A short course on the differential geometry of curves and surfaces in Euclidean spaces

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Contents

Chapter 1

Curves

1.1 Regular curves

A *regular parameterized curve* in $\mathbb{R}^n$ is a continuously differentiable map $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, such that $\gamma'(t) \neq 0$ for $t \in I$. This condition implies that γ admits a tangent line at every point. A *regular curve* is an equivalence class of regular parameterized curves, where $\gamma \sim \eta$ if and only if $\eta = \gamma \circ \varphi$ for a continuously differentiable $\varphi : J \to I$, $\varphi' > 0$. We shall normally deal with curves satisfying some higher differentiability condition, like class $\mathcal{C}^k$ for $k \in \{1, 2, \ldots, \infty\}$.

Examples 1.1 1. A *line* $\gamma(t) = p + tv = (x_0 + at, y_o + bt, z_0 + ct)$, where $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ is a point and $v = (a, b, c) \in \mathbb{R}^n$ is a vector.

2. The *circle* $\gamma(t) = (\cos t, \sin t)$ in the plane, or, more generally, $\gamma(t) = (x_0 + R\cos\omega t, y_0 + R\sin\omega t)$.

3. The *helix* $\gamma(t) = (a\cos t, a\sin t, bt)$, where a, $b \neq 0$.

4. The *semi-cubical parabola* $\gamma(t) = (t^2, t^3)$.

5. The *cathenary* $\gamma(t) = (t, \cosh(at))$, where $a > 0$.

6. The *tractrix* $\gamma(t) = (e^{-t}, \int_0^t \sqrt{1 - e^{-2\xi}}\, d\xi)$.

The *length* of a regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ is

$$L(\gamma) = \int_a^b ||\gamma'(t)||\, dt.$$

It is invariant under reparameterization.

Lemma 1.2 *Every regular curve $\gamma : [a, b] \to \mathbb{R}^n$ admits a reparameterization by arc length, that is, $\eta : [0, \ell] \to \mathbb{R}^n$, where $\ell = L(\gamma)$, such that $L(\eta|_{[0,t]}) = t$; equivalently, $||\eta'|| \equiv 1$, and we say that γ has unit speed.*

Proof. Define

$$\psi(t) = \int_a^t ||\gamma'(\xi)||\, d\xi.$$

Then $\psi : [a, b] \to [0, \ell]$, $\psi' > 0$ and we can take $\varphi = \psi^{-1}$, $\eta = \gamma \circ \varphi$. $\square$

Unless explicit mention to the contrary, we shall generally assume that our curves are parameterized by arc-length.

1.2 Plane curves

Let $\gamma : I \to \mathbb{R}^2$ be a curve parameterized by arc-length of class $\mathcal{C}^2$. Then $||\gamma'(s)|| = 1$ for all s. The curvature of γ is the rate of change of the direction of γ. Namely, let

$$\mathbf{t}(s) = \gamma'(s)$$

be the unit tangent vector at time s, and complete it to a positively oriented orthonormal base $\mathbf{t}(s)$, $\mathbf{n}(s)$ of $\mathbb{R}^2$. Then $\langle \mathbf{t}, \mathbf{t} \rangle = 1$ implies $\langle \mathbf{t}, \mathbf{t}' \rangle = 0$, so $\mathbf{t}' = \kappa \mathbf{n}$ for some continuous function $\kappa : I \to \mathbb{R}$. Similarly, $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ yields $\mathbf{n}' = -\kappa \mathbf{t}$. We can write

$$\left(\begin{array}{c} \mathbf{t} \\ \mathbf{n} \end{array} \right)' = \left(\begin{array}{cc} 0 & \kappa \\ -\kappa & 0 \end{array} \right) \left(\begin{array}{c} \mathbf{t} \\ \mathbf{n} \end{array} \right),$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^2$.

Proposition 1.3 *Suppose $\gamma : I \to \mathbb{R}^2$ is a curve parameterized by arc-length. Then κ is constant if and only γ is either part of a circle (if $\kappa \neq 0$) or part of a line (if $\kappa = 0$).*

Proof. If κ is identically zero, then the Frenet-Serret equations give $\mathbf{t}' = 0$ so $\gamma' = \mathbf{t}$ is a constant vector $\mathbf{t}_0$ in the plane and $\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} \gamma'(\xi)\, d\xi = \gamma(s_0) + (s - s_0)v_0$, for $s_0 \in I$.

If κ is a nonzero constant, by changing the orientaion we may assume that $\kappa > 0$. We first show that

$$c(s) := \gamma(s) + \frac{1}{\kappa}\mathbf{n}(s)$$

is a constant curve. In fact,

$$\begin{array}{rcl} c' & = & \gamma' + \dfrac{1}{\kappa}\mathbf{n}' \\ & = & \mathbf{t} + \dfrac{1}{\kappa}(-\kappa \mathbf{t}) \\ & = & 0. \end{array}$$

Set $c(s) = c_0$ for all $s \in I$. Since

$$||\gamma(s) - c_0|| = \frac{1}{\kappa}$$

for all $s \in I$, we deduce that γ is part of the circle of center c_0 and radius $1/\kappa$, as wished. $\square$

Theorem 1.4 (Fundamental theorem of plane curves) *The curvature is a complete invariant of plane curves, up to rigid motion. More precisely, given a continuous function $\alpha : [a, b] \to \mathbb{R}$ there is a unique curve in the plane defined on $[a, b]$, parametrized by arc-length, whose curvature at time $s \in [a, b]$ is $\alpha(s)$, up to a translation and rotation of the plane.*

Proof. For the existence, set $\gamma(s) = (x(s), y(s))$, where

$$x(s) = \int_a^s \cos\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta,\ y(s) = \int_a^s \sin\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta$$

for $s \in [a, b]$. Then γ has curvature function given by α.

Conversely, suppose $\gamma : [a, b] \to \mathbb{R}$, $\gamma(s) = (x(s), y(s))$ is parameterized by arc-length and has curvature α. The Frenet-Serret frame t, n along γ can be written

$$t(s) = (\cos\theta(s), \sin\theta(s)),\ n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now

$$\alpha(s) = \langle t'(s), n(s)\rangle = \theta'(s),$$

so

$$\theta(s) = \theta(a) + \int_a^s \alpha(\xi)\, d\xi.$$

Also, $t = (x', y')$ yields

$$x(s) = x(a) + \int_a^s \cos(\theta(\tau))\, d\tau,\ y(s) = y(a) + \int_a^s \sin(\theta(\tau))\, d\tau.$$

This determines completely γ up to the values of $x(a)$, $y(a)$, $\theta(a)$, that is, up to translation and rotation. □

Regular curves in $\mathbb{R}^2$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$ and the curvature κ of γ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\kappa}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

for $t \in I$. Denote the velocity of γ by $\nu(t) = ||\gamma'(t)||$ and recall that $\varphi' = \nu$ (Lemma 1.2). The function ν is the appropriate correction term when we want to write Frenet-Serret equations for a curve γ of arbitrary speed, as we show in the sequel.

Since $\tilde{\gamma}$ is of unit speed, we have the Frenet-Serret equations $\tilde{\mathbf{t}}' = \tilde{\kappa}\tilde{\mathbf{n}}$, $\tilde{\mathbf{n}}' = -\tilde{\kappa}\tilde{\mathbf{t}}$. Now

$$\begin{aligned}\mathbf{t}'(t) &= \tilde{\mathbf{t}}'(t)\varphi'(t)\\ &= \kappa(\varphi(t))\nu(t)\tilde{\mathbf{n}}(\varphi(t))\\ &= \kappa(t)\nu(t)\mathbf{n}(t),\end{aligned}$$

and similarly

$$\mathbf{n}'(t) = -\kappa(t)\nu(t)\mathbf{t}(t).$$

So we have the following Frenet-Serret equations for γ:

$$\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix}' = \begin{pmatrix}0 & \kappa\nu\\ -\kappa\nu & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix}.$$

In particular,

$$\kappa = \frac{1}{\nu}\langle \mathbf{t}', \mathbf{n}\rangle. \tag{1.5}$$

In practice, sometimes can be hard to find the explicit reparameterization by arc-length of a given regular curve, so equation (1.5) comes in handy to compute the curvature in such cases.

Example 1.6 We compute the curvature of the catenary $\gamma(t) = (t, \cosh t)$. We have $\gamma'(t) = (1, \sinh t)$ and $\nu(t) = (1 + \sinh^2 t)^{1/2} = \cosh t$, so γ has variable speed. We first seek to reparameterize γ by arc-length. We have $\varphi = \int \nu$ yields $\varphi(t) = \sinh t$ and $\varphi^{-1}(s) = \operatorname{arcsinh} s = \log(s + \sqrt{s^2+1})$, so

$$\tilde{\gamma}(s) = \gamma(\varphi^{-1}(s)) = (\log(s + \sqrt{s^2+1}), \sqrt{1+s^2})$$

is a reparameteriztion by arc-length. Pursuing this line of reasoning would require us to differentiate $\tilde{\gamma}$ twice (in the end we would still need to change back to the variable t), which is possible but not worth it. Instead, we start again and use (1.5). We have

$$\mathbf{t}(t) = \frac{1}{\nu(t)}\gamma'(t) = (\mathrm{sech} t, \tanh t),$$

so

$$\mathbf{n}(t) = (-\tanh t, \mathrm{sech} t),$$

$$\mathbf{t}'(t) = (-\tanh t\ \mathrm{sech}\, t, \mathrm{sech}^2\, t),$$

and

$$\kappa(t) = \frac{1}{\cosh t}(\tanh^2 t\, \mathrm{sech} t + \mathrm{sech}^3 t) = \mathrm{sech}^2 t.$$

Example 1.7 We can generalize Example 1.6. If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve of arbitrary speed and $\gamma(t) = (x(t), y(t))$, then $\nu = (x'^2 + y'^2)^{1/2}$, so

$$\mathbf{t} = \frac{1}{\nu}\gamma' = \frac{1}{(x'^2+y'^2)^{1/2}}(x', y'),$$

and

$$\mathbf{n} = \frac{1}{(x'^2 + y'^2)^{1/2}}(-y', x').$$

Now

$$\begin{aligned}
\mathbf{t}' &= -\frac{x'x'' + y'y''}{(x'^2 + y'^2)^{3/2}}(x', y') + \frac{1}{(x'^2 + y'^2)^{1/2}}(x'' + y'') \\
&= \frac{1}{(x'^2 + y'^2)^{3/2}}(x''y'^2 - x'y'y'', x'^2y'' - x'x''y').
\end{aligned}$$

So

$$\begin{aligned}
\langle \mathbf{t}', \mathbf{n} \rangle &= \frac{1}{(x'^2 + y'^2)^2}(-x''y'^3 + x'y'^2y'' + x'^3y'' - x'^2x''y') \\
&= \frac{1}{x'^2 + y'^2}(x'y'' - x''y'),
\end{aligned}$$

and (1.5) yields

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

Remark 1.8 (i) If $\gamma_1 : I_1 \to \mathbb{R}^2$ and $\gamma_2 : I_2 \to \mathbb{R}^2$ are two unit speed reparameterizations preserving the orientation of a given regular parameterized curve γ then $\gamma_1(t) = \gamma_2(\varphi(t))$ for some $\varphi : I_1 \to I_2$ with $\varphi' > 0$. Then $\gamma_1' = \varphi'\gamma_2'$ with $||\gamma_1'|| = ||\gamma_2'|| = 1$, so $\varphi' \equiv 1$ implying that $\varphi(t) = t + t_0$ for some $t_0 \in \mathbb{R}$. We deduce that the definition of curvature of a regular parameterized curve γ of arbitrary speed does not depend on the reparameterization by arc-length that we choose.

(ii) The curvature of a regular parameterized curve in the plane thus defined is invariant under reparameterization preserving the orientation.

(iii) The curvature of a regular paparemeterized curve in the plane changes sign under a change of orientation.

1.3 Space curves

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of class $\mathcal{C}^3$ and assume that $\gamma'' \neq 0$ everywhere. Then we can associate an adapted trihedron to $\gamma(s)$ for each $s \in I$. We put:

$$t = \gamma' \ (\textit{tangent}), n = \frac{\gamma''}{||\gamma''||} \ (\textit{normal}), b = t \times n \ (\textit{binormal}).$$

The *curvature* is $\kappa = ||\gamma''||$. It follows that $t' = \kappa n$. Since $n(s)$ is a unit vector for all s, $n' \perp n$ so

$$\begin{aligned}
n' &= \langle n', t \rangle t + \langle n', b \rangle b \\
&= -\langle n, t' \rangle t + \langle n', b \rangle b.
\end{aligned}$$

We define the *torsion* $\tau = \langle n', b \rangle$. Now

$$n' = -\kappa t + \tau b.$$

Finally,

$$\begin{aligned} b' &= t' \times n + t \times n' \\ &= \kappa n \times n + t \times (-\kappa t + \tau b) \\ &= -\tau n. \end{aligned}$$

We summarize this discussion in matrix notation:

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^3$.

Remark 1.9 A space curve with nonzero curvature is planar if and only if $\tau \equiv 0$.

Example 1.10 We compute the curvature and torsion of the helix

$$\gamma(s) = (a\cos(s/c), a\sin(s/c), b(s/c)),\ s \in \mathbb{R},$$

for $a > 0$, $b \in \mathbb{R}$ and $c \neq 0$. We have

$$\gamma'(s) = (-(a/c)\sin(s/c), (a/c)\cos(s/c), b/c),$$

so γ is parameterized by arc-length precisely when

$$a^2 + b^2 = c^2, \tag{1.11}$$

and then $t(s) = \gamma'(s)$. Further,

$$\gamma''(s) = (-(a/c^2)\cos(s/c), -(a/c^2)\sin(s/c), 0),$$

so

$$n(s) = (-\cos(s/c), -\sin(s/c), 0)$$

and

$$b(s) = ((b/c)\sin(s/c), -(b/c)\cos(s/c), a/c).$$

We compute

$$n'(s) = ((1/c)\sin(s/c), -(1/c)\cos(s/c), 0)$$

and

$$b'(s) = ((b/c^2)\cos(s/c), (b/c^2)\sin(s/c), 0).$$

It follows that

$$\kappa(s) = ||\gamma''(s)|| = a/c^2$$

and

$$\tau(s) = \langle n'(s), b(s) \rangle = b/c^2$$

are constant functions. Moreover $\kappa^2 + \tau^2 = 1/c^2$, so

$$a = \frac{\kappa}{\kappa^2 + \tau^2} \quad \text{and} \quad b = \frac{\tau}{\kappa^2 + \tau^2}. \tag{1.12}$$

Therefore, given κ, τ, we can solve equations (1.11), (1.12) for a, b, c and obtain a unique helix with curvature κ and torsion τ.

Theorem 1.13 (Fundamental theorem of space curves) *The curvature and torsion are complete invariants of space curves. More precisely, given continuous functions α, $\beta : [a, b] \to \mathbb{R}$ with $\alpha(s) > 0$ for all s, there exists a unique regular curve in $\mathbb{R}^3$ defined on $[a, b]$, parameterized by arc-length, of class C^3, whose curvature and torsion at time $s \in [a, b]$ are respectively given by $\alpha(s)$ and $\beta(s)$, up to a translation and rotation of $\mathbb{R}^3$.*

Proof. Consider

$$A = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}$$

as a matrix-valued function $[a, b] \to \mathbb{R}^{3\times3}$. We consider the first order system of linear differential equations

$$F' = AF$$

for a matrix-valued $F : [a, b] \to \mathbb{R}^{3\times3}$, given by the Frenet-Serret equations. Here the lines of F will yield the Frenet-Serret frame of our curve γ to be constructed, namely, $F(s) = (t(s), n(s), b(s))$. For a given initial condition $F(a) = (e_1, e_2, e_3)$, which is a positively oriented orthonormal basis of $\mathbb{R}^3$, the system has a unique solution $F(s)$ of class $\mathcal{C}^3$ defined for $s \in [a, b]$.

We claim that $F(s)$ is an ortogonal matrix of determinant 1 for all $s \in [a, b]$. The crucial fact involved here is that $A(s)$ is a skew-symmetric matrix. In fact, set $G = FF^t$. Then $G(a) = I$ and

$$\begin{aligned} G' &= (FF^t)' \\ &= F'F^t + F(F^t)' \\ &= F'F^t + F(F')^t \\ &= AFF^t + FF^tA^t \\ &= AG + GA^t. \end{aligned}$$

Since the constant function given by the identity matrix also satisfies the differential equation $G' = AG + GA^t$, due to the fact that $A(s) + A^t(s) = 0$ for all s, by the uniqueness theorem of solutions of first order ODE, $G(s) = I$ for all s. This proves that $F(s)$ is an orthogonal matrix and hence $\det F(s) = \pm 1$ for all

s. Since the determinant is a continuous function and $\det F(0) = 1$, we deduce that $\det F(s) = 1$ for all s.

Now $F(s) = (t(s), n(s), b(s))$ is a trihedron for all s. For a given initial point $\gamma(a) = p \in \mathbb{R}^3$, the curve is completely determined by

$$\gamma(s) = p + \int_a^s t(\xi)\, d\xi.$$

From the equation $F' = AF$ we see that (t, n, b) is the Frenet-Serret frame along γ and α, β are its curvature and torsion respectively. Note that the ambiguity in the construction of γ precisely amounts to the choices of point p and positive orthonormal basis (e_1, e_2, e_3), so any two choices differ by a translation and a rotation. $\square$

Remark 1.14 (Local form) Let $\gamma : I \to \mathbb{R}^3$ be a regular curve of class $\mathcal{C}^3$ parameterized by arc-length and suppose that $\kappa > 0$ so that the Frenet-Serret frame is well-defined. We may assume that $0 \in I$, $\gamma(0) = 0$ and $(t(0), n(0), b(0))$ is the canonical basis of $\mathbb{R}^3$. Then the Taylor expansion of $\gamma(s) = (x(s), y(s), z(s))$ at $s = 0$ yields:

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)^2}{6} s^3 + R_x, \\ y(s) &= \frac{\kappa(0)}{2} s^2 + \frac{\kappa'(0)}{6} s^3 + R_y, \\ z(s) &= \frac{\kappa(0)\tau(0)}{6} s^3 + R_z, \end{aligned}$$

where $\lim_{s\to 0} \frac{1}{s^3}(R_x, R_y, R_z) = 0$. Therefore the projections of γ in the (t, n)-plane (*osculating plane*), (n, b)-plane (*normal plane*, (t, b)-plane (*rectifying plane plane*) has the form of a parabola, semi-cubical parabola (if $\tau(0) \neq 0$), cubical parabola (if $\tau(0) \neq 0$), respectively, up to third order.

Regular curves in $\mathbb{R}^3$ of arbitrary speed

If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve not necessarily of unit speed, we first find $\varphi : I \to J$ with $\varphi' > 0$ so that $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ is of unit speed and then set the Frenet-Serret frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the curvature κ and the torsion τ of γ to be the objects $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$, $\tilde{\mathbf{b}}$, $\tilde{\kappa}$, $\tilde{\tau}$ associated to $\tilde{\gamma}$ in the corresponding point, namely,

$$\mathbf{t}(t) = \tilde{\mathbf{t}}(\varphi(t)),$$

$$\mathbf{n}(t) = \tilde{\mathbf{n}}(\varphi(t)),$$

$$\mathbf{b}(t) = \tilde{\mathbf{b}}(\varphi(t)),$$

$$\kappa(t) = \tilde{\kappa}(\varphi(t))$$

$$\tau(t) = \tilde{\tau}(\varphi(t))$$

for $t \in I$.

Set $\nu = ||\gamma'||$. As in the case of $\mathbb{R}^2$, we deduce the Frenet-Serret equations for γ:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

In the following, we find formuale for κ and τ in terms of the first three derivatives of γ. We have

$$\gamma' = \nu\mathbf{t},$$

$$\begin{aligned} \gamma'' &= \nu'\mathbf{t} + \nu\mathbf{t}' \\ &= \nu'\mathbf{t} + \kappa\nu^2\mathbf{n}, \end{aligned}$$

and

$$\begin{aligned} \gamma''' &= \nu''\mathbf{t} + \nu'(\kappa\mathbf{n}) + (\kappa'\nu^2 + 2\kappa\nu\nu')\mathbf{n} + \kappa\nu^2(-\kappa\nu\mathbf{t} + \tau\nu\mathbf{b}) \\ &= (\nu'' - \kappa^2\nu^2)\mathbf{t} + (\kappa'\nu^2 + 3\kappa\nu\nu')\mathbf{n} + \kappa\tau\nu^3\mathbf{b}. \end{aligned}$$

We deduce that $\gamma \times \gamma'' = \kappa\nu^3\mathbf{b}$, so

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}. \tag{1.15}$$

Further $||\gamma \times \gamma''|| = \kappa\nu^3$ and $(\gamma' \times \gamma'') \cdot \gamma''' = \kappa^2\tau\nu^6$, so

$$\tau = \frac{\gamma' \times \gamma'' \cdot \gamma'''}{||\gamma' \times \gamma''||^2}. \tag{1.16}$$

Example 1.17 It is very easy to compute curvature and torsion of the space curve $\gamma(t) = (t, t^2, t^3)$ using (1.15) and (1.16), as opposed to the moethod of finding a reparameteriztion by arc-length. Indeed

$$\gamma'(t) = (1, 2t, 3t^2),$$

$$\gamma''(t) = (0, 2, 6t)$$

and

$$\gamma'''(t) = (0, 0, 6).$$

Now

$$\gamma' \times \gamma'' = (6t^2, -6t, 2),$$

and

$$\kappa = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{(9t^4 + 4t^2 + 1)^3}},$$

$$\tau = \frac{3}{9t^4 + 9t^2 + 1}.$$

1.4 Global theory

Our discussion so far has been mostly local in nature, that is, we have studied properties of curves that depend on the behavior of the curve in a neighborhood of a point, like for instance the curvature and the torsion. The global differential geometry of curves studies curves as whole objects. For example, length is a global concept, and the property of being a closed curve is a global property.

In this section we survey on some of the most famous and interesting questions pertaining to the global differential geometry of curves. We neither intend to present an exhaustive treatment nor to delve into the details of proofs, but just sketch a few geometric ideas.

1.4.1 The rotation index

A regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^n$ of class $\mathcal{C}^k$ is said to be *closed* if γ and its derivatives up to order k coincide at a and b:

$$\gamma(a) = \gamma(b), \ldots, \gamma^{(k)}(a) = \gamma^{(k)}(b).$$

Equivalently, γ extends to a $\mathcal{C}^k$ map $\mathbb{R} \to \mathbb{R}^n$.

A closed regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^n$ is called *simple* if it has no self-intersections, that is, $\gamma|_{(a,b)}$ is injective as a map.

For simplicity, hereafter we consider only curves with class C^∞. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed curve parameterized by arc length in the plane. Let $\theta(s)$ be an angle determination of its tangent direction. On the one hand, we have seen that $\theta'(s) = \kappa(s)$, so

$$\int_a^b \kappa(s)\, ds = \theta(b) - \theta(a).$$

On the other hand, since γ is closed,

$$\theta(b) - \theta(a) = 2\pi k$$

for some $k \in \mathbb{Z}$. The integer k is called the *rotation index* of γ. It is clear that the rotation index of a closed curves changes sign if we change the orientation of the curve. For a closed regular parameterized curve $\gamma : [a,b] \to \mathbb{R}^2$ of arbitrary speed, its rotation index is defined as the rotation index of a reparameterization by arc-length, so that it equals

$$\frac{1}{2\pi} \int_a^b \kappa(t) ||\gamma'(t)||\, dt.$$

Hence we can talk of the rotation index of a regular curve in $\mathbb{R}^2$ (without reference to parameterization).

Theorem 1.18 (Hopf's Umlaufsatz 1935) *The rotation index of a simple closed regular curve is* ± 1.

The proof is not very difficult, but we omit it.

Two regular parameterized curves γ_0, $\gamma_1 : [a,b] \to \mathbb{R}^n$ are called *regularly homotopic* if there exists a map $F : [a,b] \times [0,1] \to \mathbb{R}^n$ of class $\mathcal{C}^\infty$ such that:

- $F(s,0) = \gamma_0(s)$ and $F(s,1) = \gamma_1(s)$ for all $s \in [a,b]$;
- if we set $\gamma_t(s) = F(s,t)$, then $f_t : [a,b] \to \mathbb{R}^n$ is a regular parameterized curve for all $t \in [0,1]$.

If, in addition, γ_0 and γ_1 are closed then γ_t is required to be closed for all t.

It is not difficult to see that if γ_0 and γ_1 are two closed regular curves which are regularly homotopic then they have the same rotation index. Conversely, we state:

Theorem 1.19 (Whitney-Graustein 1937) *Two closed regular curves in the plane are homotopic if and only if they have the same rotation index.*

1.4.2 Total absolute curvature

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a closed regular parameterized curve. Recall that

$$\int_a^b \kappa(t)||\gamma'(t)||\,dt = \int_a^b \kappa(s)\,ds$$

equals 2π times the rotation index of γ, where s is arc-length parameter. The *total absolute curvature* of γ is

$$\int_a^b |\kappa(t)|||\gamma'(t)||\,dt = \int_a^b |\kappa(s)|\,ds$$

(note that the absolute value in the integrand is important only in case $n = 2$).

A closed regular curve $\gamma : [a,b] \to \mathbb{R}^2$ is called *convex* if, for all $s \in [a,b]$, $\gamma([a,b])$ is contained in one half-plane determined by the tangent line at s.

Theorem 1.20 (Fenchel 1929) *The total absolute curvature of a regular curve in* $\mathbb{R}^3$ *is bounded below by* 2π*, and equality holds if and only if the curve is planar and convex.*

Proof. We work with a parameterization by arc-length. The total absolute curvature of a curve $\gamma : [a,b] \to \mathbb{R}^3$ parameterized by arc-length equals the length of its spherical image $\alpha : [a,b] \to S^2(1)$, where $\alpha(s) = t(s) = \gamma'(s)$.

If α is contained in a hemisphere then $\alpha(s) \cdot v \geq 0$ for all s and some unit vector v. But

$$0 = (\gamma(b) - \gamma(a)) \cdot v = \int_a^b \alpha(s) \cdot v\,ds \geq 0,$$

so γ must be planar.

If α is not contained in a hemisphere, let $s_0 \in [a, b]$ divide α into two curves of the same length, $\alpha_1 = \alpha|_{[a,s_0]}$ and $\alpha_2 = \alpha|_{[s_0,b]}$. Up to a rotation, we may assume $\alpha(0)$ and $\alpha(s_0)$ are symmetric with respect to the north pole. One of α_0, α_1 crosses the equator, say α_0 crosses the equator at p. Reflect t_0 on the plane through $\alpha(0)$, the north pole and $\alpha(s_0)$ to obtain a closed curve α_2 in $S^2(1)$ passing through p and $-p$.

Since α_2 is closed and passes through two antipodal points, clearly $L(\alpha_2) \geq 2\pi$, with equality holding only in case α is contained in the equator. On the other hand,

$$L(\alpha_2) = 2L(\alpha_0) = L(\alpha_0) + L(\alpha_1) = L(\alpha),$$

as desired.

To finish, note that a simple closed curve in $\mathbb{R}^2$ has curvature κ not changing sign if and only if it is convex. $\square$

Theorem 1.21 (Fary-Milnor) *The total absolute curvature of a non-trivially knotted regular curve in $\mathbb{R}^3$ is strictly bounded below by 4π.*

Chapter 2

Surfaces: basic definitions

A *regular parameterized surface* is a smooth mapping $\varphi : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$, of maximal rank. This is equivalent to saying that the rank of φ is 2 or that φ is an immersion. Such a φ is called a *parameterization*.

Let (u, v) be coordinates in $\mathbb{R}^2$, (x, y, z) be coordinates in $\mathbb{R}^3$. Then

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$x(u, v)$, $y(u, v)$, $z(u, v)$ admit partial derivatives of all orders and the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank two. This is equivalent to

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(y, z)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(z, x)}{\partial(u, v)} \neq 0,$$

or to the columns of the Jacobian matrix, denoted φ_u and φ_v, to be linearly independent.

A *surface* is a subset S of $\mathbb{R}^3$ satisfying:

(1) $S = \cup_{i \in I}$, where V_i is an open subset of S and $\varphi_i : U_i \subset \mathbb{R}^2 \to \varphi_i(U_i) = V_i$ is a parameterization for all $i \in I$. In other words, every point $p \in S$ lies in an open subset $W \subset \mathbb{R}^3$ such that $W \cap S$ is the image of a smooth immersion of an open subset of $\mathbb{R}^2$ into $\mathbb{R}^3$.

(2) Each $\varphi_i : U_i \to V_i$ is a homeomorphism. The continuity of φ_i^{-1} means that for given $i \in I$, $q \in V_i$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi_i^{-1}(\underbrace{B(q, \delta)}_{\text{ball in } \mathbf{R}^3} \cap V_i) \subset \underbrace{B(\varphi_i^{-1}(q), \epsilon)}_{\text{ball in } \mathbf{R}^2}.$$

2.1 Examples

1. The graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open, is a regular parameterized surface, where the parameterization is given by $\varphi(u,v) = (u, v, f(u,v))$. Note that

$$(d\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

has rank two.

2. If $S \subset \mathbb{R}^3$ is a subset such that any one of its points lies in a open subset of S which is a graph as in (1) (with respect to any one of the three coordinate planes), then S is a surface. It only remains to check that the parameterizations constructed in (1) are homeomorphisms. But this follows from the fact that $\varphi^{-1} = \pi|_{\varphi(U)}$ is continuous, where $\pi(x,y,z) = (x,y)$ is continuous.

3. The unit sphere is defined as

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Any point of S^2 lies in one of the following six open subsets, which are graphs, given by $z = \pm\sqrt{1 - x^2 - y^2}$, $y = \pm\sqrt{1 - x^2 - z^2}$, $x = \pm\sqrt{1 - y^2 - z^2}$.

2.2 Inverse images of regular values

Let $F : W \to \mathbb{R}$ be a smooth map, where $W \subset \mathbb{R}^3$ is open. A point $p \in W$ is called a *critical point* of F is $dF_p = 0$; otherwise, it is called a *regular point*. A point $q \in \mathbb{R}$ is called a *critical value* of F if there exists a critical point of F in $F^{-1}(q)$; otherwise, it is called a *regular value*. Note that a point $q \in \mathbb{R}$ lying outside the image of F is automatically a regular value of F.

Theorem 2.1 *If q is a regular value of F and $F^{-1}(q) \neq \varnothing$, then $S = F^{-1}(q)$ is a surface.*

Proof. It suffices to show that every point of S lies in an open subset of S which is a graph. Let $p = (x_0, y_0, z_0) \in S$. Then $dF_p = \begin{pmatrix} \frac{\partial F}{\partial x}(p) \\ \frac{\partial F}{\partial y}(p) \\ \frac{\partial F}{\partial z}(p) \end{pmatrix} \neq 0$ by the assumption. Without loss of generality, assume that $\frac{\partial F}{\partial z}(p) \neq 0$. By the implicit function theorem, there exist open neighborhoods $\tilde{V}$ of (x_0, y_0, z_0) in $\mathbb{R}^3$ and U of (x_0, y_0) in $\mathbb{R}^2$ and a smooth function $f : U \to \mathbb{R}$ such that $F(x,y,z) = q$, $(x,y,z) \in \tilde{V}$ if and only if $z = f(x,y)$, $(x,y) \in U$. Hence $V = \tilde{V} \cap S$ is the graph of f and an open neighborhood of p in S. $\square$

2.3 More examples

4. Spheres can also be seen as inverse images of regular values. Let $F(x,y,z) = x^2+y^2+z^2$. Then $(dF_{(x,y,z)})^t = (2x\ 2y\ 2z) = (0\ 0\ 0)$ if and only if $(x,y,z) = (0,0,0)$. Since $(0,0,0) \notin F^{-1}(r^2)$ for $r>0$, we have that the sphere $F^{-1}(r^2)$ of radius $r>0$ is a surface. Similarly, the ellipsoids $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (a, b, $c>0$) are surfaces.

5. The hyperboloids $x^2+y^2-z^2=r^2$ (one sheet) and $x^2+y^2-z^2=-r^2$ (two sheets) are surfaces, $r>0$. The cone $x^2+y^2-z^2=0$ is not a surface in a neighborhood of its vertex $(0,0,0)$.

6. The tori of revolution are surfaces given by the equation $z^2+(\sqrt{x^2+y^2}-a)^2=r^2$, where $a, r>0$.

7. More generally, one can consider surfaces of revolution. Let $\gamma(t) = (f(t),0,g(t))$ be a regular parameterized curve, $t\in(a,b)$. Define

$$\varphi(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u,v)\in(a,b)\times(v_0,v_0+2\pi)$. One can cover the surface by varying v_0 in $\mathbb{R}$. But there are conditions on γ for φ to be an immersion. One has

$$\frac{\partial(x,y)}{\partial(u,v)} = ff', \quad \frac{\partial(y,z)}{\partial(u,v)} = -fg'\cos v, \quad \frac{\partial(z,x)}{\partial(u,v)} = -fg'\sin v,$$

so

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2+\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 = f^2||\dot\gamma||^2,$$

and φ is an immersion if and only if $f>0$. Note also that φ is injective if and only if γ is injective. One also checks that φ^{-1} is continuous by writing its explicit expression.

8. Let $\xi:[0,2\pi]\to S^2$ be the smooth curve given in spherical coordinates as $\varphi=\theta/2$ (θ: longitude, φ; co-latitude) Then

$$\xi(t) = (\cos t\sin(t/2), \sin t\sin(t/s), \cos(t/2)),$$

and we can parameterize the Möbius band as

$$\varphi(u,v) = \alpha(u)+v\xi(u),$$

where $\alpha(t) = (\cos t,\sin t,0)$ is the unit circle in $\mathbb{R}^2$. Since $\varphi_u\cdot\varphi_v=0$, φ is an immersion.

2.4 Change of parameters

Theorem 2.2 *Let $S\subset\mathbb{R}^3$ be a surface and let $\varphi:U\to\varphi(U)$, $\psi:V\to\psi(V)$ be two parameterizations of S, where U, $V\subset\mathbb{R}^2$ are open. Then the* change of parameters *$h=\varphi^{-1}\circ\psi:\psi^{-1}(\varphi(U))\to\varphi^{-1}(\psi(V))$ is a diffeomorphism between open sets of $\mathbb{R}^2$.*

Proof. h is a homeomorphism because it is the composite map of two homeomorphisms. Note that a similar argument cannot be used to say that h is smooth, because it does not make sense (yet) to say that φ^{-1} is smooth.

Let $p = (u_0, v_0) \in U$, $q \in V$, $\varphi(p) = \psi(q)$. Since φ is an immersion, $d\varphi$ has rank two and we may assume WLOG that $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$. Write $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in U$ and define

$$\Phi(u, v, w) = (x(u, v), y(u, v), z(u, v) + w),$$

where $(u, v, w) \in U \times \mathbb{R}$. Then $\Phi : U \times \mathbb{R} \to \mathbb{R}^3$ is smooth and

$$\det(d\Phi_{(u_0,v_0,0)}) = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{array} \right|_{(u_0,v_0,0)} = \frac{\partial(x, y)}{\partial(u, v)}(p) \neq 0.$$

Since $d\Phi_{(u_0,v_0,0)}$ is non-singular, by the inverse function theorem, Φ^{-1} is defined and is smooth on some open neighborhood W of $\varphi(p)$ in $\mathbb{R}^3$. Since $\Phi|_{U\times\{0\}} = \varphi$, we have that $\Phi^{-1}_{\varphi(U)\cap W} = \varphi^{-1}|_{\varphi(U)\cap W}$. Since $W \cap \varphi(U)$ is open in S and ψ is a homeomorphism, $\psi^{-1}(W \cap \varphi(U)) \subset V$ is open. Now

$$h|_{\psi^{-1}(W)} = \varphi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))} = \Phi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))}$$

is smooth, because it is the composite map of smooth maps.

Similarly, one sees that h^{-1} is smooth by reversing the rôles of φ and ψ in the argument above. Hence h is a diffeomorphism. □

Corollary 2.3 *Let $S \subset \mathbb{R}^3$ be a surface and suppose $f : W \to \mathbb{R}^3$ is a smooth map defined on the open subset $W \subset \mathbb{R}^m$ such that $f(W) \subset S$. Then $\varphi^{-1} \circ f : W \to \mathbb{R}^2$ is smooth for every parameterization $\varphi : U \to \varphi(U)$ of S.*

Proof. If Φ is as in the proof of the theorem, we have that $\varphi^{-1} \circ f = \Phi^{-1} \circ f$ is the composite of smooth maps between Euclidean spaces. □

As an application of the smoothness of change of parameters, we can make the following definition. Let S be a surface. An application $f : S \to \mathbb{R}^n$ is smooth at a point $p \in S$ if $f \circ \varphi : U \to \mathbb{R}^n$ is smooth at $\varphi^{-1}(p) \in U$, for some parameterization $\varphi : U \to \varphi(U)$ of S with $p \in \varphi(U)$. Note that if $\psi : V \to \psi(V)$ is another parameterization of S with $p \in \psi(V)$, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is smooth at $\varphi^{-1}(p)$, because

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi^{-1} \circ \psi)$$

and the change of parameters $\varphi^{-1} \circ \psi$ is smooth.

Example 2.4 If S is a surface and $F : \mathbb{R}^3 \to \mathbb{R}$ is smooth, then the restriction $f = F|_S : S \to \mathbb{R}$ is smooth. In fact, $f \circ \varphi = F \circ \varphi$ is smooth for any parameterization φ of S. As special cases, we can take the *height function* relative to a, $F(x) = \langle x, a\rangle$, where $a \in \mathbb{R}^3$ is a fixed vector; or the *distance function* from q, $F(x) = ||x - q||^2$, where $q \in \mathbb{R}^3$ is a fixed point.

In particular, if $f : S \to \mathbb{R}^3$ is smooth at $p \in S$ and $\tilde{S} \subset \mathbb{R}^3$ is a surface such that $f(S) \subset \tilde{S}$, then we say that $f : S \to \tilde{S}$ is smooth at p.

2.5 Tangent plane

Let $S \subset \mathbb{R}^3$ be a surface. Recall that a smooth curve $\gamma : I \subset \mathbb{R} \to S$ is simply a smooth curve $\gamma : I \to \mathbb{R}^3$ such that $\gamma(I) \subset S$. Fix a point $p \in S$. A *tangent vector* to S at p is the tangent vector $\dot\gamma(0)$ to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. The *tangent plane* to S at p is the collection of all tangent vectors to S at p.

Proposition 2.5 *The tangent space T_pS is the image of the differential*

$$d\varphi_a : \mathbb{R}^2 \to \mathbb{R}^3, \tag{2.6}$$

where $\varphi : U \to \varphi(U)$ is any parameterization of S with $p = \varphi(a)$ and $a \in U$.

Proof. Any vector in the image of (2.6) is of the form $d\varphi_a(w_0)$ for some $w_0 \in \mathbb{R}^2$ and therefore is the tangent vector at 0 of the smooth curve $t \mapsto \varphi(a + tw_0)$.

Conversely, suppose $w = \dot\gamma(0)$ is tangent to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. By Corollary 2.3, $\eta := \varphi^{-1} \circ \gamma : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$ is a smooth curve in $\mathbb{R}^2$ with $\eta(0) = a$. Note that $\gamma = \varphi \circ \eta$. By the chain rule

$$v = d\varphi_a(\dot\eta(0)) \tag{2.7}$$

lies in the image of (2.6). □

Corollary 2.8 *The tangent plane T_pS is a 2-dimensional vector subspace of $\mathbb{R}^3$. For any parameterization $\varphi : U \to \varphi(U)$ of S with $p = \varphi(a)$, $a \in U$,*

$$\{\tfrac{\partial\varphi}{\partial u}(a), \tfrac{\partial\varphi}{\partial v}(a)\} \tag{2.9}$$

is a basis of T_pS.

It is also convenient to write $\varphi_u := \frac{\partial\varphi}{\partial u}$ and $\varphi_v := \frac{\partial\varphi}{\partial v}$.

Consider a tangent vector $w \in T_pS$, say $w = \dot\gamma(0)$ where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, as in the proof of Proposition 2.5. Then $\eta = \varphi^{-1} \circ \gamma$ is a smooth curve in $\mathbb{R}^2$ which we may write as $\eta(t) = (u(t), v(t))$. Since $\dot\eta(0) = (u'(0), v'(0))$, eqn. (2.7) yields that

$$w = u'(0)\varphi_u(a) + v'(0)\varphi_v(a),$$

namely, $u'(0)$, $v'(0)$ are the coordinates of w in the basis (2.9). This remark also shows that another smooth curve $\bar\gamma : (-\epsilon, \epsilon) \to S$ represents the same w if and only if $\bar\eta(t) = \varphi^{-1} \circ \bar\gamma(t) = (\bar u(t), \bar v(t))$ satisfies $(\bar u'(0), \bar v'(0)) = (u'(0), v'(0))$.

With the same notation as above, suppose now that $f : S \to \tilde S$ is a smooth map at $p \in S$. Note that $f \circ \gamma$ is a smooth curve in $\tilde S$. The *differential* of f at p is the map

$$df_p : T_pS \to T_{f(p)}\tilde S$$

that maps $w = \dot\gamma(0) \in T_pS$ to the tangent vector $\dot{\tilde\gamma}(0)$, where $\tilde\gamma = f \circ \gamma$. We check that $df_p(w)$ does not depend on the choice of curve γ. Let $\varphi : U \to \varphi(U) = V$,

$\tilde{\varphi} : \tilde{U} \to \tilde{\varphi}(\tilde{U}) = \tilde{V}$ be parameterizations of S, $\tilde{S}$, resp., with $p = \varphi(a)$, $a \in U$, $f(p) = \tilde{\varphi}(\tilde{a})$, $\tilde{a} \in \tilde{U}$, and such that $f(V) \subset \tilde{V}$. Consider the local representation of f,

$$g = \tilde{\varphi}^{-1} \circ f \circ \varphi : U \to \tilde{U},$$

and write

$$g(u, v) = (g_1(u, v), g_2(u, v))$$

for $(u, v) \in U \subset \mathbb{R}^2$. Then

$$\tilde{\gamma}(t) = \tilde{\varphi}\left(g_1(u(t), v(t)), g_2(u(t), v(t)) \right),$$

so

$$\dot{\tilde{\gamma}}(0) = \left(\tfrac{\partial g_1}{\partial u}\, u'(0) + \tfrac{\partial g_1}{\partial v}\, v'(0) \right) \tilde{\varphi}_{\tilde{u}} + \left(\tfrac{\partial g_2}{\partial u}\, u'(0) + \tfrac{\partial g_2}{\partial v}\, v'(0) \right) \tilde{\varphi}_{\tilde{v}}.$$

This relation shows that $\dot{\tilde{\gamma}}(0)$ depends only on $u'(0)$, $v'(0)$ and hence has the same value for any smooth curve representing w. This relation can also be rewritten as

$$df_p(w) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix},$$

which shows that df_p is a linear map whose matrix with respect to the bases $\{\varphi_u, \varphi_v\}$, $\{\tilde{\varphi}_{\tilde{u}}, \varphi_{\tilde{v}}\}$ is the 2 by 2 matrix above.

Example 2.10 If S is a surface given as the inverse image under $F : \mathbb{R}^3 \to \mathbb{R}$ of a regular value, then $T_pS = \ker(dF_p)$ for every $p \in S$. In fact, if $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, then $F(\gamma(t))$ is constant for $t \in (-\epsilon, \epsilon)$. By the chain rule, $dF_p(\dot{\gamma}(0)) = 0$. This proves the inclusion $T_pS \subset \ker(dF_p)$ and hence the equality by dimensional reasons.

Example 2.11 Let S be a surface and consider the height function $h : S \to \mathbb{R}$ for a fixed unit vector $\xi \in S^2$, given by $h(x) = x \cdot \xi$ for $x \in S$. We compute the differential $dh_p : T_pS \to \mathbb{R}$ at $p \in S$. Given $w \in T_pS$, there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = w$. Now

$$dh_p(w) = \frac{d}{dt}\Big|_{t=0} h(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \gamma(t) \cdot \xi = \dot{\gamma}(0) \cdot \xi = w \cdot \xi.$$

In particular, $p \in S$ is a critical point of h if and only if ξ is normal to S at p.