= -\sum_{k,l} h_{ik} g^{kl} \langle X_l, X_j \rangle + \sum_k \Gamma^k_{ij} \langle \nu, X_k \rangle + h_{ij} \langle \nu, \nu \rangle \\
\frac{\partial}{\partial u^k}\langle X_i, X_j \rangle &= \langle \frac{\partial X_i}{\partial u^i}, X_j \rangle + \langle X_i, \frac{\partial X_j}{\partial u^i} \rangle \\
&= \sum_r \Gamma^r_{ik} \langle X_r, X_j \rangle + h_{ik} \langle \nu, X_j \rangle \\
&\qquad + \sum_s \Gamma^s_{jk} \langle X_s, X_i \rangle + h_{jk} \langle \nu, X_i \rangle.
\end{aligned}$$

These identities show that the functions $\langle \nu, \nu \rangle$, $\langle \nu, X_i \rangle$, $\langle X_i, X_j \rangle$ satisfy a system of PDE's in U. It is easy to check that the functions $1, 0, g_{ij}$ satisfy the same system of PDE's. (Do it!) Since the values of the two triples of functions coincide at the point u_0, by the uniqueness part of Theorem 4.25, the equations (4.32) are satisfied on a neighborhood of u_0.

The final step is to solve (4.30) in φ. The integrability conditions $\frac{\partial X_j}{\partial u^i} = \frac{\partial X_i}{\partial u^j}$ are satisfied because $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $h_{ij} = h_{ji}$. Therefore there exists a smaller neighborhood V of u_0 and a unique solution $\varphi : V \to \mathbb{R}^3$ with $\varphi(u_0) = p_0$. Since $\langle \nu, \nu \rangle = 1$, $\langle \nu, \frac{\partial \varphi}{\partial u^i} \rangle = \langle \nu, X_i \rangle = 0$, ν is a unit normal vector field along φ. Moreover, the fundamental forms of φ are

$$\langle \tfrac{\partial \varphi}{\partial u^i}, \tfrac{\partial \varphi}{\partial u^j} \rangle = \langle X_i, X_j \rangle = g_{ij}$$

and

$$\left\langle \frac{\partial^2 \varphi}{\partial u^i \partial u^j}, \nu \right\rangle = \left\langle \sum_k \Gamma^k_{ij} \frac{\partial^\varphi}{\partial u^k} + h_{ij}\nu, \nu \right\rangle = h_{ij},$$

as we wished. As a final remark, note that

$$\nu = \pm \frac{\frac{\partial \varphi}{\partial u^1} \times \frac{\partial \varphi}{\partial u^2}}{|| \frac{\partial \varphi}{\partial u^1} \times \frac{\partial \varphi}{\partial u^2} ||}$$

where the sign is "+" or "−" according to whether $\{X_{1,0}, X_{2,0}, \nu_0\}$ is a positive basis of $\mathbb{R}^3$ or not. □

4.11 Differential forms

Let U be an open set of $\mathbb{R}^n$. A *differential* 1-*form* (or a *differential form of degree* 1) on U is a map

$$\omega : p \in U \to (\mathbb{R}^n)^*;$$

ω is said to be smooth if, for all i, the function $\omega(e_i) : U \to \mathbb{R}$ given by $\omega(e_i)(p) = \omega_p(e_i)$ is smooth, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Suppose $\{X_1, \ldots, X_n\}$ is a smooth orthonormal frame on U, that is, the X_i are smooth vector fields and $\{X_1(p), \ldots, X_n(p)\}$ is an orthonormal basis for $p \in U$. Then we can consider the *dual coframe* $\{\omega^1, \ldots, \omega^n\}$ by specifying $\omega^i(X_j) = \delta^i_j$ (Kronecker delta). For instance, the dual coframe to the canonical basis of $\mathbb{R}^n$ is usually denoted $\{dx^1, \ldots, dx^n\}$. Of course, dx^i is just the linear projection onto the x^i-axis of $\mathbb{R}^n$.

Example 4.33 On $U = \mathbb{R}^2 \setminus \{(0,0)\}$, consider the orthonormal frame

$$\begin{aligned} X_1 &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}} e_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} e_2, \\ X_2 &= \frac{-x_2}{\sqrt{x_1^2 + x_2^2}} e_1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} e_2. \end{aligned}$$

Then the dual coframe has

$$\begin{aligned} \omega^1 &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\, dx^1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\, dx^2, \\ \omega^2 &= \frac{-x_2}{\sqrt{x_1^2 + x_2^2}}\, dx^1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\, dx^2. \end{aligned}$$

We will also consider differential forms of degree 2. A *differential* 2-*form* on $U \subset \mathbb{R}^n$ is a map Ω that takes $p \in U$ to a skew-symmetric bilinear map

$$\Omega_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R};$$

Ω is said to be smooth if $\Omega(e_i, e_j) : U \to \mathbb{R}$ is a smooth function for all i, j. There are two important ways of manufacturing 2-forms starting with 1-forms.

The first one is the exterior product. If ω, η are 1-forms on U, their *exterior product* is defined to be the 2-form $\omega \wedge \eta$ given by

$$(\omega \wedge \eta)_p(u, v) = \omega_p(u)\eta_p(v) - \omega_p(v)\eta_p(u),$$

where $p \in U$ and u, $v \in \mathbb{R}^n$. It is immediate to see that $(\omega \wedge \eta)_p$ is skew-symmetric and bilinear. It is also easy to see that $\omega \wedge \eta$ is smooth if ω, η are smooth. For future use, we note the following properties:

1. $\omega \wedge \eta = -\eta \wedge \omega$;

2. $\omega \wedge \omega = 0$.

The second way of constructing 2-forms from 1-forms is exterior derivation. If ω is a 1-from on U, we can write $\omega = \sum_{i=1}^n a^i\,dx_i$, where $a_i : U \to \mathbb{R}$ are smooth functions. The *exterior derivative* of ω is the 2-form

$$\begin{aligned} d\omega &= \sum_{i=1}^n da^i \wedge dx_i \\ &= \sum_{i,j=1}^n \tfrac{\partial a^i}{\partial x^j} dx^j \wedge dx^i. \end{aligned}$$

Example 4.34 Referring to Example 4.33, we have

$$d\omega^1 = 0 \quad \text{and} \quad d\omega^2 = \frac{1}{\sqrt{x_1^2 + x_2^2}}\, dx_1 \wedge dx_2.$$

The next lemma shows how to compute $d\omega$ without invoking coordinates. However, note that on left-hand side of (4.36), X and Y need to be *vector fields*, whereas $(d\omega)_p(X, Y)$ makes sense even if X, Y are just vectors.

Lemma 4.35 *If ω is a smooth 1-form on an open set U of $\mathbb{R}^n$, and X, Y are smooth vector fields on U, then*

$$d\omega(X, Y) = D_X(\omega(Y)) - D_Y(\omega(X)) - \omega([X, Y]). \tag{4.36}$$

Proof. Write $\omega = \sum_i a^i\,dx_i$. Since both hand sides of (4.36) are linear in ω, we may assume that $\omega = a\,dx_i$, where $a : U \to \mathbb{R}$ is smooth. We write $X = \sum_i X^i e_i$, $Y = \sum_i Y^i e_i$ and compute

$$\begin{aligned} &D_X(\omega(Y)) - D_Y(\omega(X)) - \omega([X, Y]) \\ &\quad= D_X(a\,dx^i(Y)) - D_Y(a\,dx^i(X)) - a\,dx^i([X, Y]) \\ &\quad= (D_X a)Y^i + aD_X Y^i - (D_Y a)X^i - aD_Y X^i - a[X, Y]^i \\ &\quad= da(X)Y^i - da(Y)X^i + a\underbrace{(D_X Y^i - D_Y X^i - [X, Y]^i)}_{=0} \\ &\quad= da \wedge dx^i(X, Y) \\ &\quad= d\omega(X, Y), \end{aligned}$$

as we wished. □

Examples 4.37 1. Every vector field X defines a 1-form via the equation $\omega(Y) = \langle X, Y\rangle$.

2. The line integral of a 1-form ω in U along a smooth curve $\gamma : [a, b] \to U$ is defined to be

$$\int_\gamma \omega = \int_a^b \omega(\dot\gamma(t))\, dt.$$

3. The differential of a smooth function $f : U \to \mathbb{R}$ is the 1-form

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

Then

$$\begin{aligned} d^2 f &= d(df) \\ &= \sum_i d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \\ &= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j \\ &= 0. \end{aligned}$$

In particular, $d(dx^i) = 0$.

4. More generally, a 1-form ω is called *exact* if there exists a function f such that $df = \omega$. A 1-form ω is called *closed* if $d\omega = 0$. We have shown that every exact 1-form is closed. The converse is true if the domain U simply-connected; this can be proven using Theorem 4.25.

4.12 Connection forms and the integrability equations

Our next goal is to express the Gauss and Codazzi-Mainardi equations in terms of differential forms. In order to express the covariant derivative of a surface in terms of differential forms, we first consider the directional derivative in the ambient space; fix a frame $\{X_1, X_2, X_3\}$ in $\mathbb{R}^3$ with dual coframe $\{\omega^1, \omega^2, \omega^3\}$ and define 1-forms by setting

$$\omega^i_j(Y) = \omega^i(D_Y X_j)$$

for $i, j = 1, 2, 3$. Then

$$D_Y X_j = \sum_{i=1}^{3} \omega^i_j(Y) X_i.$$

We also have that

$$\omega^i_j + \omega^j_i = 0$$

for all i, j, because

$$\begin{aligned} \omega^i_j(Y) + \omega^j_i(Y) &= \langle D_Y X_j, X_i \rangle + \langle D_Y X_i, X_j \rangle \\ &= D_Y \langle X_i, X_j \rangle \\ &= 0, \end{aligned}$$

since $\langle X_i, X_j \rangle$ is constant.

Henceforth we suppose that a surface S in $\mathbb{R}^3$ is given and we take an *adapted orthonormal frame*, namely, assume that X_1, X_2 are tangent to S and

$X_3 = \nu$ is normal to S. Let $\{\omega^1, \omega^2, \omega^3\}$ denote the dual coframe. Since the covariant derivative is the tangential component of the directional derivative, for a tangent vector Y and $i, j = 1, 2$, we have that

$$\omega^i_j(Y) = \omega^i(D_Y X_j) = \omega^i(\nabla_Y X_j).$$

The restrictions of the ω^i_j (i, j = 1, 2) to the tangent spaces of S are called *connection forms* of S. They determine (and are determined by) the covariant derivative. Since ω^i_j is skew-symmetric in i, j, there is in fact only one curvature form ω^1_2.

If Y is tangent to S and $j = 1, 2$, then

$$\nabla_Y X_j = (D_Y X_j)^\top = \sum_{i=1}^{2} \omega^i_j(Y) X_i \tag{4.38}$$

and

$$\omega^3_j(Y) = \langle D_Y X_j, X_3\rangle = -\langle X_j, D_Y \nu\rangle = \langle X_j, AY\rangle = II(X_j, Y) \tag{4.39}$$

Equations (4.38), (4.39) are the Gauss formula (4.17) and Weingarten equation (4.18) written in the language of differential forms. In order to do the same with the integrability equations (4.21) and (4.22), let us first prove the following lemma.

Lemma 4.40 *If X, Y, Z are smooth vector fields defined on an open set U of $\mathbb{R}^n$, then*

$$D_X D_Y Z - D_Y D_X Z = D_[X, Y]Z.$$

(Equivalently, $[D_X, D_Y] = D_{[X,Y]}$ as operators)

Remark 4.41 In case $X = e_i$, $Y = e_j$, the lemma follows form the equality of mixed second partial derivatives. Indeed, $D_{e_i} D_{e_j} Z - D_{e_j} D_{e_i} Z = \frac{\partial^2 Z}{\partial x^i \partial x^j} - \frac{\partial^2 Z}{\partial x^j \partial x^i} = 0$ and $D_{[e_i, e_j]} = 0$ (since $[e_i, e_j] = 0$).

Proof of lemma 4.40. Write $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$. Then

$$\begin{aligned}
D_X D_Y Z - D_Y D_X Z &= \sum_i X^i D_{e_i}\left(\sum_j Y^j e_j\right) - \sum_j Y^j D_{e_j}\left(\sum_i X^i e_i\right) \\
&= \sum_{i,j} X^i Y^j D_{e_i} D_{e_j} Z + \sum_{i,j} X^i \tfrac{\partial Y^j}{\partial x^i} D_{e_j} Z \\
&\qquad - \sum_{i,j} Y^j X^i D_{e_j} D_{e_i} Z + \sum_{i,j} Y^j \tfrac{\partial X^i}{\partial x^j} D_{e_i} Z \\
&= \sum_{i,j}\left(X^i \tfrac{\partial Y^j}{\partial x^i} - Y^i \tfrac{\partial X^j}{\partial x^i}\right) D_{e_j} Z \\
&= \sum_j [X, Y]^j D_{e_j} Z \\
&= D_{[X,Y]} Z,
\end{aligned}$$

as desired. □

Proposition 4.42 (Maurer-Cartan structural equations) *For all $i,$, j, we have that*

$$d\omega_j^i + \sum_{k=1}^{3} \omega_k^i \wedge \omega_j^k = 0.$$

In particular

$$\begin{array}{rcll} d\omega_2^1 + \omega_3^1 \wedge \omega_2^3 & = & 0 & \textit{(Gauss eqn.)} \\ d\omega_3^1 + \omega_2^1 \wedge \omega_3^2 & = & 0 & \textit{(Codazzi-Mainardi eqn.)} \end{array}$$

Proof. If X, Y are smooth vector fields on $\mathbb{R}^3$, then

$$\begin{aligned} \left(d\omega_j^i + \sum_{k=1}^{3} \omega_k^i \wedge \omega_j^k\right)(X,Y) &= d\omega_j^i(X,Y) + \sum_{k=1}^{3}(\omega_k^i \wedge \omega_j^k)(X,Y) \\ &= D_X(\omega_j^i(Y)) - D_Y(\omega_j^i(X)) - \omega_j^i([X,Y]) \\ &\qquad + \sum_k (\omega_k^i(X)\omega_j^k(Y) - \omega_k^i(Y)\omega_j^k(X)) \\ &= \sum_k D_X(\omega_j^k(Y))\omega^i(X_k) - \sum_k D_Y(\omega_j^k(Y))\omega^i(X_k) \\ &\qquad - \sum_k \omega_j^k([X,Y])\omega^i(X_k) \\ &\qquad + \sum_k \omega_j^k(Y)\omega^i(D_X X_k) - \sum_k \omega_j^k(X)\omega^i(D_Y X_k) \\ &= \omega^i\big(D_X(\sum_k \omega_j^k(Y)X_k) - D_Y(\sum_k \omega_j^k(X)X_k) \\ &\qquad - \sum_k \omega_j^k([X,Y])X_k\big) \\ &= \omega^i(D_X D_Y X_j - D_Y D_X X_j - D_{[X,Y]}X_j) \\ &= 0, \end{aligned}$$

using Lemma 4.40, as we wished. □

The *curvature form* of S is the 2-form *defined on S* (meaning that it is restricted to the tangent spaces of S)

$$\Omega_2^1 = d\omega_2^1.$$

The relation to (Gaussian) curvature is expressed by the following proposition.

Proposition 4.43 *We have that*

$$\Omega_2^1 = K\,\omega^1 \wedge \omega^2,$$

where K is the Gaussian curvature of S.

Proof. By skew-symmetry, he only nontrivial ordered pair of vectors on which to evaluate a 2-form on S is (X_1, X_2). On one hand,

$$\begin{aligned}
\Omega^1_2(X_1, X_2) &= d\omega^1_2(X_1, X_2) \\
&= -\omega^1_3 \wedge \omega^3_2(X_1, X_2) \quad \text{(by Propostion 4.42)} \\
&= -\omega^1_3(X_1)\omega^3_2(X_2) + \omega^1_3(X_2)\omega^3_2(X_1) \\
&= \omega^3_1(X_1)\omega^3_2(X_2) - \omega^3_1(X_2)\omega^3_2(X_1) \\
&= II(X_1, X_1)II(X_2, X_2) - II(X_1, X_2)II(X_2, X_1) \\
&= \det(II) \\
&= K \quad \text{(since } \{X_1, X_2\} \text{ is orthonormal).}
\end{aligned}$$

On the other hand,

$$\omega^1 \wedge \omega^2(X_1, X_2) = 1.$$

Finally, $\Omega^1_2(X_1, X_2) = K\omega^1 \wedge \omega^2(X_1, X_2)$ implies that $\Omega^1_2 = K\, \omega^1 \wedge \omega^2$. □

Remark 4.44 The reasoning in the proof of Proposition 4.43 shows in fact that any smooth 2-form Ω on S can be written $\Omega = f\omega^1 \wedge \omega^2$ for some smooth function $f : S \to \mathbb{R}$. Suppose $\varphi : U \to S$ is a parametrization and $B \subset U$ is compact. The integral of the 2-form Ω on $\tilde{B} = \varphi(B)$ is defined to be the surface integral of f on $\tilde{B}$:

$$\int_{\tilde{B}} \Omega := \int\int_{\tilde{B}} f\, dA = \int\int_B f \circ \varphi \sqrt{EG - F^2}\, dudv,$$

and its value is known not to depend on the parametrization. On the other hand, by taking $\{X_1, X_2\}$ to be the result of the Gram-Schmidt orthonormalization of $\{\frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial v}\}$,

$$\begin{aligned}
\omega^1 \wedge \omega^2(\tfrac{\partial\varphi}{\partial u}, \tfrac{\partial\varphi}{\partial v}) &= \langle X_1, \tfrac{\partial\varphi}{\partial u}\rangle\langle X_2, \tfrac{\partial\varphi}{\partial v}\rangle - \langle X_1, \tfrac{\partial\varphi}{\partial v}\rangle\langle X_2, \tfrac{\partial\varphi}{\partial u}\rangle \\
&= \sqrt{EG - F^2}.
\end{aligned}$$

It is also easy to see that for any other positively oriented orthonormal frame $\{X_1', X_2'\}$, the dual coframe $\{\omega^{1\prime}, \omega^{2\prime}\}$ satisfies $\omega^{1\prime} \wedge \omega^{2\prime} = \omega^1 \wedge \omega^2$. For this reason, $\omega^1 \wedge \omega^2$ is called the *area element* of S and written $\omega^1 \wedge \omega^2 = dA$.

4.13 The Gauss-Bonnet theorem

The Gauss-Bonnet is one of the most important theorems in Differential Geometry. The global version expresses the invariance the total (Gaussian) curvature of a closed orientable surface under deformations in the ambient space preserving the topology. For this reason, it is said that this theorem relates the geometry and topology of a closed surface.

We start our discussion with the notion of geodesic curvature. Let $\gamma : I \to S$ be a curve parametrized by arc-length whose image lies in a regular parametrized surface $\varphi : U \to S$. We want to consider the curvature of γ from the point of view of an observer in S. We construct a frame along γ which is adapted to S: take $e_1 = \gamma'$ and $e_2 = \pm e_1 \times \nu$, where the sign is chosen so that $\{e_1, e_2, \nu\}$ is a positive basis of $\mathbb{R}^3$. The curvature κ of γ as a curve in $\mathbb{R}^3$ is of course

$$\kappa = ||e_1'|| = ||D_{e_1} e_1||.$$

Since e_1 has constant length 1, $\langle D_{e_1} e_1, e_1 \rangle = 0$. The normal component is the *normal curvature*,

$$\kappa_\nu = \langle D_{e_1} e_1, \nu \rangle = -\langle e_1, D_{e_1} \nu \rangle = \langle e_1, A e_1 \rangle = II(e_1, e_1),$$

and the tanegential component is the *geodesic curvature*,

$$\kappa_g = \langle D_{e_1} e_1, e_2 \rangle.$$

In particular,

$$\kappa^2 = \kappa_\nu^2 + \kappa_g^2.$$

Note that $\kappa_g = 0$ if and only if $\langle \nabla_{e_1} e_1, e_2 \rangle = 0$ (since e_2 is tangent) if and only if $\nabla_{e_1} e_1 = 0$ (since $\langle \nabla_{e_1} e_1, e_1 \rangle = 0$), which is the same as $\nabla_{\gamma'} \gamma' = 0$, namely, γ is a geodesic. This shows that the geodesic curvature is a measure of how far from being a geodesic the curve is. Plainly, we also have the equations

$$\begin{aligned} \nabla_{e_1} e_1 &= \kappa_g e_2 \\ \nabla_{e_1} e_2 &= -\kappa_g e_1. \end{aligned}$$

We can now state the first theorem.

Theorem 4.45 (Gauss-Bonnet, first local version) *Let $\varphi : U \subset S$ be a regular parametrized surface, and consider a subset $B \subset U$ diffeomorphic to a closed disk, where the boundary ∂B in oriented in the counter-clockwise sense. Then*

$$\int_{\varphi(B)} K\, dA + \int_{\varphi(\partial B)} \kappa_g\, ds = 2\pi.$$

Examples 4.46 1. For a disk of radius r in the plane $\mathbb{R}^2 \subset \mathbb{R}^3$, we have $K = 0$ and $\kappa_g = \frac{1}{r}$, so $\int K\, dA + \int \kappa_g\, ds = \frac{1}{r} \int ds = \frac{1}{r}\, 2\pi r = 2\pi$.

2. For the closed hemisphere

$$S_+^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

we have $K = 1$, $\kappa_g = 0$ (the equator is a geodesic), so $\int K\, dA + \int \kappa_g\, ds = \int dA = \text{area}(S_+^2) = \frac{1}{2}\, 4\pi = 2\pi$.

Proof of Theorem 4.45. Recall that S is oriented by the unit normal ν coming from φ. Consider a parametrization $\gamma : [a, b] \to \partial B$ (in the counter-clockwise sense) so that $\varphi \circ \gamma : [a, b] \to \varphi(\partial B)$ is a parametrization by arc-length. To compute the geodesic curvature, we need an adapted positive orthonormal frame $\{e_1, e_2\}$ along where $e_1 = (\varphi \circ \gamma)'$. We also consider $\{X_1, X_2\}$, positive orthonormal frame on S which is the result of the Gram-Schmidt process to $\{\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v}\}$; in particular, $X_1 = \frac{\partial \varphi}{\partial u} / ||\frac{\partial \varphi}{\partial u}||$. Then there exists a smooth function $\theta : [a, b] \to \mathbb{R}$ such that

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Writing $\gamma(s) = (x(s), y(s))$, we have

$$\cos \angle(e_1, X_1) = \frac{I((\varphi \circ \gamma)', \frac{\partial \varphi}{\partial u})}{||(\varphi \circ \gamma)'||||\frac{\partial \varphi}{\partial u}||} = \frac{x' g_{11} + y' g_{12}}{\sqrt{g_{11}}}.$$

We define a one-parameter family of inner products

$$g^t_{ij} = (1 - t) g_{ij} + t \delta_{ij},$$

continuous deformation of the first fundamental form of S into the standard inner product of $\mathbb{R}^2$. The angle $\angle(e_1, X_1)(s, t)$ is continuous as function of s and t. Since the difference $\angle(e_1, X_1)(b, t) - \angle(e_1, X_1)(a, t)$ is always an integral multiple of 2π, it must be a constant function of t. By the Umlaufsatz, $\angle(e_1, X_1)(b, 1) - \angle(e_1, X_1)(a, 1) = 2\pi$. Hence $\theta(b) - \theta(a) = \angle(e_1, X_1)(b, 0) - \angle(e_1, X_1)(a, 0) = 2\pi$.

We compute

$$\begin{aligned}
2\pi &= \theta(b) - \theta(a) \\
&= \int_a^b \frac{d\theta}{ds}\, ds \\
&= -\int_a^b \frac{1}{\sin\theta} (\langle \nabla_{e_1} e_1, X_1 \rangle + \langle e_1, \nabla_{e_1} X_1 \rangle)\, ds \\
&= -\int_a^b \frac{1}{\sin\theta} (\cos\theta \underbrace{\langle \nabla_{e_1} e_1, e_1 \rangle}_{=0} - \sin\theta \langle \nabla_{e_1} e_1, e_2 \rangle \\
&\qquad + \cos\theta \underbrace{\langle X_1, \nabla_{e_1} X_1 \rangle}_{=0} + \sin\theta \langle X_2, \nabla_{e_1} X_1 \rangle)\, ds \\
&= \int_a^b (\kappa_g + \omega^1_2(e_1))\, ds \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi \partial B)} \omega^1_2 \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi(B)} d\omega^1_2 \qquad \text{(by Stokes theorem)} \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi(B)} K\, dA,
\end{aligned}$$

as desired. □

In the second version of the local Gauss-Bonnet theorem, we alllow ∂B to have corners.

Theorem 4.47 (Gauss-Bonnet, second local version) *Same assumptions as in 4.45, except that now B is only homeomorphic to a closed disk and ∂B is piecewise smooth. Let α_i be the exterior angle at the ith vertex of ∂B. Then*

$$\int_{\varphi(B)} K\,dA + \int_{\varphi(\partial B)} \kappa_g\,ds + \sum_i \alpha_i = 2\pi.$$

We only make some remarks about what needs to be changed in the proof of this theorem in relation to Theorem 4.45. The first ingredient is the Umalufsatz for piecewise smooth closed curves. Suppose $\gamma : [a,b] \to \partial B$ is continuous, closed ($\gamma(a) = \gamma(b)$), and piecewise smooth in the sense that there exists a partition $a = s_0 < s_1 < \dots s_{n+1} = b$ such that $\gamma|_{[s_i,s_{i+1}]}$ is smooth for $i = 0,\dots,n$. For the sake of convenience, we also assume that $\gamma(s_0) = \gamma(s_n)$ is not a vertex. By smoothing γ near its vertices, one shows that the Umlaufsatz remains valid, te index of rotation of γ is 1. Now we can write

$$\begin{aligned} 2\pi &= \theta(b) - \theta(a) \\ &= \sum_{i=0}^{n}(\theta(s_{i+1}-) - \theta(s_i+)) + \sum_{i=1}^{n}(\theta(s_i+) - \theta(s_i-)) \\ &= \sum_{i=0}^{n}\int_{s_i}^{s_{i+1}} \frac{d\theta}{ds}\,ds + \sum_{i=1}^{n}\alpha_i, \end{aligned}$$

and the rest of the proof goes as before.

Corollary 4.48 (Geodesic n-gon) *If the sides of ∂B are geodesic segments, then*

$$\int_{\varphi(B)} K\,dA = 2\pi - \sum_{i=1}^{n}\alpha_i.$$

Corollary 4.49 (Theorema Elegantissimum, Gauss) *For a geodesic triangle ($n = 3$), we have that*

$$\int_{\varphi(B)} K\,dA = \beta_1 + \beta_2 + \beta_3 - \pi,$$

where $\beta_i = \pi - \alpha_i$ is the interior angle.

Corollary 4.50 *The sum of the interior angles of a geodesic triangle in a surface S is* $\left\{\begin{array}{l} > \pi \\ = \pi \\ < \pi \end{array}\right.$ *if* $\left\{\begin{array}{l} K > 0 \\ K = 0 \\ K < 0 \end{array}\right.$*, resp.*

As an application of the second version of the local Gauss-Bonnt theorem, we have the following proposition.

Proposition 4.51 *If the Gaussian curvature $K \leq 0$ on a simply-connected surface S, then two geodesics that start at a point $p \in S$ cannot meet again (i.e. there is no geodesic 2-gon in S).*

Proof. Suppose the geodesics meet again. Then they bound a region $\mathcal{R}$ diffeomorphic to a disk, by simply-connectedness of S. By Theorem 4.47, $\int\int_{\mathcal{R}} K\,dA + \alpha_1 + \alpha_2 = 2\pi$. Note that α_1, $\alpha < \pi$ since two distinct geodesics cannot be tangent at a point, and the integral term is nonpositive, so we get a contradiction. □

In particular (case $\alpha_1 = \alpha = 2$):

Corollary 4.52 *There is no simple closed geodesic in a simply-connected nonpositively curved surface.*

Of course, a cilinder is not simply-connected and violates the conclusion of the preceding corollary.

Theorem 4.53 (Gauss-Bonnet, global version) *Let S be a compact orientable surface. Then*

$$\int\int_S K\,dA = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

Remark 4.54 (Digression on the Euler characteristic of a compact surface) A *triangulation* $\mathcal{T}$ of a compact surface S is a decomposition $S = \cup\Delta_i$ into finitely many triangles such that a non-empty intersection $\Delta_i \cap \Delta_j$ $(i \neq j)$ consists of one common side or one common vertex. Radó proved in 1925 that every compact surface admits a triangulation. The Euler characteristic of S with respect to $\mathcal{T}$ is defined to be $\chi(S, \mathcal{T}) = V_{\mathcal{T}} - E_{\mathcal{T}} + F_{\mathcal{T}}$, where $V_{\mathcal{T}}$, $E_{\mathcal{T}}$, $F_{\mathcal{T}}$ denote respectively the total numbers of vertices, edges, faces of triangles in $\mathcal{T}$. It is not difficult to see that $\chi(S, \mathcal{T}) = \chi(S, \mathcal{T}')$ for two triangulations $\mathcal{T}$, $\mathcal{T}'$ of S. This can be proved in two stages: first one checks that it is true in case $\mathcal{T}'$ si a refinement of $\mathcal{T}$; in the general case, one sees that $\mathcal{T}$, $\mathcal{T}'$ admit a commn refinement. This being so, one can define the *Euler characteristic* of S as $\chi(S) = \chi(S, \mathcal{T})$ for any triangulation $\mathcal{T}$ of S. In fact, Poincaré showed that $\chi(S)$ is a topological invariant of S, that is, two homemorphic surfaces have the same Euler characteristic. Theorem 4.53 then says that the total curvature $\int\int_S K\,dA$ is a topological invariant.

The sphere S^2 has $\chi(S^2) = 2$, as is easily seen. This relation is reminiscent of formulas by Descartes and Euler (Euler's relation for convex polyhedra). Similarly, one sees that the torus T^2 has $\chi(T^2) = 0$. More generally, every compact orientable surface is homeomorphic to a sphere with g handles (cylinders) attached; the number is a topological invariant of the surface called *genus*. So $g = 0$ for the sphere and $g = 1$ for the torus. Also, the relation $\chi = 2 - 2g$ is easily checked.

Proof of Theorem 4.53. It is possible to choose a triangulation $\mathcal{T}$ of S such that each triangle Δ_i is contained in the image of a parametrization compatible with the orientation of S. By Theorem 4.47,

$$\int\int_{\Delta_i} K\,dA + \int_{\partial\Delta_i} \kappa_g\,ds + \sum_{j=1}^{3} \alpha_{ij} = 2\pi,$$

where α_{ij} are the external angles of Δ_i. Summing over $i = 1, \ldots, F$, we get

$$\int\int_{S} K\,dA + \sum_{i=1}^{F} \int_{\partial\Delta_i} \kappa_g\,ds + \sum_{i=1}^{F}\sum_{j=1}^{3} \alpha_{ij} = 2\pi\,F.$$

The second term vanishes, because each edge is counted twice, each time with a different orientation induced by the corresponding triangle. Moreover, if $\beta_{ij} = \pi - \alpha_{ij}$ is the internal angle,

$$\begin{aligned}\sum_{i=1}^{F}\sum_{j=1}^{3} \alpha_{ij} &= \sum_{i=1}^{F}\sum_{j=1}^{3} \pi - \sum_{i=1}^{F}\sum_{j=1}^{3} \beta_{ij} \\ &= 3F\,\pi - 2\pi\,V,\end{aligned}$$

since the sum of the internal angles around a vertex is 2π. The proof is completed by noting that $3F = 2E$ since each edge is shared by two faces. □

As an easy application, we have:

Corollary 4.55 *A compact orientable surface in $\mathbb{R}^3$ with constant (Gaussian) curvature is homeomorphic to the sphere.*

Proof. Indeed, a compact surface S in $\mathbb{R}^3$ admits a point p with $K(p) > 0$, so the constant curvature must be positive. By Gauss-Bonnet, $2 - 2g = \chi(S) > 0$ implying $g = 0$. □

Indeed Liebmann's theorem (1899) says that S must be a round sphere of radius $1/\sqrt{K}$.