# A short course on the differential geometry of curves and surfaces in Euclidean spaces

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# Contents

# Chapter 1

# Curves

## 1.1 Regular curves

A *regular parameterized curve* in $\mathbb{R}^n$ is a continuously differentiable map $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, such that $\gamma'(t) \neq 0$ for $t \in I$. This condition implies that $\gamma$ admits a tangent line at every point. A *regular curve* is an equivalence class of regular parameterized curves, where $\gamma \sim \eta$ if and only if $\eta = \gamma \circ \varphi$ for a continuously differentiable $\varphi : J \to I$, $\varphi' > 0$. We shall normally deal with curves satisfying some higher differentiability condition, like class $\mathcal{C}^k$ for $k \in \{1, 2, \dots, \infty\}$.

**Examples 1.1** 1. A *line* $\gamma(t) = p + tv = (x_0 + at, y_o + bt, z_0 + ct)$, where $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ is a point and $v = (a, b, c) \in \mathbb{R}^n$ is a vector.

2. The *circle* $\gamma(t) = (\cos t, \sin t)$ in the plane, or, more generally, $\gamma(t) = (x_0 + R\cos \omega t, y_0 + R \sin \omega t)$.

3. The *helix* $\gamma(t) = (a \cos t, a \sin t, bt)$, where $a$, $b \neq 0$.

4. The *semi-cubical parabola* $\gamma(t) = (t^2, t^3)$.

5. The *cathenary* $\gamma(t) = (t, \cosh(at))$, where $a > 0$.

6. The *tractrix* $\gamma(t) = (e^{-t}, \int_0^t \sqrt{1 - e^{-2\xi}}\, d\xi)$.

The *length* of a regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ is

$$L(\gamma) = \int_a^b ||\gamma'(t)||\, dt.$$

It is invariant under reparameterization.

**Lemma 1.2** *Every regular curve $\gamma : [a, b] \to \mathbb{R}^n$ admits a reparameterization by arc length, that is, $\eta : [0, \ell] \to \mathbb{R}^n$, where $\ell = L(\gamma)$, such that $L(\eta|_{[0,t]}) = t$; equivalently, $||\eta'|| \equiv 1$, and we say that $\gamma$ has unit speed.*

*Proof.* Define

$$\psi(t) = \int_a^t ||\gamma'(\xi)||\, d\xi.$$

Then $\psi : [a, b] \to [0, \ell]$, $\psi' > 0$ and we can take $\varphi = \psi^{-1}$, $\eta = \gamma \circ \varphi$. $\square$

*Unless explicit mention to the contrary, we shall generally assume that our curves are parameterized by arc-length.*

## 1.2 Plane curves

Let $\gamma : I \to \mathbb{R}^2$ be a curve parameterized by arc-length of class $\mathcal{C}^2$. Then $||\gamma'(s)|| = 1$ for all $s$. The curvature of $\gamma$ is the rate of change of the direction of $\gamma$. Namely, let

$$\mathbf{t}(s) = \gamma'(s)$$

be the unit tangent vector at time $s$, and complete it to a positively oriented orthonormal base $\mathbf{t}(s)$, $\mathbf{n}(s)$ of $\mathbb{R}^2$. Then $\langle \mathbf{t}, \mathbf{t} \rangle = 1$ implies $\langle \mathbf{t}, \mathbf{t}' \rangle = 0$, so $\mathbf{t}' = \kappa \mathbf{n}$ for some continuous function $\kappa : I \to \mathbb{R}$. Similarly, $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ yields $\mathbf{n}' = -\kappa \mathbf{t}$. We can write

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^2$.

**Proposition 1.3** *Suppose $\gamma : I \to \mathbb{R}^2$ is a curve parameterized by arc-length. Then $\kappa$ is constant if and only $\gamma$ is either part of a circle (if $\kappa \neq 0$) or part of a line (if $\kappa = 0$).*

*Proof.* If $\kappa$ is identically zero, then the Frenet-Serret equations give $\mathbf{t}' = 0$ so $\gamma' = \mathbf{t}$ is a constant vector $\mathbf{t}_0$ in the plane and $\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} \gamma'(\xi)\, d\xi = \gamma(s_0) + (s - s_0)v_0$, for $s_0 \in I$.

If $\kappa$ is a nonzero constant, by changing the orientaion we may assume that $\kappa > 0$. We first show that

$$c(s) := \gamma(s) + \frac{1}{\kappa}\mathbf{n}(s)$$

is a constant curve. In fact,

$$\begin{aligned} c' &= \gamma' + \frac{1}{\kappa}\mathbf{n}' \\ &= \mathbf{t} + \frac{1}{\kappa}(-\kappa \mathbf{t}) \\ &= 0. \end{aligned}$$

Set $c(s) = c_0$ for all $s \in I$. Since

$$||\gamma(s) - c_0|| = \frac{1}{\kappa}$$

for all $s \in I$, we deduce that $\gamma$ is part of the circle of center $c_0$ and radius $1/\kappa$, as wished. $\square$

**Theorem 1.4 (Fundamental theorem of plane curves)** *The curvature is a complete invariant of plane curves, up to rigid motion. More precisely, given a continuous function $\alpha : [a, b] \to \mathbb{R}$ there is a unique curve in the plane defined on $[a, b]$, parametrized by arc-length, whose curvature at time $s \in [a, b]$ is $\alpha(s)$, up to a translation and rotation of the plane.*

*Proof.* For the existence, set $\gamma(s) = (x(s), y(s))$, where

$$x(s) = \int_a^s \cos\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta,\ y(s) = \int_a^s \sin\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta$$

for $s \in [a, b]$. Then $\gamma$ has curvature function given by $\alpha$.

Conversely, suppose $\gamma : [a, b] \to \mathbb{R}$, $\gamma(s) = (x(s), y(s))$ is parameterized by arc-length and has curvature $\alpha$. The Frenet-Serret frame $t$, $n$ along $\gamma$ can be written

$$t(s) = (\cos\theta(s), \sin\theta(s)),\ n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now

$$\alpha(s) = \langle t'(s), n(s)\rangle = \theta'(s),$$

so

$$\theta(s) = \theta(a) + \int_a^s \alpha(\xi)\, d\xi.$$

Also, $t = (x', y')$ yields

$$x(s) = x(a) + \int_a^s \cos(\theta(\tau))\, d\tau,\ y(s) = y(a) + \int_a^s \sin(\theta(\tau))\, d\tau.$$

This determines completely $\gamma$ up to the values of $x(a)$, $y(a)$, $\theta(a)$, that is, up to translation and rotation. □

## Regular curves in $\mathbb{R}^2$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$ and the curvature $\kappa$ of $\gamma$ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\kappa}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

for $t \in I$. Denote the velocity of $\gamma$ by $\nu(t) = ||\gamma'(t)||$ and recall that $\varphi' = \nu$ (Lemma 1.2). The function $\nu$ is the appropriate correction term when we want to write Frenet-Serret equations for a curve $\gamma$ of arbitrary speed, as we show in the sequel.

Since $\tilde{\gamma}$ is of unit speed, we have the Frenet-Serret equations $\tilde{\mathbf{t}}' = \tilde{\kappa}\tilde{\mathbf{n}}$, $\tilde{\mathbf{n}}' = -\tilde{\kappa}\tilde{\mathbf{t}}$. Now

$$\begin{aligned} \mathbf{t}'(t) &= \tilde{\mathbf{t}}'(t)\varphi'(t) \\ &= \kappa(\varphi(t))\nu(t)\tilde{\mathbf{n}}(\varphi(t)) \\ &= \kappa(t)\nu(t)\mathbf{n}(t), \end{aligned}$$

and similarly

$$\mathbf{n}'(t) = -\kappa(t)\nu(t)\mathbf{t}(t).$$

So we have the following Frenet-Serret equations for $\gamma$:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa\nu \\ -\kappa\nu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}.$$

In particular,

$$\kappa = \frac{1}{\nu}\langle \mathbf{t}', \mathbf{n}\rangle. \tag{1.5}$$

In practice, sometimes can be hard to find the explicit reparameterization by arc-length of a given regular curve, so equation (1.5) comes in handy to compute the curvature in such cases.

**Example 1.6** We compute the curvature of the catenary $\gamma(t) = (t, \cosh t)$. We have $\gamma'(t) = (1, \sinh t)$ and $\nu(t) = (1 + \sinh^2 t)^{1/2} = \cosh t$, so $\gamma$ has variable speed. We first seek to reparameterize $\gamma$ by arc-length. We have $\varphi = \int \nu$ yields $\varphi(t) = \sinh t$ and $\varphi^{-1}(s) = \operatorname{arcsinh} s = \log(s + \sqrt{s^2+1})$, so

$$\tilde{\gamma}(s) = \gamma(\varphi^{-1}(s)) = (\log(s + \sqrt{s^2+1}), \sqrt{1+s^2})$$

is a reparameteriztion by arc-length. Pursuing this line of reasoning would require us to differentiate $\tilde{\gamma}$ twice (in the end we would still need to change back to the variable $t$), which is possible but not worth it. Instead, we start again and use (1.5). We have

$$\mathbf{t}(t) = \frac{1}{\nu(t)}\gamma'(t) = (\mathrm{sech} t, \tanh t),$$

so

$$\mathbf{n}(t) = (-\tanh t, \mathrm{sech} t),$$

$$\mathbf{t}'(t) = (-\tanh t\ \mathrm{sech}\, t, \mathrm{sech}^2\, t),$$

and

$$\kappa(t) = \frac{1}{\cosh t}(\tanh^2 t\, \mathrm{sech} t + \mathrm{sech}^3 t) = \mathrm{sech}^2 t.$$

**Example 1.7** We can generalize Example 1.6. If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve of arbitrary speed and $\gamma(t) = (x(t), y(t))$, then $\nu = (x'^2 + y'^2)^{1/2}$, so

$$\mathbf{t} = \frac{1}{\nu}\gamma' = \frac{1}{(x'^2 + y'^2)^{1/2}}(x', y'),$$

and

$$\mathbf{n} = \frac{1}{(x'^2 + y'^2)^{1/2}}(-y', x').$$

Now

$$\begin{aligned} \mathbf{t}' &= -\frac{x'x'' + y'y''}{(x'^2 + y'^2)^{3/2}}(x', y') + \frac{1}{(x'^2 + y'^2)^{1/2}}(x'' + y'') \\ &= \frac{1}{(x'^2 + y'^2)^{3/2}}(x''y'^2 - x'y'y'', x'^2y'' - x'x''y'). \end{aligned}$$

So

$$\begin{aligned} \langle \mathbf{t}', \mathbf{n} \rangle &= \frac{1}{(x'^2 + y'^2)^2}(-x''y'^3 + x'y'^2y'' + x'^3y'' - x'^2x''y') \\ &= \frac{1}{x'^2 + y'^2}(x'y'' - x''y'), \end{aligned}$$

and (1.5) yields

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

**Remark 1.8** (i) If $\gamma_1 : I_1 \to \mathbb{R}^2$ and $\gamma_2 : I_2 \to \mathbb{R}^2$ are two unit speed reparameterizations preserving the orientation of a given regular parameterized curve $\gamma$ then $\gamma_1(t) = \gamma_2(\varphi(t))$ for some $\varphi : I_1 \to I_2$ with $\varphi' > 0$. Then $\gamma_1' = \varphi'\gamma_2'$ with $||\gamma_1'|| = ||\gamma_2'|| = 1$, so $\varphi' \equiv 1$ implying that $\varphi(t) = t + t_0$ for some $t_0 \in \mathbb{R}$. We deduce that the definition of curvature of a regular parameterized curve $\gamma$ of arbitrary speed does not depend on the reparameterization by arc-length that we choose.

(ii) The curvature of a regular parameterized curve in the plane thus defined is invariant under reparameterization preserving the orientation.

(iii) The curvature of a regular paparemeterized curve in the plane changes sign under a change of orientation.

## 1.3 Space curves

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of class $\mathcal{C}^3$ and assume that $\gamma'' \neq 0$ everywhere. Then we can associate an adapted trihedron to $\gamma(s)$ for each $s \in I$. We put:

$$t = \gamma' \ (\textit{tangent}), n = \frac{\gamma''}{||\gamma''||} \ (\textit{normal}), b = t \times n \ (\textit{binormal}).$$

The *curvature* is $\kappa = ||\gamma''||$. It follows that $t' = \kappa n$. Since $n(s)$ is a unit vector for all $s$, $n' \perp n$ so

$$\begin{aligned} n' &= \langle n', t \rangle t + \langle n', b \rangle b \\ &= -\langle n, t' \rangle t + \langle n', b \rangle b. \end{aligned}$$

We define the *torsion* $\tau = \langle n', b\rangle$. Now

$$n' = -\kappa t + \tau b.$$

Finally,

$$\begin{aligned} b' &= t' \times n + t \times n' \\ &= \kappa n \times n + t \times (-\kappa t + \tau b) \\ &= -\tau n. \end{aligned}$$

We summarize this discussion in matrix notation:

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^3$.

**Remark 1.9** A space curve with nonzero curvature is planar if and only if $\tau \equiv 0$.

**Example 1.10** We compute the curvature and torsion of the helix

$$\gamma(s) = (a\cos(s/c), a\sin(s/c), b(s/c)),\ s \in \mathbb{R},$$

for $a > 0$, $b \in \mathbb{R}$ and $c \neq 0$. We have

$$\gamma'(s) = (-(a/c)\sin(s/c), (a/c)\cos(s/c), b/c),$$

so $\gamma$ is parameterized by arc-length precisely when

$$a^2 + b^2 = c^2, \tag{1.11}$$

and then $t(s) = \gamma'(s)$. Further,

$$\gamma''(s) = (-(a/c^2)\cos(s/c), -(a/c^2)\sin(s/c), 0),$$

so

$$n(s) = (-\cos(s/c), -\sin(s/c), 0)$$

and

$$b(s) = ((b/c)\sin(s/c), -(b/c)\cos(s/c), a/c).$$

We compute

$$n'(s) = ((1/c)\sin(s/c), -(1/c)\cos(s/c), 0)$$

and

$$b'(s) = ((b/c^2)\cos(s/c), (b/c^2)\sin(s/c), 0).$$

It follows that

$$\kappa(s) = ||\gamma''(s)|| = a/c^2$$

and

$$\tau(s) = \langle n'(s), b(s) \rangle = b/c^2$$

are constant functions. Moreover $\kappa^2 + \tau^2 = 1/c^2$, so

$$a = \frac{\kappa}{\kappa^2 + \tau^2} \quad \text{and} \quad b = \frac{\tau}{\kappa^2 + \tau^2}. \tag{1.12}$$

Therefore, given $\kappa$, $\tau$, we can solve equations (1.11), (1.12) for $a$, $b$, $c$ and obtain a unique helix with curvature $\kappa$ and torsion $\tau$.

**Theorem 1.13 (Fundamental theorem of space curves)** *The curvature and torsion are complete invariants of space curves. More precisely, given continuous functions $\alpha$, $\beta : [a,b] \to \mathbb{R}$ with $\alpha(s) > 0$ for all $s$, there exists a unique regular curve in $\mathbb{R}^3$ defined on $[a,b]$, parameterized by arc-length, of class $C^3$, whose curvature and torsion at time $s \in [a,b]$ are respectively given by $\alpha(s)$ and $\beta(s)$, up to a translation and rotation of $\mathbb{R}^3$.*

*Proof.* Consider

$$A = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}$$

as a matrix-valued function $[a,b] \to \mathbb{R}^{3\times 3}$. We consider the first order system of linear differential equations

$$F' = AF$$

for a matrix-valued $F : [a,b] \to \mathbb{R}^{3\times 3}$, given by the Frenet-Serret equations. Here the lines of $F$ will yield the Frenet-Serret frame of our curve $\gamma$ to be constructed, namely, $F(s) = (t(s), n(s), b(s))$. For a given initial condition $F(a) = (e_1, e_2, e_3)$, which is a positively oriented orthonormal basis of $\mathbb{R}^3$, the system has a unique solution $F(s)$ of class $\mathcal{C}^3$ defined for $s \in [a,b]$.

We claim that $F(s)$ is an ortogonal matrix of determinant $1$ for all $s \in [a,b]$. The crucial fact involved here is that $A(s)$ is a skew-symmetric matrix. In fact, set $G = FF^t$. Then $G(a) = I$ and

$$\begin{aligned} G' &= (FF^t)' \\ &= F'F^t + F(F^t)' \\ &= F'F^t + F(F')^t \\ &= AFF^t + FF^tA^t \\ &= AG + GA^t. \end{aligned}$$

Since the constant function given by the identity matrix also satisfies the differential equation $G' = AG + GA^t$, due to the fact that $A(s) + A^t(s) = 0$ for all $s$, by the uniqueness theorem of solutions of first order ODE, $G(s) = I$ for all $s$. This proves that $F(s)$ is an orthogonal matrix and hence $\det F(s) = \pm 1$ for all

$s$. Since the determinant is a continuous function and $\det F(0) = 1$, we deduce that $\det F(s) = 1$ for all $s$.

Now $F(s) = (t(s), n(s), b(s))$ is a trihedron for all $s$. For a given initial point $\gamma(a) = p \in \mathbb{R}^3$, the curve is completely determined by

$$\gamma(s) = p + \int_a^s t(\xi)\, d\xi.$$

From the equation $F' = AF$ we see that $(t, n, b)$ is the Frenet-Serret frame along $\gamma$ and $\alpha$, $\beta$ are its curvature and torsion respectively. Note that the ambiguity in the construction of $\gamma$ precisely amounts to the choices of point $p$ and positive orthonormal basis $(e_1, e_2, e_3)$, so any two choices differ by a translation and a rotation. □

**Remark 1.14 (Local form)** Let $\gamma : I \to \mathbb{R}^3$ be a regular curve of class $\mathcal{C}^3$ parameterized by arc-length and suppose that $\kappa > 0$ so that the Frenet-Serret frame is well-defined. We may assume that $0 \in I$, $\gamma(0) = 0$ and $(t(0), n(0), b(0))$ is the canonical basis of $\mathbb{R}^3$. Then the Taylor expansion of $\gamma(s) = (x(s), y(s), z(s))$ at $s = 0$ yields:

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)^2}{6} s^3 + R_x, \\ y(s) &= \frac{\kappa(0)}{2} s^2 + \frac{\kappa'(0)}{6} s^3 + R_y, \\ z(s) &= \frac{\kappa(0)\tau(0)}{6} s^3 + R_z, \end{aligned}$$

where $\lim_{s\to 0} \frac{1}{s^3}(R_x, R_y, R_z) = 0$. Therefore the projections of $\gamma$ in the $(t, n)$-plane (*osculating plane*), $(n, b)$-plane (*normal plane*, $(t, b)$-plane (*rectifying plane plane*) has the form of a parabola, semi-cubical parabola (if $\tau(0) \neq 0$), cubical parabola (if $\tau(0) \neq 0$), respectively, up to third order.

## Regular curves in $\mathbb{R}^3$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the curvature $\kappa$ and the torsion $\tau$ of $\gamma$ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\mathbf{b}}$, $\tilde{\kappa}$, $\tilde{\tau}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\mathbf{b}(t) = \tilde{\mathbf{b}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

$$\tau(t) = \tilde{\tau}(\varphi(t))$$

for $t \in I$.

Set $\nu = ||\gamma'||$. As in the case of $\mathbb{R}^2$, we deduce the Frenet-Serret equations for $\gamma$:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

In the following, we find formuale for $\kappa$ and $\tau$ in terms of the first three derivatives of $\gamma$. We have

$$\gamma' = \nu\mathbf{t},$$

$$\begin{aligned} \gamma'' &= \nu'\mathbf{t} + \nu\mathbf{t}' \\ &= \nu'\mathbf{t} + \kappa\nu^2\mathbf{n}, \end{aligned}$$

and

$$\begin{aligned} \gamma''' &= \nu''\mathbf{t} + \nu'(\kappa\mathbf{n}) + (\kappa'\nu^2 + 2\kappa\nu\nu')\mathbf{n} + \kappa\nu^2(-\kappa\nu\mathbf{t} + \tau\nu\mathbf{b}) \\ &= (\nu'' - \kappa^2\nu^2)\mathbf{t} + (\kappa'\nu^2 + 3\kappa\nu\nu')\mathbf{n} + \kappa\tau\nu^3\mathbf{b}. \end{aligned}$$

We deduce that $\gamma \times \gamma'' = \kappa\nu^3\mathbf{b}$, so

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}. \tag{1.15}$$

Further $||\gamma \times \gamma''|| = \kappa\nu^3$ and $(\gamma' \times \gamma'') \cdot \gamma''' = \kappa^2\tau\nu^6$, so

$$\tau = \frac{\gamma' \times \gamma'' \cdot \gamma'''}{||\gamma' \times \gamma''||^2}. \tag{1.16}$$

**Example 1.17** It is very easy to compute curvature and torsion of the space curve $\gamma(t) = (t, t^2, t^3)$ using (1.15) and (1.16), as opposed to the moethod of finding a reparameteriztion by arc-length. Indeed

$$\gamma'(t) = (1, 2t, 3t^2),$$

$$\gamma''(t) = (0, 2, 6t)$$

and

$$\gamma'''(t) = (0, 0, 6).$$

Now

$$\gamma' \times \gamma'' = (6t^2, -6t, 2),$$

and

$$\kappa = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{(9t^4 + 4t^2 + 1)^3}},$$

$$\tau = \frac{3}{9t^4 + 9t^2 + 1}.$$

## 1.4 Global theory

Our discussion so far has been mostly local in nature, that is, we have studied properties of curves that depend on the behavior of the curve in a neighborhood of a point, like for instance the curvature and the torsion. The global differential geometry of curves studies curves as whole objects. For example, length is a global concept, and the property of being a closed curve is a global property.

In this section we survey on some of the most famous and interesting questions pertaining to the global differential geometry of curves. We neither intend to present an exhaustive treatment nor to delve into the details of proofs, but just sketch a few geometric ideas.

### 1.4.1 The rotation index

A regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^n$ of class $\mathcal{C}^k$ is said to be *closed* if $\gamma$ and its derivatives up to order $k$ coincide at $a$ and $b$:

$$\gamma(a) = \gamma(b), \ldots, \gamma^{(k)}(a) = \gamma^{(k)}(b).$$

Equivalently, $\gamma$ extends to a $\mathcal{C}^k$ map $\mathbb{R} \to \mathbb{R}^n$.

A closed regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^n$ is called *simple* if it has no self-intersections, that is, $\gamma|_{(a,b)}$ is injective as a map.

For simplicity, hereafter we consider only curves with class $C^\infty$. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed curve parameterized by arc length in the plane. Let $\theta(s)$ be an angle determination of its tangent direction. On the one hand, we have seen that $\theta'(s) = \kappa(s)$, so

$$\int_a^b \kappa(s)\,ds = \theta(b) - \theta(a).$$

On the other hand, since $\gamma$ is closed,

$$\theta(b) - \theta(a) = 2\pi k$$

for some $k \in \mathbb{Z}$. The integer $k$ is called the *rotation index* of $\gamma$. It is clear that the rotation index of a closed curves changes sign if we change the orientation of the curve. For a closed regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^2$ of arbitrary speed, its rotation index is defined as the rotation index of a reparameterization by arc-length, so that it equals

$$\frac{1}{2\pi}\int_a^b \kappa(t)||\gamma'(t)||\,dt.$$

Hence we can talk of the rotation index of a regular curve in $\mathbb{R}^2$ (without reference to parameterization).

**Theorem 1.18 (Hopf's Umlaufsatz 1935)** *The rotation index of a simple closed regular curve is* $\pm 1$*.*

The proof is not very difficult, but we omit it.

Two regular parameterized curves $\gamma_0$, $\gamma_1 : [a,b] \to \mathbb{R}^n$ are called *regularly homotopic* if there exists a map $F : [a,b] \times [0,1] \to \mathbb{R}^n$ of class $\mathcal{C}^\infty$ such that:

- $F(s,0) = \gamma_0(s)$ and $F(s,1) = \gamma_1(s)$ for all $s \in [a,b]$;
- if we set $\gamma_t(s) = F(s,t)$, then $f_t : [a,b] \to \mathbb{R}^n$ is a regular parameterized curve for all $t \in [0,1]$.

If, in addition, $\gamma_0$ and $\gamma_1$ are closed then $\gamma_t$ is required to be closed for all $t$.

It is not difficult to see that if $\gamma_0$ and $\gamma_1$ are two closed regular curves which are regularly homotopic then they have the same rotation index. Conversely, we state:

**Theorem 1.19 (Whitney-Graustein 1937)** *Two closed regular curves in the plane are homotopic if and only if they have the same rotation index.*

### 1.4.2 Total absolute curvature

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a closed regular parameterized curve. Recall that

$$\int_a^b \kappa(t)||\gamma'(t)||\,dt = \int_a^b \kappa(s)\,ds$$

equals $2\pi$ times the rotation index of $\gamma$, where $s$ is arc-length parameter. The *total absolute curvature* of $\gamma$ is

$$\int_a^b |\kappa(t)|||\gamma'(t)||\,dt = \int_a^b |\kappa(s)|\,ds$$

(note that the absolute value in the integrand is important only in case $n = 2$).

A closed regular curve $\gamma : [a,b] \to \mathbb{R}^2$ is called *convex* if, for all $s \in [a,b]$, $\gamma([a,b])$ is contained in one half-plane determined by the tangent line at $s$.

**Theorem 1.20 (Fenchel 1929)** *The total absolute curvature of a regular curve in* $\mathbb{R}^3$ *is bounded below by* $2\pi$*, and equality holds if and only if the curve is planar and convex.*

*Proof.* We work with a parameterization by arc-length. The total absolute curvature of a curve $\gamma : [a,b] \to \mathbb{R}^3$ parameterized by arc-length equals the length of its spherical image $\alpha : [a,b] \to S^2(1)$, where $\alpha(s) = t(s) = \gamma'(s)$.

If $\alpha$ is contained in a hemisphere then $\alpha(s) \cdot v \geq 0$ for all $s$ and some unit vector $v$. But

$$0 = (\gamma(b) - \gamma(a)) \cdot v = \int_a^b \alpha(s) \cdot v\,ds \geq 0,$$

so $\gamma$ must be planar.

If $\alpha$ is not contained in a hemisphere, let $s_0 \in [a, b]$ divide $\alpha$ into two curves of the same length, $\alpha_1 = \alpha|_{[a,s_0]}$ and $\alpha_2 = \alpha|_{[s_0,b]}$. Up to a rotation, we may assume $\alpha(0)$ and $\alpha(s_0)$ are symmetric with respect to the north pole. One of $\alpha_0$, $\alpha_1$ crosses the equator, say $\alpha_0$ crosses the equator at $p$. Reflect $t_0$ on the plane through $\alpha(0)$, the north pole and $\alpha(s_0)$ to obtain a closed curve $\alpha_2$ in $S^2(1)$ passing through $p$ and $-p$.

Since $\alpha_2$ is closed and passes through two antipodal points, clearly $L(\alpha_2) \geq 2\pi$, with equality holding only in case $\alpha$ is contained in the equator. On the other hand,

$$L(\alpha_2) = 2L(\alpha_0) = L(\alpha_0) + L(\alpha_1) = L(\alpha),$$

as desired.

To finish, note that a simple closed curve in $\mathbb{R}^2$ has curvature $\kappa$ not changing sign if and only if it is convex. $\square$

**Theorem 1.21 (Fary-Milnor)** *The total absolute curvature of a non-trivially knotted regular curve in $\mathbb{R}^3$ is strictly bounded below by $4\pi$.*

# Chapter 2

# Surfaces: basic definitions

A *regular parameterized surface* is a smooth mapping $\varphi : U \to \mathbb{R}^3$, where $U$ is an open subset of $\mathbb{R}^2$, of maximal rank. This is equivalent to saying that the rank of $\varphi$ is 2 or that $\varphi$ is an immersion. Such a $\varphi$ is called a *parameterization*.

Let $(u, v)$ be coordinates in $\mathbb{R}^2$, $(x, y, z)$ be coordinates in $\mathbb{R}^3$. Then

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$x(u, v)$, $y(u, v)$, $z(u, v)$ admit partial derivatives of all orders and the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank two. This is equivalent to

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(y, z)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(z, x)}{\partial(u, v)} \neq 0,$$

or to the columns of the Jacobian matrix, denoted $\varphi_u$ and $\varphi_v$, to be linearly independent.

A *surface* is a subset $S$ of $\mathbb{R}^3$ satisfying:

(1) $S = \cup_{i \in I}$, where $V_i$ is an open subset of $S$ and $\varphi_i : U_i \subset \mathbb{R}^2 \to \varphi_i(U_i) = V_i$ is a parameterization for all $i \in I$. In other words, every point $p \in S$ lies in an open subset $W \subset \mathbb{R}^3$ such that $W \cap S$ is the image of a smooth immersion of an open subset of $\mathbb{R}^2$ into $\mathbb{R}^3$.

(2) Each $\varphi_i : U_i \to V_i$ is a homeomorphism. The continuity of $\varphi_i^{-1}$ means that for given $i \in I$, $q \in V_i$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi_i^{-1}( \underbrace{B(q, \delta)}_{\text{ball in } \mathbf{R}^3} \cap V_i) \subset \underbrace{B(\varphi_i^{-1}(q), \epsilon)}_{\text{ball in } \mathbf{R}^2}.$$

## 2.1 Examples

1. The graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open, is a regular parameterized surface, where the parameterization is given by $\varphi(u, v) = (u, v, f(u, v))$. Note that

$$(d\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

has rank two.

2. If $S \subset \mathbb{R}^3$ is a subset such that any one of its points lies in a open subset of $S$ which is a graph as in (1) (with respect to any one of the three coordinate planes), then $S$ is a surface. It only remains to check that the parameterizations constructed in (1) are homeomorphisms. But this follows from the fact that $\varphi^{-1} = \pi|_{\varphi(U)}$ is continuous, where $\pi(x, y, z) = (x, y)$ is continuous.

3. The unit sphere is defined as

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Any point of $S^2$ lies in one of the following six open subsets, which are graphs, given by $z = \pm\sqrt{1 - x^2 - y^2}$, $y = \pm\sqrt{1 - x^2 - z^2}$, $x = \pm\sqrt{1 - y^2 - z^2}$.

## 2.2 Inverse images of regular values

Let $F : W \to \mathbb{R}$ be a smooth map, where $W \subset \mathbb{R}^3$ is open. A point $p \in W$ is called a *critical point* of $F$ is $dF_p = 0$; otherwise, it is called a *regular point*. A point $q \in \mathbb{R}$ is called a *critical value* of $F$ if there exists a critical point of $F$ in $F^{-1}(q)$; otherwise, it is called a *regular value*. Note that a point $q \in \mathbb{R}$ lying outside the image of $F$ is automatically a regular value of $F$.

**Theorem 2.1** *If $q$ is a regular value of $F$ and $F^{-1}(q) \neq \varnothing$, then $S = F^{-1}(q)$ is a surface.*

*Proof.* It suffices to show that every point of $S$ lies in an open subset of $S$ which is a graph. Let $p = (x_0, y_0, z_0) \in S$. Then $dF_p = \begin{pmatrix} \frac{\partial F}{\partial x}(p) \\ \frac{\partial F}{\partial y}(p) \\ \frac{\partial F}{\partial z}(p) \end{pmatrix} \neq 0$ by the assumption. Without loss of generality, assume that $\frac{\partial F}{\partial z}(p) \neq 0$. By the implicit function theorem, there exist open neighborhoods $\tilde{V}$ of $(x_0, y_0, z_0)$ in $\mathbb{R}^3$ and $U$ of $(x_0, y_0)$ in $\mathbb{R}^2$ and a smooth function $f : U \to \mathbb{R}$ such that $F(x, y, z) = q$, $(x, y, z) \in \tilde{V}$ if and only if $z = f(x, y)$, $(x, y) \in U$. Hence $V = \tilde{V} \cap S$ is the graph of $f$ and an open neighborhood of $p$ in $S$. $\square$

## 2.3 More examples

4. Spheres can also be seen as inverse images of regular values. Let $F(x,y,z) = x^2+y^2+z^2$. Then $(dF_{(x,y,z)})^t = (2x\ 2y\ 2z) = (0\ 0\ 0)$ if and only if $(x,y,z) = (0,0,0)$. Since $(0,0,0) \notin F^{-1}(r^2)$ for $r > 0$, we have that the sphere $F^{-1}(r^2)$ of radius $r > 0$ is a surface. Similarly, the ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ($a$, $b$, $c > 0$) are surfaces.

5. The hyperboloids $x^2+y^2-z^2 = r^2$ (one sheet) and $x^2+y^2-z^2 = -r^2$ (two sheets) are surfaces, $r > 0$. The cone $x^2+y^2-z^2 = 0$ is not a surface in a neighborhood of its vertex $(0,0,0)$.

6. The tori of revolution are surfaces given by the equation $z^2+(\sqrt{x^2+y^2}-a)^2 = r^2$, where $a, r > 0$.

7. More generally, one can consider surfaces of revolution. Let $\gamma(t) = (f(t), 0, g(t))$ be a regular parameterized curve, $t \in (a,b)$. Define

$$\varphi(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u,v) \in (a,b) \times (v_0, v_0+2\pi)$. One can cover the surface by varying $v_0$ in $\mathbb{R}$. But there are conditions on $\gamma$ for $\varphi$ to be an immersion. One has

$$\frac{\partial(x,y)}{\partial(u,v)} = ff', \quad \frac{\partial(y,z)}{\partial(u,v)} = -fg'\cos v, \quad \frac{\partial(z,x)}{\partial(u,v)} = -fg'\sin v,$$

so

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 = f^2||\dot{\gamma}||^2,$$

and $\varphi$ is an immersion if and only if $f > 0$. Note also that $\varphi$ is injective if and only if $\gamma$ is injective. One also checks that $\varphi^{-1}$ is continuous by writing its explicit expression.

8. Let $\xi : [0,2\pi] \to S^2$ be the smooth curve given in spherical coordinates as $\varphi = \theta/2$ ($\theta$: longitude, $\varphi$; co-latitude) Then

$$\xi(t) = (\cos t \sin(t/2), \sin t \sin(t/s), \cos(t/2)),$$

and we can parameterize the Möbius band as

$$\varphi(u,v) = \alpha(u) + v\xi(u),$$

where $\alpha(t) = (\cos t, \sin t, 0)$ is the unit circle in $\mathbb{R}^2$. Since $\varphi_u \cdot \varphi_v = 0$, $\varphi$ is an immersion.

### 2.3.1 Graphs

According to Example 2.1(2), every subset $S \subset \mathbb{R}^3$ which is locally a graph is a surface. Conversely:

**Proposition 2.2** *If $S \subset \mathbb{R}^3$ is a surface then it is locally a graph, in the sense that, for every $p \in S$, there is an open subset $W$ of $\mathbb{R}^3$ containing $p$ such that $W \cap S$ is the graph of one of the forms $z = f(x, y)$, $y = f(x, z)$ or $x = f(y, z)$, where $f$ is a mooth function on a open subset $V$ of $\mathbb{R}^2$.*

*Proof.* Let $\varphi : U \to \mathbb{R}^3$ be a parameterization of $S$ around $p = \varphi(u_0, v_0)$, where $(u_0, v_0) \in U$. Since $\varphi$ has rank 2 at $(u_0, v_0)$, there is a non-zero $2 \times 2$ minor of its Jacobian matrix at $(u_0, v_0)$. Without loss of generality, let us say that

$$\frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) \neq 0.$$

Denote the orthogonal projection from $\mathbb{R}^3$ onto the plane $z = 0$ by $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Then

$$J(\pi \circ \varphi)_{(u_0,v_0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \circ (J\varphi)_{(u_o,v_0)} = \frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) \neq 0.$$

Therefore we can apply the Inverse Mapping Theorem to $\pi \circ \varphi : U \to \mathbb{R}2$ at $(u_0, v_0)$. It says there exist an open neighborhood $\tilde{U}$ of $(u_0, v_0)$ in $\mathbb{R}^2$ and an open neighborhoof $V$ of $\pi \circ \varphi(u_0, v_0)$ in $\mathbb{R}^2$ such that $\pi \circ \varphi : \tilde{U} \to V$ is a bijection and its inverse is smooth.

We claim that $\varphi(\tilde{U})$ is the graph of the map

$$f = \pi' \circ \varphi \circ (\pi \circ \varphi|_{\tilde{U}})^{-1} : V \to \mathbb{R},$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}$ is the orthogonal projection onto the $z$-axis. Indeed: $(x, y, z)$ lies in the graph of $f$ if and only if $(x, y) \in V$ and $z = f(x, y)$. The latter is equivalent to

$$\begin{aligned} \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y) &= \left(\pi \circ \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y), \pi' \circ \varphi(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y)\right) \\ &= (x, y, z). \end{aligned}$$

But this says that $(x, y, z)$ is the image of $(\pi \circ \varphi|_{\tilde{U}})^{-1}(x, y)$ (an arbitrary point of $\tilde{U}$) under $\varphi$. This proves the claim and finishes the proof. □

**Example 2.3** The cone $C$ given by $z = \sqrt{x^2 + y^2}$ is not a surface near $(0, 0, 0)$. In fact, if $C$ were a surface near the origin, it would be locally graph there. It cannot be a graph of the form $x = f(y, z)$ or $y = f(x, z)$, since the projection of $C$ to the planes $x = 0$ and $y = 0$ are not injective. If it were a graph of the form $z = f(x, y)$, then we would need to have $f(x, y) = \sqrt{x^2 + y^2}$. But this function $f$ is not even differentiable at $0$.

## 2.4 Change of parameters

**Theorem 2.4** *Let $S \subset \mathbb{R}^3$ be a surface and let $\varphi : U \to \varphi(U)$, $\psi : V \to \psi(V)$ be two parameterizations of $S$, where $U, V \subset \mathbb{R}^2$ are open. Then the* change of parameters *$h = \varphi^{-1} \circ \psi : \psi^{-1}(\varphi(U)) \to \varphi^{-1}(\psi(V))$ is a diffeomorphism between open sets of $\mathbb{R}^2$.*

*Proof.* $h$ is a homeomorphism because it is the composite map of two homeomorphisms. Note that a similar argument cannot be used to say that $h$ is smooth, because it does not make sense (yet) to say that $\varphi^{-1}$ is smooth.

Let $p = (u_0, v_0) \in U$, $q \in V$, $\varphi(p) = \psi(q)$. Since $\varphi$ is an immersion, $d\varphi$ has rank two and we may assume WLOG that $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$. Write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in U$ and define

$$\Phi(u,v,w) = (x(u,v), y(u,v), z(u,v) + w),$$

where $(u,v,w) \in U \times \mathbb{R}$. Then $\Phi : U \times \mathbb{R} \to \mathbb{R}^3$ is smooth and

$$\det(d\Phi_{(u_0,v_0,0)}) = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{array} \right|_{(u_0,v_0,0)} = \frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0.$$

Since $d\Phi_{(u_0,v_0,0)}$ is non-singular, by the inverse function theorem, $\Phi^{-1}$ is defined and is smooth on some open neighborhood $W$ of $\varphi(p)$ in $\mathbb{R}^3$. Since $\Phi|_{U\times\{0\}} = \varphi$, we have that $\Phi^{-1}_{\varphi(U)\cap W} = \varphi^{-1}|_{\varphi(U)\cap W}$. Since $W \cap \varphi(U)$ is open in $S$ and $\psi$ is a homeomorphism, $\psi^{-1}(W \cap \varphi(U)) \subset V$ is open. Now

$$h|_{\psi^{-1}(W)} = \varphi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))} = \Phi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))}$$

is smooth, because it is the composite map of smooth maps.

Similarly, one sees that $h^{-1}$ is smooth by reversing the rôles of $\varphi$ and $\psi$ in the argument above. Hence $h$ is a diffeomorphism. □

**Corollary 2.5** *Let $S \subset \mathbb{R}^3$ be a surface and suppose $f : W \to \mathbb{R}^3$ is a smooth map defined on the open subset $W \subset \mathbb{R}^m$ such that $f(W) \subset S$. Then $\varphi^{-1} \circ f : W \to \mathbb{R}^2$ is smooth for every parameterization $\varphi : U \to \varphi(U)$ of $S$.*

*Proof.* If $\Phi$ is as in the proof of the theorem, we have that $\varphi^{-1} \circ f = \Phi^{-1} \circ f$ is the composite of smooth maps between Euclidean spaces. □

As an application of the smoothness of change of parameters, we can make the following definition. Let $S$ be a surface. An application $f : S \to \mathbb{R}^n$ is smooth at a point $p \in S$ if $f \circ \varphi : U \to \mathbb{R}^n$ is smooth at $\varphi^{-1}(p) \in U$, for some parameterization $\varphi : U \to \varphi(U)$ of $S$ with $p \in \varphi(U)$. Note that if $\psi : V \to \psi(V)$ is another parameterization of $S$ with $p \in \psi(V)$, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is smooth at $\varphi^{-1}(p)$, because

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi^{-1} \circ \psi)$$

and the change of parameters $\varphi^{-1} \circ \psi$ is smooth.

**Example 2.6** If $S$ is a surface and $F : \mathbb{R}^3 \to \mathbb{R}$ is smooth, then the restriction $f = F|_S : S \to \mathbb{R}$ is smooth. In fact, $f \circ \varphi = F \circ \varphi$ is smooth for any parameterization $\varphi$ of $S$. As special cases, we can take the *height function* relative to $a$, $F(x) = \langle x, a \rangle$, where $a \in \mathbb{R}^3$ is a fixed vector; or the *distance function* from $q$, $F(x) = ||x - q||^2$, where $q \in \mathbb{R}^3$ is a fixed point.

In particular, if $f : S \to \mathbb{R}^3$ is smooth at $p \in S$ and $\tilde{S} \subset \mathbb{R}^3$ is a surface such that $f(S) \subset \tilde{S}$, then we say that $f : S \to \tilde{S}$ is smooth at $p$.

## 2.5 Tangent plane

Let $S \subset \mathbb{R}^3$ be a surface. Recall that a smooth curve $\gamma : I \subset \mathbb{R} \to S$ is simply a smooth curve $\gamma : I \to \mathbb{R}^3$ such that $\gamma(I) \subset S$. Fix a point $p \in S$. A *tangent vector* to $S$ at $p$ is the tangent vector $\dot{\gamma}(0)$ to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. The *tangent plane* to $S$ at $p$ is the collection of all tangent vectors to $S$ at $p$.

**Proposition 2.7** *The tangent space $T_pS$ is the image of the differential*

$$d\varphi_a : \mathbb{R}^2 \to \mathbb{R}^3, \tag{2.8}$$

*where $\varphi : U \to \varphi(U)$ is any parameterization of $S$ with $p = \varphi(a)$ and $a \in U$.*

*Proof.* Any vector in the image of (2.8) is of the form $d\varphi_a(w_0)$ for some $w_0 \in \mathbb{R}^2$ and therefore is the tangent vector at $0$ of the smooth curve $t \mapsto \varphi(a + tw_0)$.

Conversely, suppose $w = \dot{\gamma}(0)$ is tangent to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. By Corollary 2.5, $\eta := \varphi^{-1} \circ \gamma : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$ is a smooth curve in $\mathbb{R}^2$ with $\eta(0) = a$. Note that $\gamma = \varphi \circ \eta$. By the chain rule

$$v = d\varphi_a(\dot{\eta}(0)) \tag{2.9}$$

lies in the image of (2.8). ☐

**Corollary 2.10** *The tangent plane $T_pS$ is a 2-dimensional vector subspace of $\mathbb{R}^3$. For any parameterization $\varphi : U \to \varphi(U)$ of $S$ with $p = \varphi(a)$, $a \in U$,*

$$\{\tfrac{\partial \varphi}{\partial u}(a), \tfrac{\partial \varphi}{\partial v}(a)\} \tag{2.11}$$

*is a basis of $T_pS$.*

It is also convenient to write $\varphi_u := \frac{\partial \varphi}{\partial u}$ and $\varphi_v := \frac{\partial \varphi}{\partial v}$.

Consider a tangent vector $w \in T_pS$, say $w = \dot{\gamma}(0)$ where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, as in the proof of Proposition 2.7. Then $\eta = \varphi^{-1} \circ \gamma$ is a smooth curve in $\mathbb{R}^2$ which we may write as $\eta(t) = (u(t), v(t))$. Since $\dot{\eta}(0) = (u'(0), v'(0))$, eqn. (2.9) yields that

$$w = u'(0)\varphi_u(a) + v'(0)\varphi_v(a),$$

namely, $u'(0)$, $v'(0)$ are the coordinates of $w$ in the basis (2.11). This remark also shows that another smooth curve $\bar{\gamma} : (-\epsilon, \epsilon) \to S$ represents the same $w$ if and only if $\bar{\eta}(t) = \varphi^{-1} \circ \bar{\gamma}(t) = (\bar{u}(t), \bar{v}(t))$ satisfies $(\bar{u}'(0), \bar{v}'(0)) = (u'(0), v'(0))$.

With the same notation as above, suppose now that $f : S \to \tilde{S}$ is a smooth map at $p \in S$. Note that $f \circ \gamma$ is a smooth curve in $\tilde{S}$. The *differential* of $f$ at $p$ is the map

$$df_p : T_pS \to T_{f(p)}\tilde{S}$$

that maps $w = \dot{\gamma}(0) \in T_pS$ to the tangent vector $\dot{\tilde{\gamma}}(0)$, where $\tilde{\gamma} = f \circ \gamma$. We check that $df_p(w)$ does not depend on the choice of curve $\gamma$. Let $\varphi : U \to \varphi(U) = V$,

$\tilde{\varphi} : \tilde{U} \to \tilde{\varphi}(\tilde{U}) = \tilde{V}$ be parameterizations of $S$, $\tilde{S}$, resp., with $p = \varphi(a)$, $a \in U$, $f(p) = \tilde{\varphi}(\tilde{a})$, $\tilde{a} \in \tilde{U}$, and such that $f(V) \subset \tilde{V}$. Consider the local representation of $f$,

$$g = \tilde{\varphi}^{-1} \circ f \circ \varphi : U \to \tilde{U},$$

and write

$$g(u, v) = (g_1(u, v), g_2(u, v))$$

for $(u, v) \in U \subset \mathbb{R}^2$. Then

$$\tilde{\gamma}(t) = \tilde{\varphi}\,(\, g_1(u(t), v(t)), g_2(u(t), v(t))\,)\,,$$

so

$$\dot{\tilde{\gamma}}(0) = \left(\tfrac{\partial g_1}{\partial u}\, u'(0) + \tfrac{\partial g_1}{\partial v}\, v'(0)\right) \tilde{\varphi}_{\tilde{u}} + \left(\tfrac{\partial g_2}{\partial u}\, u'(0) + \tfrac{\partial g_2}{\partial v}\, v'(0)\right) \tilde{\varphi}_{\tilde{v}}.$$

This relation shows that $\dot{\tilde{\gamma}}(0)$ depends only on $u'(0)$, $v'(0)$ and hence has the same value for any smooth curve representing $w$. This relation can also be rewritten as

$$df_p(w) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix},$$

which shows that $df_p$ is a linear map whose matrix with respect to the bases $\{\varphi_u, \varphi_v\}$, $\{\tilde{\varphi}_{\tilde{u}}, \varphi_{\tilde{v}}\}$ is the 2 by 2 matrix above.

**Example 2.12** If $S$ is a surface given as the inverse image under $F : \mathbb{R}^3 \to \mathbb{R}$ of a regular value, then $T_pS = \ker(dF_p)$ for every $p \in S$. In fact, if $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, then $F(\gamma(t))$ is constant for $t \in (-\epsilon, \epsilon)$. By the chain rule, $dF_p(\dot{\gamma}(0)) = 0$. This proves the inclusion $T_pS \subset \ker(dF_p)$ and hence the equality by dimensional reasons.

**Example 2.13** Let $S$ be a surface and consider the height function $h : S \to \mathbb{R}$ for a fixed unit vector $\xi \in S^2$, given by $h(x) = x \cdot \xi$ for $x \in S$. We compute the differential $dh_p : T_pS \to \mathbb{R}$ at $p \in S$. Given $w \in T_pS$, there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = w$. Now

$$dh_p(w) = \frac{d}{dt}\Big|_{t=0} h(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \gamma(t) \cdot \xi = \dot{\gamma}(0) \cdot \xi = w \cdot \xi.$$

In particular, $p \in S$ is a critical point of $h$ if and only if $\xi$ is normal to $S$ at $p$.

# Chapter 3

# Surfaces: local theory

## 3.1 The first fundamental form

Let $S \subset \mathbb{R}^3$ be a surface. The *first fundamental form* of $S$ is just the restriction $I$ of the dot product of $\mathbb{R}^3$ to the tangent spaces of $S$:

$$I_p = \langle \cdot, \cdot \rangle|_{T_pS \times T_pS}.$$

If $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) = V \subset S$ is a parametrization, we already know that $\{\varphi_u(u,v), \varphi_v(u,v)\}$ is a basis of $T_{\varphi(u,v)}S$. Then any tangent vector to $S$ at a point in $V$ can be written $w = a\varphi_u + b\varphi_v$ and thus

$$I(w,w) = a^2 \underbrace{I(\varphi_u,\varphi_u)}_{E(u,v)} + 2ab \underbrace{I(\varphi_u,\varphi_v)}_{F(u,v)} + b^2 \underbrace{I(\varphi_v,\varphi_v)}_{G(u,v)}.$$

$E$, $F$, $G$ are smooth functions on $U$, the so called coefficients of the first fundamental form. If $\{du, dv\}$ denotes the dual basis of $\{\varphi_u, \varphi_v\}$, then $a = du(w)$, $b = dv(w)$ and we can write

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

This is the local expression of $I$ with respect to $\varphi$. The matrix associated to this bilinear form is

$$(I) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

**Examples 3.1** 1. A plane can be parametrized by $\varphi(u,v) = p + uw_1 + vw_2$, where $p$ is a point in $\mathbb{R}^3$, $\{w_1, w_2\}$ is an orthonormal set of vectors in $\mathbb{R}^3$, and $(u,v) \in \mathbb{R}^2$. We have $\varphi_u = w_1$, $\varphi_v = w_2$, so $E = G = 1$, $F = 0$ and $I = du^2 + dv^2$.

2. The (right circular) cilinder can be parametrized by $\varphi(u,v) = (\cos u, \sin u, v)$, where $u_o < u < u_0 + 2\pi$ and $v \in \mathbb{R}$. In this case, $\varphi_u = (-\sin u, \cos u, 0)$, $\varphi_v = (0,0,1)$, so $E = G = 1$, $F = 0$ and $I = du^2 + dv^2$.

3. The helicoid is the union of horizontal lines joining the $z$-axis to the points of an helix, namely, it is the image of the parametrization $\varphi(u,v) = (v\cos u, v\sin u, au)$ $(a>0)$. We see that $I = (a^2+v^2)du^2 + dv^2$.

4. Spherical coordinates $\varphi(u,v) = (\sin u, \cos v, \sin u \sin v, \cos u)$, where $0 < u < \pi/2$, $v_0 < v < v_0 + 2\pi$, yield that $I = du^2 + \sin^2 u dv^2$ on the sphere.

**Remark 3.2** If two surfaces $S_1$, $S_2$ can be covered by open sets which are images of parametrizations such that the local expressions of the first fundamental forms of $S_1$, $S_2$ coincide on corresponding open sets (like in the cases of the plane and the cilinder), then $S_1$ and $S_2$ are called *locally isometric*.

Using the first fundamental form, we can define:
**Length** of a smooth curve $\gamma : (a,b) \to S$ by

$$L(\gamma) = \int_a^b I(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt.$$

**Angle** between two vectors $w_1, w_2 \in T_pS$ by

$$\cos\angle(w_1,w_2) = \frac{I_p(w_1,w_2)}{I_p(w_1,w_1)^{1/2} I_p(w_2,w_2)^{1/2}}.$$

**Surface integral** of a compactly supported function $f : S \to \mathbb{R}$. If the support $D$ of $f$ is contained in the image of a parametrization $\varphi : U \to S$, then

$$\int\int_D f\, dS = \int\int_{\varphi^{-1}(D)} f(\varphi(u,v))\, ||\varphi_u \times \varphi_v||\, dudv,$$

where the left hand side is a double integral. Taking another parametrization $\bar\varphi : \bar U \to S$, the change of paramters $(u,v) = (\varphi^{-1} \circ \bar\varphi)(\bar u, \bar v)$ is smooth with Jacobian determinant $\frac{\partial(u,v)}{\partial(\bar u,\bar v)}$. Note that

$$||\bar\varphi_{\bar u} \times \bar\varphi_{\bar v}|| = \left|\frac{\partial(u,v)}{\partial(\bar u,\bar v)}\right| ||\varphi_u \times \varphi_v||\, dudv.$$

The formula of change of variables in the double integral yields that

$$\begin{aligned} &\int\int_{\bar\varphi^{-1}(D)} f(\bar\varphi(\bar u,\bar v))\, ||\bar\varphi_{\bar u} \times \bar\varphi_{\bar v}||\, d\bar u d\bar v \\ = &\int\int_{\bar\varphi^{-1}(D)} f(\bar\varphi(\bar u,\bar v))\, ||\varphi_u \times \varphi_v|| \left|\frac{\partial(u,v)}{\partial(\bar u,\bar v)}\right| d\bar u d\bar v \\ = &\int\int_{\varphi^{-1}(D)} f(\varphi(u,v))\, ||\varphi_u \times \varphi_v||\, dudv, \end{aligned}$$

so the definition is independent of the choice of parametrization. In general, onde needs to cover the support of $f$ by finitely many parametrizations and define the surface integral as a sum of double integrals. The proof of independence of the choices involved is more complicated in this case, and we will not go into details. The relation to the first fundamental form is that

$$||\varphi_u \times \varphi_v||^2 = ||\varphi_u||^2||\varphi_v||^2 - \langle \varphi_u, \varphi_v \rangle^2,$$

so

$$||\varphi_u \times \varphi_v|| = \sqrt{EG - F^2}.$$

**Area** of a compact domain $D \subset S$ is

$$\text{area}(D) = \int\int_D 1\, dS.$$

In particular, if $D$ is contained in the image of $\varphi$,

$$\text{area}(D) = \int\int_{\varphi^{-1}(D)} \sqrt{EG - F^2}\, dudv.$$

## 3.2 The Gauss map and the second fundamental form

Let $S \subset \mathbb{R}^3$ be a surface. For each $p \in S$, we want to assign a unit vector $\nu(p)$ which is normal to $T_pS$; note that there are exactly two possible choices. If it is possible to make such an assigment continuously along the whole of $S$, we say that $S$ is *orientable*. The resulting map

$$\nu : S \to S^2$$

into the unit sphere is called the *Gauss map*. We will always assume that the Gauss map is continuous.

**Examples 3.3** 1. If $\varphi : U \to S$ is a parametrization, then we can take

$$\nu = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}.$$

This construction shows that every surface is locally orientable. It also shows that the Gauss map is smooth.

2. If $S$ is given as the inverse image under $F$ of regular value, then we can take

$$\nu = \frac{\nabla F}{||\nabla F||}.$$

The differential

$$d\nu_p : T_pS \to T_{\nu(p)}S^2 = T_pS$$

is an operator on $T_pS$, since $T_qS^2$ is always normal to $q$, for $q \in S^2$. The operator

$$-d\nu_p : T_pS \to T_pS$$

is called the *Weingarten operator*.

**Proposition 3.4** *The Weingarten operator is symmetric:*

$$\langle d\nu_p(w_1), w_2\rangle = \langle w_1, d\nu_p(w_2)\rangle,$$

*where* $w_1$, $w_2 \in T_pS$.

*Proof*. By linearity, it suffices to check the relation for a basis of $T_pS$. Let $\varphi : U \to S$ be a parametrization; then $\{\varphi_u, \varphi_v\}$ is a tangent frame. Set $N = \nu\circ\varphi$. Then

$$d\nu(\varphi_u) = d\nu\left(\tfrac{\partial\varphi}{\partial u}\right) = \tfrac{\partial}{\partial u}(\nu\circ\varphi) = \tfrac{\partial N}{\partial u}.$$

Similarly $d\nu(\varphi_v) = \frac{\partial N}{\partial v}$. Since $\langle\frac{\partial\varphi}{\partial v}, N\rangle = 0$ on $U$, differentiating with respect to $u$,

$$\langle\frac{\partial^2\varphi}{\partial u\partial v}, N\rangle + \langle\tfrac{\partial\varphi}{\partial v}, \tfrac{\partial N}{\partial u}\rangle = 0,$$

and similarly

$$\langle\frac{\partial^2\varphi}{\partial v\partial v}, N\rangle + \langle\tfrac{\partial\varphi}{\partial u}, \tfrac{\partial N}{\partial v}\rangle = 0,$$

Taking the difference of these equations,

$$\langle\tfrac{\partial\varphi}{\partial v}, \tfrac{\partial N}{\partial u}\rangle - \langle\tfrac{\partial\varphi}{\partial u}, \tfrac{\partial N}{\partial v}\rangle = 0,$$

which says that

$$\langle\varphi_v, d\nu(\varphi_v)\rangle - \langle\varphi_u, d\nu(\varphi_v)\rangle = 0,$$

as we wished. □

The associated symmetric bilinear form

$$II_p(w_1, w_2) = -\langle d\nu_p(w_1), w_2\rangle$$

is called the *second fundamental form*.

**Proposition 3.5** *Let* $\gamma : (-\epsilon, \epsilon) \to S$ *be a smooth curve parametrized by arc-length,* $\gamma(0) = p$, $\gamma'(0) = w \in T_pS$. *Then*

$$\langle\gamma''(0), \nu(p)\rangle = II_p(w, w).$$

*Proof*. Start with the equation $\langle\gamma'(s), \nu(\gamma(s))\rangle = 0$ and differentiate it at $s = 0$ to obtain

$$\langle\gamma''(0), \nu(p)\rangle + \langle w, \frac{d}{ds}|_{s=0}\nu(\gamma(s))\rangle = 0.$$

Then $\langle\gamma''(0), \nu(p)\rangle = -\langle w, d\nu_p(w)\rangle = II_p(w, w)$, as desired. □

There is a geometric interpretation of last proposition (Meusnier). Given a unit vector $w \in T_pS$, the affine plane through $p$ spanned by $w$ and $\nu(p)$ meets $S$ transversally along a curve which is called the *normal section of* $S$ along $w$. If $\gamma$

is a parametrization by arc-length of this normal section as in the proposition, then the curvature of $\gamma$ at $p = \gamma(0)$ is

$$\kappa_w = \langle \gamma''(0), \nu(p) \rangle = II_p(w, w),$$

where we view $\gamma$ as a plane curve and we view its supporting plane as oriented by $\{w, \nu(p)\}$.

Since the Weingarten operator $-d\nu_p$ is symmetric, there exists an orthonormal basis $\{e_1, e_2\}$ of $T_pS$ such that

$$-d\nu_p(e_1) = \kappa_1 e_1, \qquad -d\nu_p(e_2) = \kappa_2 e_2.$$

The eigenvalues $\kappa_1$, $\kappa_2$ are called *principal curvatures* at $p$, and the eigenvectors $e_1$, $e_2$ are called *principal directions* at $p$. Of course, $II(e_1, e_1) = \kappa_1$, $II(e_2, e_2) = \kappa_2$, $II(e_1, e_2) = 0$. It follows that for a unit vector $w = \cos\theta e_1 + \sin\theta e_2 \in T_pS$ we have *Euler's formula*:

$$\kappa_w = I_p(w, w) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Since this a convex linear combination ($\sin^2\theta + \cos^2\theta = 1$), Euler's formula also shows that $\kappa_1$, $\kappa_2$ are the extrema of the curvatures of the normal sections through $p$.

In general, a smooth curve $\gamma$ in $S$ is called a *line of curvature* if $\dot\gamma(t)$ is a principal direction at $\gamma(t)$ for all $t$. Note that if $\kappa_1(p) \neq \kappa_2(p)$, then the principal directions at $p$ are uniquely defined, but not otherwise. If $\kappa_1(p) = \kappa_2(p)$, we say that $p$ is an *umbilic point* of $S$.

**Proposition 3.6** *If all the points of a connected surface $S$ are umbilic, then $S$ is contained in a plane or a sphere.*

*Proof*. We first prove the result in case $S$ is the image $V$ of a parametrization $\varphi : U \to V$. Set $N = \nu \circ \varphi$. By assumption, $d\nu = \lambda \cdot I$, where $\lambda : V \to \mathbb{R}$ is a smooth function. It follows that

$$\begin{aligned} N_u = d\nu(\varphi_u) &= (\lambda \circ \varphi)\varphi_u, \\ N_v = d\nu(\varphi_v) &= (\lambda \circ \varphi)\varphi_v. \end{aligned}$$

We differentiate the first (resp. second) of these equations with respect to $v$ (resp. $u$) to get

$$\begin{aligned} N_{uv} &= (\lambda \circ \varphi)_v\varphi_u + (\lambda \circ \varphi)\varphi_{uv}, \\ N_{vu} &= (\lambda \circ \varphi)_u\varphi_v + (\lambda \circ \varphi)\varphi_{vu}. \end{aligned}$$

Taking the difference,

$$0 = (\lambda \circ \varphi)_v\varphi_u - (\lambda \circ \varphi)_u\varphi_v.$$

Since $\{\varphi_u, \varphi_v\}$ is linearly independent, the partial derivatives of $\lambda \circ \varphi$ on $U$ are zero. Since $U$ is connected, $\lambda \circ \varphi$ is constant.

Next, we consider two cases. If $\lambda \circ \varphi = 0$, then $N_u = N_v = 0$ so $N$ is constant. This implies $\frac{\partial}{\partial u}\langle \varphi, N\rangle = \frac{\partial}{\partial v}\langle \varphi, N\rangle = 0$ and hence $V$ is contained in an affine plane parallel to $\langle N\rangle^\perp$. On the other hand, if $\lambda \circ \varphi = R \neq 0$, then $\varphi - \frac{1}{R} N$ is a constant $q \in \mathbb{R}^3$ and hence $V$ is contained in the sphere of center $q$ and radius $1/|R|$.

In the case of arbitrary $S$, fix a point $p \in S$ and a parametrized neighborhood $V_0$ of $p$. By the previous case, $V_0$ is contained in a plane or a sphere. Given $x \in S$, by connectedness of $S$, there exists a continuous curve $\gamma : [0,1] \to S$ joining $p$ to $x$ ($S$ is locally arcwise connected, so it is arcwise connected). For any $t \in [0,1]$, there exists a parametrized neighborhood of $\gamma(t)$ which is contained in a plane or a sphere. By compactness of $\gamma([0,1])$, it is possible to cover it by open sets $V_0, V_1, \ldots, V_n$ such that each $V_i$ is contained in a plane or a sphere and $V_i \cap V_{i+1} \neq \varnothing$ (check this!); the latter condition implies that $V_{i+1}$ is contained in the same plane or sphere that contains $V_i$. The result follows. □

## 3.3 Curvature of surfaces

Let $S \subset \mathbb{R}^3$ and consider its Weingarten operator $-d\nu_p : T_pS \to T_pS$. Recall that $-d\nu_p$ is symmetric and its eigenvalues $\kappa_1(p)$, $\kappa_2(p)$ are the principal curvatures of $S$ at $p$. We define:

$$\begin{aligned} \textit{Gaussian curvature} : K(p) &= \det(-d\nu_p) = \kappa_1(p)\cdot\kappa_2(p), \\ \textit{Mean curvature} : H(p) &= \frac{1}{2}\mathrm{trace}\,(-d\nu_p) = \frac{1}{2}\left(\kappa_1(p)+\kappa_2(p)\right). \end{aligned}$$

Note that $\kappa_1, \kappa_2 = H \pm \sqrt{H^2 - K}$; we will soon see that $H$ and $K$ are smooth functions on $S$, and so it follows from this equation that $\kappa_1$, $\kappa_2$ are continuous functions on $S$ which are smooth away from umbilic points (points characterized by $H^2 = K$).

If we change $\nu$ to $-\nu$, then $H$ is changed to $-H$ but $K$ is unchanged. The next example analyses the meaning of the sign of $K$.

**Examples 3.7** 1. Let us compute the Gaussian curvature of the graph $S$ of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open. In general, for a parametrization $\varphi$ and $N = \nu \circ \varphi$,

$$II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u\rangle = -\langle N_u, \varphi_u\rangle = \langle N, \varphi_{uu}\rangle.$$

Similarly,

$$II(\varphi_u, \varphi_v) = -\langle N_u, \varphi_v\rangle = \langle N, \varphi_{uv}\rangle$$

and

$$II(\varphi_v, \varphi_v) = -\langle N_v, \varphi_v\rangle = \langle N, \varphi_{vv}\rangle.$$

In our case,

$$\begin{aligned} \varphi_u &= (1,0,f_u), \\ \varphi_v &= (0,1,f_v), \\ \varphi_{uu} &= (0,0,f_{uu}), \\ \varphi_{uv} &= (0,0,f_{uv}), \\ \varphi_{vv} &= (0,0,f_{vv}), \\ N &= \frac{(-f_u,-f_v,1)}{\sqrt{1+f_u^2+f_v^2}}. \end{aligned}$$

Hence

$$\begin{aligned} (II) &= \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \\ &= \frac{1}{\sqrt{1+f_u^2+f_v^2}} \left(\mathrm{Hess}(f)\right). \end{aligned}$$

We specialize to the case $p=(0,0,0)=f(0,0)$ and $T_pS$ is the $xy$-plane. Then $f_u(0,0)=f_v(0,0)=0$ and $II_p=\mathrm{Hess}_{(0,0)}(f)$. In particular, if $f(u,v)=au^2+bv^2$, then

$$(II)=\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

It follows that $K(p)=4ab$ is positive (resp. negative) if $a$ and $b$ have the same sign (resp. opposite signs), and it is zero if one of $a$, $b$ is zero.

Another interesting case is $f(u,v)=u^4+v^4$. We get $II=0$.

2. Consider the sphere $S^2(R)$ of radius $R>0$. We can take $\nu(p)=-\frac{1}{R}\,p$, so $-d\nu_p=\frac{1}{R}\,\mathrm{id}_{T_pS}$ and $K(p)=\frac{1}{R^2}>0$, $H(p)=\frac{1}{R}$.

A point $p$ in a surface $S$ is called *elliptic* (resp. *hyperbolic*, *parabolic*) if $K(p)>0$ (resp. $K(p)<0$, $K(p)=0$).

## 3.4 Local expressions for $K$, $H$

Fix a parametrization $\varphi$ of $S$. Then $\{\varphi_u,\varphi_v\}$ is a tangent frame with respect to which we consider the matrices of the fundamental forms and the Weingarten operator and introduce a new (index) notation for the coefficients.

$$(I)=\begin{pmatrix} \langle\varphi_u,\varphi_u\rangle & \langle\varphi_u,\varphi_v\rangle \\ \langle\varphi_v,\varphi_u\rangle & \langle\varphi_v,\varphi_v\rangle \end{pmatrix}=\begin{pmatrix} E & F \\ F & G \end{pmatrix}=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

$$(II)=\begin{pmatrix} \langle N,\varphi_{uu}\rangle & \langle N,\varphi_{uv}\rangle \\ \langle N,\varphi_{vu}\rangle & \langle N,\varphi_{vv}\rangle \end{pmatrix}=\begin{pmatrix} \ell & m \\ m & n \end{pmatrix}=\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$(-d\nu)=\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}.$$

We have that

$$h_{11} = II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u\rangle = \langle h_1^1\varphi_u + h_1^2\varphi_v, \varphi_u\rangle,$$

so

$$h_{11} = h_1^1 g_{11} + h_1^2 g_{21}.$$

Similarly,

$$\begin{aligned} h_{12} &= h_1^1 g_{12} + h_1^2 g_{22}, \\ h_{21} &= h_2^1 g_{11} + h_2^2 g_{21}, \\ h_{22} &= h_2^1 g_{12} + h_2^2 g_{22}. \end{aligned}$$

In matrix form.

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Recall that $(I)$ is invertible since it is positive definite. Thus

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.$$

Back to the classical notation,

$$\begin{aligned} (-d\nu) &= \frac{1}{EG - F^2} \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} \ell G - mF & -\ell F + mE \\ mG - nF & -mF + nE \end{pmatrix}. \end{aligned}$$

Hence

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{\det(II)}{\det(I)}$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)}.$$

It follows from these expressions that $K$, $H$ are smooth on $S$.

## 3.5 Surfaces of revolution

Consider the parametrized surface of revolution

$$\varphi(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u, v) \in (a, b) \times (v_0, v_0 + 2\pi)$ and the geratrix $\gamma(s) = (f(s), 0, g(s))$ is parametrized by arc length. Then

$$\varphi_u = (f'\cos v, f'\sin v, g'), \qquad \varphi_v = (-f\sin v, f\cos v, 0),$$

so

$$E = f'^2 + g'^2 = 1, \quad F = 0, \quad G = f^2.$$

Moreover,

$$\begin{aligned} \varphi_{uu} &= (f'' \cos v, f'' \sin v, g''), \\ \varphi_{uv} &= (-f' \sin v, f' \cos v, 0), \\ \varphi_{vv} &= (-f \cos v, -f \sin v, 0). \end{aligned}$$

We compute the coefficients of $II$.

$$\begin{aligned} \ell &= \langle N, \varphi_{uu} \rangle = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uu} \rangle}{||\varphi_u \times \varphi_v||} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} f' \cos v & f' \sin v & g' \\ -f \sin v & f \cos v & 0 \\ f'' \cos v & f'' \sin v & g'' \end{vmatrix} \\ &= f'g'' - f''g'. \end{aligned}$$

Similarly,

$$m = \langle N, \varphi_{uv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f' \sin v & f' \cos v & 0 \end{vmatrix} = 0$$

and

$$n = \langle N, \varphi_{vv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f \cos v & -f \sin v & 0 \end{vmatrix} = fg'.$$

Note that $f'g'' - f''g'$ is the signed curvature $\kappa_\gamma$ of $\gamma$, for $\kappa_\gamma = \langle \gamma'', (-g', 0, f') \rangle$. Now

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{(f'g'' - f''g')g'}{f} = \kappa_\gamma \frac{g'}{f} \tag{3.8}$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{1}{2} \left( \kappa_\gamma + \frac{g'}{f} \right). \tag{3.9}$$

It follows that the principal curvatures

$$\kappa_1 = \kappa_\gamma, \qquad \kappa_2 = \frac{g'}{f}.$$

The identities $F = m = 0$ mean that the fundamental forms are diagonalized in the frame $\{\varphi_u, \varphi_v\}$. In particular, $\varphi_u$, $\varphi_v$ are always principal directions and thus the curves $u$-constant (parallels) and $v$-constant (meridians) are lines of curvature.

We can also derive some other useful formulas for $K$, $H$. Differentiation of $f'^2 + g'^2 = 1$ gives $f'f'' + g'g'' = 0$. Substituting this identity into (3.8) yields

$$K = \frac{-f''}{f}.$$

The identity also gives $\kappa_1 = \kappa_\gamma = \frac{g''}{f'}$ if $f' \neq 0$, so

$$H = \frac{1}{2}\left(\frac{g''}{f'} + \frac{g'}{f}\right) = \frac{(fg')'}{(f^2)'}. \tag{3.10}$$

We use this formula to prove the following theorem. A surface satisfying $H \equiv 0$ is called *minimal*. This terminology will be explained in section 3.7.

**Theorem 3.11** *The only minimal surfaces of revolution are the plane and the catenoid (the surface of revolution generated by a catenary, which is the graph of hyperbolic cosine).*

*Proof.* We use the above notation. If $f' = 0$ on an interval, then eqn. (3.9) gives $g' = 0$, which is a contradiction to the fact that $\gamma$ is regular. Therefore we can assume that $f'$ is never zero. By formula (3.10), we need to solve the equation $fg' = k$, where $k$ is a constant. Using $g' = \pm\sqrt{1 - f'^2}$, we get

$$f' = \pm\sqrt{1 - (k/f)^2}.$$

Note that $|f| \geq |k|$ is a necessary condition. This equation can be easily integrated by rewriting it as

$$\frac{f\,df}{\sqrt{f^2 - k^2}} = \pm ds.$$

We get

$$f(s) = \pm\sqrt{k^2 + (s + c_1)^2}.$$

The constant $c_1$ can be chosen to be zero by redefining the instant $s = 0$, and we recall that $f > 0$, so we have

$$f(s) = \sqrt{k^2 + s^2}.$$

If $k = 0$ then $f(s) = \pm s$ and $g$ is constant, which corresponds to the case of the plane. Suppose $k \neq 0$ and integrate $g' = k/f$ to get

$$g(s) = k \log(s + \sqrt{k^2 + s^2}) + c_2.$$

We choose the constant $c_2 = -k \log|k|$ so that $\gamma(0) = (|k|, 0, 0)$. Changing the sign of $k$ is equivalent to changing the sign of $g$, which corresponds to a reflection on the plane $z = 0$, so we may assume $k > 0$. Finally, we make the change of variable

$$t = g(s) = k \log(\sqrt{1 + (s/k)^2} + s/k)$$

to get

$$\gamma(t) = (k \cosh(t/k), 0, t)$$

which is a catenary. □

## 3.6 Ruled surfaces

A ruled surface is a surface generated by a smooth one-parameter family of lines. More precisely, a (nonnecessarily regular) parametrized surface $\varphi : U \subset \mathbb{R}^2 \to S$ is called a *ruled surface* if there exist a smooth curves $\gamma : I \to \mathbb{R}^3$ and $w : I \to S^2$ such that

$$\varphi(u, v) = \gamma(u) + v\, w(u)$$

where $(u, v) \in I \times \mathbb{R} = U$. The curve $\gamma$ is called a *directrix* and the lines $\mathbf{R}w(u)$ are called the *rulings*.

Obvious examples of ruled surfaces are planes, cilinders and cones. Other examples are the helicoid, the one-sheeted hyperboloid and the hyperbolic paraboloid (given by the equation $z = xy$ in $\mathbb{R}^3$).

We make some local considerations. Assume that $w'(u) \neq 0$ for all $u$, in other words, $w$ is regular; this condition is sometimes expressed by saying that the ruled surface is *noncylindrical*. Then it is possible to introduce the so called *standard parameters* on $S$.

**Proposition 3.12** *There exists a unique reparametrization*

$$\tilde\varphi(\tilde u, \tilde v) = \tilde\gamma(\tilde u) + \tilde v\, \tilde w(\tilde u)$$

*such that* $||\tilde w'|| = 1$ *and* $\langle \tilde\gamma', \tilde w' \rangle = 0$.

*Proof*. Since $w$ is regular, we can introduce arc-length parameter $\tilde u$ so that $\tilde w(\tilde u) = w(u(\tilde u))$, and then $||\tilde w'|| = 1$. Next, we write $\tilde\gamma(\tilde u) = \gamma(\tilde u) - \tilde v(\tilde u)\tilde w(\tilde u)$ and impose the condition $\langle \tilde\gamma', \tilde w' \rangle = 0$ to get $\tilde v(\tilde u) = -\langle \frac{d}{d\tilde u}\gamma, \tilde w'(u)\rangle$. □

The curve $\tilde\gamma$ is called the *striction line* of the surface and its points are called *central points* of the surface; note that $\tilde\gamma$ is not necessarily regular.

Using the standard parametrization, we can compute the curvature of a ruled surface

$$\varphi(u, v) = \gamma(u) + v\, w(u)$$

where $||w|| = ||w'|| = 1$, $\langle \gamma', w' \rangle = 0$. We have

$$\varphi_u = \gamma' + v\, w', \qquad \varphi_v = w,$$

so

$$E = ||\gamma'||^2 + v^2, \quad F = \langle \gamma', w \rangle, \quad G = 1.$$

Since $w'$ is orthogonal to $w$ and $\gamma'$, there exists a smooth function $\lambda = \lambda(u)$, called the *distribution parameter*, such that

$$\gamma' \times w = \lambda\, w'. \tag{3.13}$$

It follows that

$$||\varphi_u \times \varphi_v|| = \sqrt{EG - F^2} = ||\gamma'||^2 - \langle \gamma', w \rangle^2 + v^2 = \lambda^2 + v^2.$$

In particular, the singular points of $\varphi$ occur along the striction line ($v = 0$) precisely when $\lambda(u) = 0$.

Next, we compute the coefficients of $II$. We have

$$\varphi_{uu} = \gamma'' + v\,w', \quad \varphi_{uv} = w', \quad \varphi_{vv} = 0.$$

This implies

$$m = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uv}\rangle}{||\varphi_u \times \varphi_v||} = \frac{\langle \lambda w' + v\,w \times w', w'\rangle}{\sqrt{\lambda^2 + v^2}} = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}$$

and

$$n = 0,$$

what is sufficient to get the formula for the Gaussian curvature:

$$K = \frac{\ell n - m^2}{EG - F^2} = -\frac{\lambda(u)^2}{(\lambda(u)^2 + v^2)^2}.$$

Note that $K \le 0$, and $K = 0$ precisely along the rulings that meet the striction line at a singular point ($\lambda(u) = 0$), except of course the singular point itself ($v \neq 0$). If $\lambda(u) \neq 0$, this formula also shows the striction line is characterized by the property that the maximum of $K$ along each ruling occurs exactly at the central point.

The computation of $\ell$ is more involved. We have

$$\begin{aligned} &\langle \varphi_u \times \varphi_v, \varphi_{uu}\rangle \\ &= \lambda\langle w', \gamma''\rangle + \lambda v\langle w', w''\rangle + v\langle w' \times w, \gamma''\rangle + v^2\langle w' \times w, w''\rangle. \end{aligned} \tag{3.14}$$

We analyse separately the four terms on the right-hand side. Introduce the parameter

$$J = \langle w \times w', w''\rangle.$$

Since $\{w, w', w \times w'\}$ is an orthonormal frame,

$$\gamma' = \langle \gamma', w\rangle\, w + \langle \gamma', w \times w'\rangle\, w \times w' = Fw + \lambda\, w \times w'$$

and

$$\langle w', \gamma''\rangle = -\langle w'', \gamma'\rangle = -F\langle w'', w\rangle - \lambda\langle w'', w \times w'\rangle = F - \lambda J,$$

where we have used $\langle w'', w\rangle = -\langle w', w'\rangle = -1$. Equation $||w'|| = 1$ also implies $\langle w', w''\rangle = 0$. In order to analyse the third term in eqn. (3.14), differentiate eqn. (3.13) to get

$$\gamma'' \times w + \gamma' \times w' = \lambda' w' + \lambda w'',$$

and multiply through by $w'$ to write

$$\langle \gamma'' \times w, w'\rangle = \lambda'.$$

Now eqn. (3.14) is

$$\langle \varphi_u \times \varphi_v, \varphi_{uu}\rangle = -Jv^2 - \lambda' v + \lambda(F - \lambda J)$$

and

$$\ell = \frac{-\lambda^2 J + \lambda' v + \lambda(F - \lambda J)}{\sqrt{\lambda^2 + v^2}}.$$

Hence

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = -\frac{Jv^2 + \lambda' v + \lambda(\lambda J + F)}{2(\lambda^2 + v^2)^{3/2}}.$$

**Example 3.15** The standard parametrization of the helicoid has $\gamma(u) = (0, 0, bu)$ and $w(u) = (\cos u, \sin u, 0)$. Since $\gamma' \times w = bw'$, the distribution parameter $\lambda = b$ is constant and $K = -b^2/(b^2 + v^2)^2$. Note that $F = 0$ and, since $w'' = -w$, we also have $J = 0$. Hence $H = 0$.

**Proposition 3.16** *The only minimal ruled surfaces are the plane and the helicoid.*

*Proof.* We have just seen that $H = 0$ says that

$$Jv^2 + \lambda' v + \lambda(\lambda J + F) = 0.$$

This is a quadratic polynomial in $v$ whose coefficients, being functions depending only on $u$, must vanish. It follows that $\lambda$ is constant and $J = \lambda F = 0$.

Since $J = 0$, $w''$ is a linear combination of $w$, $w'$. But $\langle w'', w' \rangle = 0$ and $\langle w'', w \rangle = -1$, so $w'' = -w$ and $w$ is a circle.

If $\lambda = 0$ then $II = 0$, which corresponds to the case of the plane. Suppose $\lambda \neq 0$. Then $F = 0$ implying that $\gamma' = \lambda\, w \times w'$. Differentiation of this equation yields $\gamma'' = 0$. It follows that $\gamma$ is a line perpendicular to the circle defined by $w$. Hence the surface is the helicoid. □

## 3.7 Minimal surfaces

Let $S \subset \mathbb{R}^3$ be a surface. We say that $S$ is a *minimal surface* if the mean curvature $H \equiv 0$. Historically speaking, this concept is related to the problem of characterizing the surface with smallest area spanned by a given boundary, a problem raised by Lagrange in 1760. The question of showing the existence of such a surface is called the *Plateau problem*, in honor of the Belgian physicist who performed experiments with soap films around 1850, and it was solved completely only in 1930, independently by Jesse Douglas and Tibor Radó.

In order to explain the relation between mean curvature and minimization of surface area, consider a parametrization $\varphi : U \subset \mathbb{R}^2 \to S$, $N = \nu \circ \varphi$ the induced unit normal, and a smooth function $f : U \to \mathbb{R}$. Then we can introduce the *normal variation* of $\varphi$ along $f$:

$$\varphi^\epsilon = \varphi + \epsilon f N.$$

Let us compute the first fundamental form of $\varphi^\epsilon$:

$$\varphi_u^\epsilon = \varphi_u + \epsilon(f_u N + f N_u), \quad \varphi_v^\epsilon = \varphi_v + \epsilon(f_v N + f N_v).$$

Since $\langle \varphi_u, N\rangle = \langle \varphi_v, N\rangle = 0$ and $\langle \varphi_u, N_u\rangle = -\ell$, $\langle \varphi_u, N_v\rangle = \langle \varphi_v, N_u\rangle = -m$, $\langle \varphi_v, N_v\rangle = -n$, we obtain

$$\begin{aligned} E^\epsilon &= E - 2\epsilon f\ell + O(\epsilon^2), \\ F^\epsilon &= F - 2\epsilon f m + O(\epsilon^2), \\ G^\epsilon &= G - 2\epsilon f n + O(\epsilon^2), \end{aligned}$$

where $O(\epsilon^2)$ denotes a continuous function satisfying $\lim_{\epsilon\to 0} O(\epsilon^2)/\epsilon = 0$. It follows that

$$\begin{aligned} E^\epsilon G^\epsilon - (F^\epsilon)^2 &= EG - F^2 - 2\epsilon f(\ell G + nE - 2mF) + O(\epsilon^2) \\ &= (EG - F^2)(1 - 4\epsilon f H) + O(\epsilon^2). \end{aligned}$$

Let now $D \subset U$ be a compact domain and introduce

$$A(\epsilon) = \mathrm{area}(\varphi^\epsilon(D)) = \int\int_D \sqrt{E^\epsilon G^\epsilon - (F^\epsilon)^2}\, dudv.$$

We have

$$\begin{aligned} A'(0) &= \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \int\int_D \left(E^\epsilon G^\epsilon - (F^\epsilon)^2\right)^{1/2} dudv \\ &= \int\int_D \frac{\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0} E^\epsilon G^\epsilon - (F^\epsilon)^2}{2(EG - F^2)^{1/2}}\, dudv. \end{aligned}$$

Hence

$$A'(0) = -2\int\int_D fH\sqrt{EG - F^2}\, dudv.$$

This formula is called *first variation of surface area*. As a corollary, we obtain the following characterization of minimal surfaces as critical points of the area functional.

**Proposition 3.17** *A surface $S$ is minimal if and only if $A'(0) = 0$ for every parametrization $\varphi : U \to S$, every normal variation of $\varphi$, and every compact domain $D \subset U$.*

*Proof.* If $H(p) \neq 0$ for some $p \in S$, we choose a compact neighborhood $\tilde{D}$ of $p$ in $S$ such that $H$ does not vanish on $\tilde{D}$ and $\tilde{D}$ is contained in the image of a parametrization $\varphi : U \to S$, set $D = \varphi^{-1}(\tilde{D})$, and take $f = H|_U$. We get

$$A'(0) = -2\int\int_D H^2\sqrt{EG - F^2}\, dudv < 0,$$

so the given condition is suficient for the minimality of $S$. That it is also necessary is obvious. ∎

### 3.7.1 Isothermal parameters

When studying minimal surfaces, it is useful to introduce special parameters. A parametrized surface $\varphi : U \to S$ is called *isothermal* if

$$E = G = \lambda^2, \qquad F = 0,$$

where $\lambda \geq 0$ is a smooth function on $U$; in this case, the parameters $(u, v) \in U$ are also called *isothermal*. Note that $\varphi$ is regular if and only if $\lambda > 0$. An isothermal parametrization $\varphi$ is also called *conformal* or *angle preserving* because angles between curves in the surface are equal to the angles between the corresponding curves in the parameter plane.

Note that the mean curvature expressed in terms of isothermal parameters becomes

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{\ell + n}{2\lambda^2}. \tag{3.18}$$

**Proposition 3.19** *If $\varphi$ is isothermal, then $\Delta\varphi = 2\lambda^2 HN$ (here $N = \nu \circ \varphi$ is the unit normal along $\varphi$).*

*Proof.* Here $\Delta$ denotes the Laplacian operator and $\Delta\varphi = \varphi_u u + \varphi_v v$. Consider the equations $\langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle$ and $\langle \varphi_u, \varphi_v \rangle = 0$; differentiating the first one with respect to $u$ and the second one with resptec to $v$, we obtain

$$\langle \varphi_{uu}, \varphi_u \rangle = \langle \varphi_{vu}, \varphi_u \rangle \quad \text{and} \quad \langle \varphi_{uv}, \varphi_v \rangle + \langle \varphi_u, \varphi_{vv} \rangle = 0.$$

Putting these together yields $\langle \Delta\varphi, \varphi_u \rangle = 0$. Similarly, differentiating the first equation with respect to $v$ and the second one with resptec to $u$, we get that $\langle \Delta\varphi, \varphi_v \rangle = 0$. This shows that $\Delta\varphi$ is a normal vector. Finally,

$$\langle \Delta\varphi, N \rangle = \langle \varphi_{uu}, N \rangle + \langle \varphi_{vv}, N \rangle = \ell + n = 2\lambda^2 H$$

by eqn. (3.18). □

**Corollary 3.20** *An isothermal regular parametrized surface $\varphi : U \to S$ is minimal if and only if the coordinate functions of $\varphi$ are harmonic functions on $U$.*

Isothermal parameters exist around any point in a surface. In the next section, we present a proof of their existence in the case of minimal surfaces.

**Theorem 3.21** *There exist no compact minimal surfaces in $\mathbb{R}^3$.*

*Proof.* Suppose, to the contrary, that $S$ is a compact minimal surface. Without loss of generality, we may assume $S$ is connected. Consider the coordinate function $x : \mathbb{R}^3 \to \mathbb{R}$. There exists a point $p \in S$ where the restriction $x|_S$ attains its maximum. Let $\varphi : U \to \varphi(U)$ be an isothermal parametrization around $p$ with $U$ connected. Then $x \circ \varphi$ is a harmonic function on $U$ which attains its maximum at an interior point $p \in U$. By the maximum principle, $x \circ \varphi$ is a constant function on $U$, or, $x \equiv x(p)$ on $\varphi(U)$.

Next, let $q \in S$ be arbitrary and choose a continuous curve $\gamma : [0,1] \to S$ joining $p$ to $q$. Cover $\gamma([0,1])$ by finitely many connected open sets $V_0 = \varphi(U)$, $V_1, \ldots, V_n$, each $V_i$ equal to the image of an isothermal parametrization $\varphi_i$, such that $V_i \cap V_{i+1} \neq \varnothing$ for all $i = 0, 1, \ldots, n$ and $q \in V_n$. Since $x|_S$ attains its maximum value along $V_0 \cap V_1 \neq \varnothing$, the maximum principle applied to $x \circ \varphi_1$ yields that this function is constant along $V_1$, namely, $x \equiv x(p)$ on $V_1$. Proceeding by induction, we get that $x \equiv x(p)$ on $V_n$ and hence $x(q) = x(p)$. Since $q$ is arbitrary, this argument proves that $x|S$ is a constant function. The same argument applied to the other coordinate functions $y$, $z : \mathbb{R}^3 \to \mathbb{R}$ finally shows that $S$ must be a point, a contradiction. $\square$

### 3.7.2 The Enneper-Weierstrass representation

We discuss now an unexpected connection beween minimal surfaces and theory of functions of one complex variable. Let $\varphi : U \to S$ be a parametrized surface. Denote by $x_1$, $x_2$, $x_3 : U \to \mathbb{R}$ the coordinate functions of $\varphi$. We introduce the complex functions ($j = 1, 2, 3$):

$$f_j(\zeta) = \frac{\partial x_j}{\partial u} - \frac{\partial x_j}{\partial v}, \qquad \text{where } \zeta = u + iv. \tag{3.22}$$

The function $f_j$ is smooth as a real function $U \subset \mathbb{R}^2 \to \mathbb{R}^2$, so a necessary and suffcient condition for $f_j$ to be holomorphic is given by the Cauchy-Riemann equations $\frac{\partial}{\partial u}\Re f_j = \frac{\partial}{\partial v}\Im f_j$, $\frac{\partial}{\partial v}\Re f_j = -\frac{\partial}{\partial u}\Im f_j$. We deduce that

(a) $f_j$ is holomorphic in $\zeta$ if and only if $x_j$ is harmonic in $u$, $v$.

Note also the identities:

$$\begin{aligned} f_1^2 + f_2^2 + f_3^2 &= \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2 - \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2 - 2i\sum_{j=1}^{3}\frac{\partial x_j}{\partial u}\cdot\frac{\partial x_j}{\partial v} \\ &= E - G - 2iF \end{aligned}$$

and

$$|f_1|^2 + |f_2|^2 + |f_3|^2 = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2 + \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2 = E + G. \tag{3.23}$$

It follows from these identities that

(b) $\varphi$ is isothermal if and only if

$$f_1^2 + f_2^2 + f_3^2 = 0. \tag{3.24}$$

(c) If $\varphi$ is isothermal, then $\varphi$ is regular if and only if

$$|f_1|^2 + |f_2|^2 + |f_3|^2 \neq 0. \tag{3.25}$$

**Proposition 3.26** *Let $\varphi : U \to S$ be an isothermal regular parametrized minimal surface. Then the functions $f_j$ defined by (3.22) are holomorphic and satisfy (3.24) and (3.25). Conversely, if $f_1$, $f_2$, $f_3$ are holomorphic functions defined on an simply-connected domain $U$ which satisfy (3.24) and (3.25), then ($j = 1, 2, 3$)*

$$x_j(\zeta) = \Re \int_{\zeta_0}^{\zeta} f_j(z)\, dz, \qquad \zeta \in U,$$

*(for fixed $\zeta_0 \in U$) are the coordinates of an isothermal regular parametrized minimal surface $\varphi : U \to S$ such that eqns. (3.22) are valid.*

*Proof.* One direction follows from assertions (a), (b), (c) above and Corollary 3.20. For the converse, note that $\zeta \mapsto \int_{\zeta_0}^{\zeta} f_j(z)\, dz$ is well defined because $U$ is simply connected and $f_j$ is holomorphic, and yields a holomorphic function on $U$ for which we can apply the Cauchy-Riemann equations:

$$\begin{aligned} \frac{d}{d\zeta} \int^{\zeta} f_j &= \frac{\partial}{\partial u} \Re \int^{\zeta} f_j + i \frac{\partial}{\partial u} \Im \int^{\zeta} f_j \\ &= \frac{\partial}{\partial u} \Re \int^{\zeta} f_j - i \frac{\partial}{\partial v} \Im \int^{\zeta} f_j, \end{aligned}$$

so eqns. (3.22) are valid; the rest now follows from (a), (b), (c) and Corollary 3.20 applied in the opposite direction. □

Note that the functions $x_j$ in the preceding proposition are defined up to an additive constant so that the surface is defined up to a translation.

Thus we see that the local study of minimal surfaces in $\mathbb{R}^3$ is reduced to solving equations (3.24) and (3.25) for triples of holomorphic functions. We next explain how this can be done. Rewrite (3.24) as

$$(f_1 + i f_2)(f_1 - i f_2) = -f_3^2. \tag{3.27}$$

Except in case $f_1 \equiv i f_2$ and $f_3 \equiv 0$ (which is easily seen to correspond to the case of a plane), the functions

$$f = f_1 - i f_2, \qquad g = \frac{f_3}{f_1 - i f_2}$$

are such that $f$ is holomorphic and $g$ is meromorphic. Clearly, $f_3 = fg$, and it follows from eqn. (3.27) that

$$f_1 + i f_2 = \frac{-f_3^2}{f_1 - i f_2} = -f g^2. \tag{3.28}$$

Hence

$$f_1 = \frac{1}{2} f(1 - g^2), \quad \text{and} \quad f_2 = \frac{i}{2} f(1 + g^2).$$

By (3.28), $fg^2$ is homolorphic and this says that at every pole of $g$, $f$ has a zero of order at least twice the order of the pole. Further, eqn. (3.25) says that $f_1$, $f_2$,

$f_3$ cannot vanish simultanelously, and this means that $f$ can only have a zero at a pole of $g$, and then the order of its zero must be exactly twice the order of the pole of $g$. We summarize this dicussion as follows.

**Theorem 3.29 (The Enneper-Weierstrass representation)** *Every minimal surface which is not a plane can be locally represented as*

$$\begin{aligned} x_1 &= \Re \int \frac{1}{2} f(\zeta)(1-g^2(\zeta)\, d\zeta \\ x_2 &= \Re \int \frac{i}{2} f(\zeta)(1+g^2(\zeta))\, d\zeta \\ x_3 &= \Re \int f(\zeta) g(\zeta)\, d\zeta, \end{aligned}$$

*where: $f$ is a holomorphic function on a simply-connected domain $U$, $g$ is meromorphic on $U$, $f$ vanishes only at the poles of $g$, and the order of its zero at such a point is exactly twice the order of the pole of $g$.*

*Conversely, every pair functions $f$, $g$ satisfying these conditions define an isothermal regular parametrized minimal surface via the above equations.*

**Examples 3.30** 1. The catenoid is given by $f(z) = -e^{-z}$, $g(z) = -e^z$.

2. The helicoid is given by $f(z) = -ie^{-z}$, $g(z) = -e^z$.

3. The minimal surface of Enneper (discovered in 1863) is given by $f(z) = 1$, $g(z) = z$. Solving for the parametrization, we obtain $x_1 = u - \frac{1}{3}u^3 + uv^2$, $x_2 = -v - u^2v + \frac{1}{3}v^3$, $x_3 = u^2 - v^2$.

4. The minimal surface of Scherk (discovered in 1834) is given by $f(z) = 4/(1-z^4)$, $g(z) = iz$. It can also be parametrized as the graph of $(x, y) \mapsto \log \frac{\cos x}{\cos y}$.

The Enneper-Weierstrass representation not only allows us to construct a great variety of minimal surfaces having interesting properties, but also serve to prove general theorems about minimal surfaces by translating the statements into corresponding statements about holomorphic functions. Unfortunately, developing this philosophy would take us beyond the scope of these notes, so we content ourselves with a small remark. Let us express the basic geometric quantities of an isothermal regular parametrized minimal surface $\varphi : U \to S$ in terms of $f$, $g$. We have

$$E = G = \lambda^2, \qquad F = 0,$$

where

$$\begin{aligned} \lambda^2 &= \frac{1}{2}\sum_{j=1}^{3} |f_j|^2 \qquad \text{by (3.23)} \\ &= \frac{1}{4}|f|^2|1+g|^2 + \frac{1}{4}|f|^2|1+g|^2 + |fg|^2 \\ &= \left(\frac{|f|(1+|g|^2)}{2}\right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned}\varphi_u \times \varphi_v &= (\,\Im\{f_2\bar{f}_3\}, \Im\{f_3\bar{f}_1\}, \Im\{f_1\bar{f}_2\}\,) \\ &= \frac{|f|^2(1+|g|^2)}{4}\,(\,2\Re g, 2\Im g, |g|^2-1\,),\end{aligned}$$

and

$$||\varphi_u \times \varphi_v|| = \sqrt{EG-F^2} = \lambda^2,$$

so

$$N = \left(\frac{2\Re g}{|g|^2+1}, \frac{2\Im g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right).$$

Recall that *stereographic projection* $\pi : S^2 \setminus \{(0,0,1)\} \to \mathbf{C}$ is the map

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1-x_3},$$

and its inverse is

$$\pi^{-1}(z) = \left(\frac{2\Re z}{|z|^2+1}, \frac{2\Im z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Hence

$$N = \pi^{-1} \circ g. \tag{3.31}$$

**Proposition 3.32** *Let $\varphi : U \to S$ be an isothermal regular parametrized minimal surface, where $U$ is the entire $\zeta$-plane. Then either $S$ lies in a plane, or the image of the Gauss map takes on all values with at most two exceptions.*

*Proof.* If $S$ does not lie in a plane, we can construct the function $g(\zeta)$ which is meromorphic on the entire $\zeta$-plane; by Picard's theorem, it either takes all values with at most two exceptions, or else it is constant. Eqn. (3.31) shows that the same alternative applies to $N$, and in the latter case $S$ lies in a plane. □

### 3.7.3 Local existence of isothermal parameters for minimal surfaces

**Lemma 3.33** *Let $S$ be a minimal surface. Then every point of $S$ lies in the image of an isothermal parametrization of $S$.*

*Proof.* Let $p \in S$. First all, we can find a neighborhood of $p$ in which $S$ is the graph of a smooth function which, by relabeling coordinates, can be assumed in the form $z = h(x,y)$ for $(x,y) \in U$ (Check!). The minimal equation for graphs is easily computed to be

$$(1+h_y^2)h_{xx} - 2h_xh_yh_{xy} + (1+h_x^2)h_{yy} = 0.$$

We then have equation

$$\frac{\partial}{\partial x}\frac{1+h_y^2}{W} = \frac{\partial}{\partial y}\frac{h_xh_y}{W}.$$

satisfied on $U$, where $W = \sqrt{1 + h_x^2 + h_y^2}$. Taking $U$ simply-connected, this implies that we can find a smooth function $\Phi$ on $U$ with

$$\frac{\partial \Phi}{\partial x} = \frac{h_x h_y}{W}, \qquad \frac{\partial \Phi}{\partial y} = \frac{1 + h_y^2}{W}.$$

Introduce new coordinates

$$\bar{x} = x, \qquad \bar{y} = \Phi(x, y).$$

One checks

$$\tfrac{\partial x}{\partial \bar{x}} = 1, \quad \tfrac{\partial x}{\partial \bar{y}} = 0, \quad \tfrac{\partial y}{\partial \bar{x}} = -\frac{h_x h_y}{1 + h_y^2}, \quad \tfrac{\partial y}{\partial \bar{y}} = \frac{W}{1 + h_y^2},$$

and the coeffcients of the second fundamental form with respect to $\bar{x}$, $\bar{y}$ are

$$\bar{E} = \bar{G} = \frac{W^2}{1 + h_y^2}, \qquad \bar{F} = 0,$$

as desired. □