A short course on the differential geometry of curves and surfaces in Euclidean spaces

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Chapter 1

Curves

1.1 Regular curves

A *regular parameterized curve* in $\mathbb{R}^n$ is a continuously differentiable map $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, such that $\gamma'(t) \neq 0$ for $t \in I$. This condition implies that γ admits a tangent line at every point. A *regular curve* is an equivalence class of regular parameterized curves, where $\gamma \sim \eta$ if and only if $\eta = \gamma \circ \varphi$ for a continuously differentiable $\varphi : J \to I$, $\varphi' > 0$. We shall normally deal with curves satisfying some higher differentiability condition, like class $\mathcal{C}^k$ for $k \in \{1, 2, \ldots, \infty\}$.

Examples 1.1 1. A *line* $\gamma(t) = p + tv = (x_0 + at, y_o + bt, z_0 + ct)$, where $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ is a point and $v = (a, b, c) \in \mathbb{R}^n$ is a vector.

2. The *circle* $\gamma(t) = (\cos t, \sin t)$ in the plane, or, more generally, $\gamma(t) = (x_0 + R\cos \omega t, y_0 + R \sin \omega t)$.

3. The *helix* $\gamma(t) = (a \cos t, a \sin t, bt)$, where a, $b \neq 0$.

4. The *semi-cubical parabola* $\gamma(t) = (t^2, t^3)$.

5. The *cathenary* $\gamma(t) = (t, \cosh(at))$, where $a > 0$.

6. The *tractrix* $\gamma(t) = (e^{-t}, \int_0^t \sqrt{1 - e^{-2\xi}}\, d\xi)$.

The *length* of a regular parameterized curve $\gamma : [a, b] \to \mathbb{R}^n$ is

$$L(\gamma) = \int_a^b ||\gamma'(t)||\, dt.$$

It is invariant under reparameterization.

Lemma 1.2 *Every regular curve $\gamma : [a, b] \to \mathbb{R}^n$ admits a reparameterization by arc length, that is, $\eta : [0, \ell] \to \mathbb{R}^n$, where $\ell = L(\gamma)$, such that $L(\eta|_{[0,t]}) = t$; equivalently, $||\eta'|| \equiv 1$.*

Proof. Define

$$\psi(t) = \int_a^t ||\gamma'(\xi)||\, d\xi.$$

Then $\psi : [a, b] \to [0, \ell]$, $\psi' > 0$ and we can take $\varphi = \psi^{-1}$, $\eta = \gamma \circ \varphi$. □

1.2 Plane curves

Let $\gamma : I \to \mathbb{R}^2$ be a regular parameterized curve of class $\mathcal{C}^2$. The curvature of γ is the rate of change of the direction of γ. Namely, let

$$t(s) = \frac{\gamma'(s)}{||\gamma(s)||}$$

be the unit tangent vector at time s, and complete it to a positively oriented orthonormal base t, n of $\mathbb{R}^2$. Then $\langle t, t \rangle = 1$ implies $\langle t, t' \rangle = 0$, som $t' = \kappa n$ for some continuous function $\kappa : I \to \mathbb{R}$. Similarly, $\langle n, n \rangle = 1$ yields $n' = -\kappa t$. We can write

$$\begin{pmatrix} t \\ n \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^2$.

Proposition 1.3 *Suppose γ is a regular curve parameterized by arc-length. Then κ is constant if and only γ is either a circle (if $\kappa \neq 0$) or a line (if $\kappa = 0$).*

Theorem 1.4 (Fundamental theorem of plane curves) *The curvature is a complete invariant of plane curves, up to rigid motion. More precisely, given a continuous function $\alpha : [a, b] \to \mathbb{R}$ there is a unique curve in the plane defined on $[a, b]$, parametrized by arc-length, whose curvature at time $s \in [a, b]$ is $\alpha(s)$, up to a translation and rotation of the plane.*

Proof. For the existence, set $\gamma(s) = (x(s), y(s))$, where

$$x(s) = \int_a^s \cos\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta,\ y(s) = \int_a^s \sin\left(\int_a^\eta \alpha(\xi)\, d\xi\right) d\eta$$

for $s \in [a, b]$. Then γ has curvature function given by α.

Conversely, suppose $\gamma : [a, b] \to \mathbb{R}$, $\gamma(s) = (x(s), y(s))$ is parameterized by arc-length and has curvature α. The Frenet-Serret frame t, n along γ can be written

$$t(s) = (\cos\theta(s), \sin\theta(s)),\ n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now

$$\alpha(s) = \langle t'(s), n(s) \rangle = \theta'(s),$$

so

$$\theta(s) = \theta(a) + \int_a^s \alpha(\xi)\, d\xi.$$

Also, $t = (x', y')$ yields

$$x(s) = x(a) + \int_a^s \cos(\theta(\tau))\, d\tau,\ y(s) = y(a) + \int_a^s \sin(\theta(\tau))\, d\tau.$$

This determines completely γ up to the values of $x(a)$, $y(a)$, $\theta(a)$, that is, up to translation and rotation. □

1.3 Space curves

Let $\gamma : I \to \mathbb{R}^3$ be a regular parameterized curve of class $\mathcal{C}^3$ and assume that $\gamma'' \neq 0$ everywhere. Then we can associate an adapted trihedron to $\gamma(s)$ for each $s \in I$. For simplicty, assume that γ is parameterized by arc-length. Then we put:

$$t = \gamma' \text{ (\textit{tangent})}, n = \frac{\gamma''}{||\gamma''||} \text{ (\textit{normal})}, b = t \times n \text{ (\textit{binormal})}.$$

The *curvature* is $\kappa = ||\gamma''||$. It follows that $t' = \kappa n$. Since $n(s)$ is a unit vector for all s, $n' \perp n$ so

$$\begin{aligned} n' &= \langle n', t\rangle t + \langle n', b\rangle b \\ &= -\langle n, t'\rangle t + \langle n', b\rangle b. \end{aligned}$$

We define the *torsion* $\tau = \langle n', b\rangle$. Now

$$n' = -\kappa t + \tau b.$$

Finally,

$$\begin{aligned} b' &= t' \times n + t \times n' \\ &= \kappa n \times n + t \times (-\kappa t + \tau b) \\ &= -\tau n. \end{aligned}$$

We summarize this discussion in matrix notation:

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},$$

the so-called *Frenet-Serret equations in* $\mathbb{R}^3$.

Remark 1.5 A space curve with nonzero curvature is planar if and only if $\tau \equiv 0$.

Example 1.6 We compute the curvature and torsion of the helix

$$\gamma(s) = (a\cos(s/c), a\sin(s/c), b(s/c)),\ s \in \mathbb{R},$$

for $a > 0$, $b \in \mathbb{R}$ and $c \neq 0$. We have

$$\gamma'(s) = (-(a/c)\sin(s/c), (a/c)\cos(s/c), b/c),$$

so γ is parameterized by arc-length precisely when

$$a^2 + b^2 = c^2, \tag{1.7}$$

and then $t(s) = \gamma'(s)$. Further,

$$\gamma''(s) = (-(a/c^2)\cos(s/c), -(a/c^2)\sin(s/c), 0),$$

so

$$n(s) = (-\cos(s/c), -\sin(s/c), 0)$$

and

$$b(s) = ((b/c)\sin(s/c), -(b/c)\cos(s/c), a/c).$$

We compute

$$n'(s) = ((1/c)\sin(s/c), -(1/c)\cos(s/c), 0)$$

and

$$b'(s) = ((b/c^2)\cos(s/c), (b/c^2)\sin(s/c), 0).$$

It follows that

$$\kappa(s) = ||\gamma''(s)|| = a/c^2$$

and

$$\tau(s) = \langle n'(s), b(s)\rangle = b/c^2$$

are constant functions. Moreover $\kappa^2 + \tau^2 = 1/c^2$, so

$$a = \frac{\kappa}{\kappa^2+\tau^2} \quad \text{and} \quad b = \frac{\tau}{\kappa^2+\tau^2}. \tag{1.8}$$

Therefore, given κ, τ, we can solve equations (1.7), (1.8) for a, b, c and obtain a unique helix with curvature κ and torsion τ.

Theorem 1.9 (Fundamental theorem of space curves) *The curvature and torsion are complete invariants of space curves. More precisely, given continuous functions α, $\beta : [a,b] \to \mathbb{R}$ with $\alpha(s) > 0$ for all s, there exists a unique regular curve in $\mathbb{R}^3$ defined on $[a,b]$, parameterized by arc-length, of class C^3, whose curvature and torsion at time $s \in [a,b]$ are respectively given by $\alpha(s)$ and $\beta(s)$, up to a translation and rotation of $\mathbb{R}^3$.*

Proof. Consider

$$A = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}$$

as a matrix-valued function $[a,b] \to \mathbb{R}^{3\times3}$. We consider the first order system of linear differential equations

$$F' = AF$$

for a matrix-valued $F : [a,b] \to \mathbb{R}^{3\times3}$, given by the Frenet-Serret equations. Here the lines of F will yield the Frenet-Serret frame of our curve γ to be constructed, namely, $F(s) = (t(s), n(s), b(s))$. For a given initial condition $F(a) = (e_1, e_2, e_3)$, which is a positively oriented orthonormal basis of $\mathbb{R}^3$, the system has a unique solution $F(s)$ of class $\mathcal{C}^3$ defined for $s \in [a,b]$.

We claim that $F(s)$ is an ortogonal matrix of determinant 1 for all $s \in [a,b]$. The crucial fact involved here is that $A(s)$ is a skew-symmetric matrix. In fact, set $G = FF^t$. Then $G(a) = I$ and

$$\begin{aligned} G' &= (FF^t)' \\ &= F'F^t + F(F^t)' \\ &= F'F^t + F(F')^t \\ &= AFF^t + FF^tA^t \\ &= AG + GA^t. \end{aligned}$$

Since the constant function given by the identity matrix also satisfies the differential equation $G' = AG + GA^t$, due to the fact that $A(s) + A^t(s) = 0$ for all s, by the uniqueness theorem of solutions of first order ODE, $G(s) = I$ for all s. This proves that $F(s)$ is an orthogonal matrix and hence $\det F(s) = \pm 1$ for all s. Since the determinant is a continuous function and $\det F(0) = 1$, we deduce that $\det F(s) = 1$ for all s.

Now $F(s) = (t(s), n(s), b(s))$ is a trihedron for all s. For a given initial point $\gamma(a) = p \in \mathbb{R}^3$, the curve is completely determined by

$$\gamma(s) = p + \int_a^s t(\xi)\, d\xi.$$

From the equation $F' = AF$ we see that (t, n, b) is the Frenet-Serret frame along γ and α, β are its curvature and torsion respectively. Note that the ambiguity in the construction of γ precisely amounts to the choices of point p and positive orthonormal basis (e_1, e_2, e_3), so any two choices differ by a translation and a rotation. □

Remark 1.10 (Local form) Let $\gamma : I \to \mathbb{R}^3$ be a regular curve of class $\mathcal{C}^3$ parameterized by arc-length and suppose that $\kappa > 0$ so that the Frenet-Serret frame is well-defined. We may assume that $0 \in I$, $\gamma(0) = 0$ and $(t(0), n(0), b(0))$ is the canonical basis of $\mathbb{R}^3$. Then the Taylor expansion of $\gamma(s) = (x(s), y(s), z(s))$ at $s = 0$ yields:

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)^2}{6} s^3 + R_x, \\ y(s) &= \frac{\kappa(0)}{2} s^2 + \frac{\kappa'(0)}{6} s^3 + R_y, \\ z(s) &= \frac{\kappa(0)\tau(0)}{6} s^3 + R_z, \end{aligned}$$

where $\lim_{s\to 0} \frac{1}{s^3}(R_x, R_y, R_z) = 0$. Therefore the projections of γ in the (t,n)-plane (*osculating plane*), (n,b)-plane (*normal plane*, (t,b)-plane (*rectifying plane plane*) has the form of a parabola, semi-cubical parabola (if $\tau(0) \neq 0$), cubical parabola (if $\tau(0) \neq 0$), respectively, up to third order.