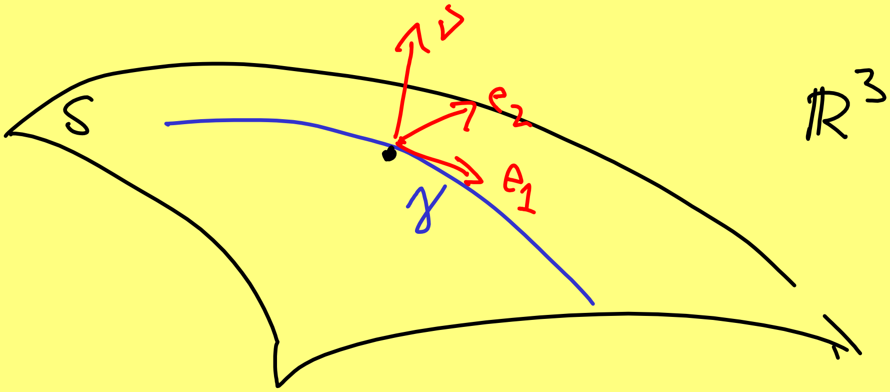
O Teorema de Gauss-Bonnet

28/11/23

$S \subset \mathbb{R}^3$ superfície $\quad \gamma : I \to S \quad C^\infty$

p.c.a.



$e_1 = \gamma'$

$e_2 = \nu \times \gamma'$

$e_3 = \nu$

e_1, e_2, e_3 base o.n. positiva de $\mathbb{R}^3$

$$k = \|e_1'\| = \|\gamma''\| = \|D_{e_1} e_1\|$$

$$\|e_1\| = 1 \Rightarrow \langle D_{e_1} e_1, e_1 \rangle = 0$$

$$k_n := \langle D_{e_1} e_1, \nu \rangle = -\langle e_1, D_{e_1} \nu \rangle = \langle e_1, A e_1 \rangle$$

$$= \mathrm{II}(e_1, e_1)$$ curvatura normal de γ

$$k_g := \overbrace{\langle D_{e_1} e_1, e_2 \rangle}^{= \langle \nabla_{e_1} e_1, e_2 \rangle}$$ curvatura geodésica de γ

$$k^2 = k_g^2 + k_n^2$$

$$k_g = 0 \iff \nabla_{e_1} e_1 = 0 \iff \frac{\nabla}{dt} \gamma' = 0 \iff \gamma \text{ é geodésica}$$

$$\begin{cases} \nabla_{e_1} e_1 = k_g e_2 \\ \nabla_{e_1} e_2 = -k_g e_1 \end{cases}$$

Teor. (1.ª versão local)

$\varphi : U \to S$ sup. param. reg.

$B \subset U$ disco fechado, ∂B orient. anti-horário

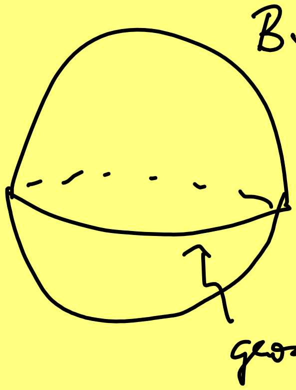
Então:
$$\boxed{\int_{\varphi(B)} K\,dA + \int_{\varphi(\partial B)} k_g\,ds = 2\pi.}$$

Ex. 1. $\mathbb{R}^2 \subset \mathbb{R}^3$ $\quad B = B(p,r)$ $\quad K = 0$

$$k_g = \frac{1}{r}$$

$$\int K\,dA + \int k_g\,ds = 0 + \frac{1}{r}\int ds = \frac{1}{r}2\pi r = 2\pi$$

2.

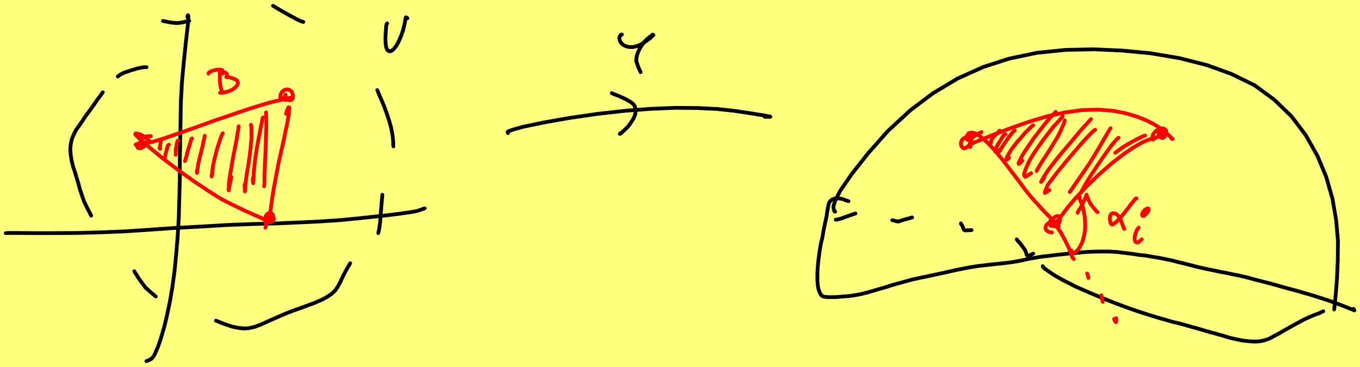


$K = \frac{1}{R^2}$ $\quad k_g = 0$

$$\int K\,dA + \int k_g\,ds = \frac{1}{R^2}\int dA + 0 = \frac{1}{R^2}2\pi R^2 = 2\pi$$

Teor. (2.ª versão local)

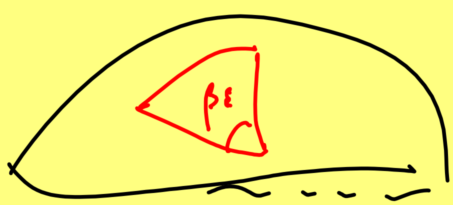
Mesmas hipóteses, mas B é apenas homeomorfo a um disco e ∂B é suave por partes.



$\Rightarrow$

$$\int_{\varphi(B)} K\,dA + \int_{\varphi(\partial B)} k_g\,ds + \sum_i \alpha_i = 2\pi$$

Cor.



triângulo geodésico

$k_g = 0$

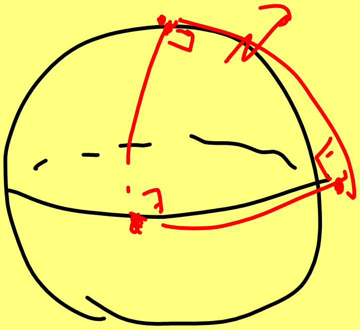
$\beta_i = \pi - \alpha_i$ ângulo interno

$$\int_{\varphi(B)} K\,dA + 0 + \underset{\pi-\beta_1}{\alpha_1} + \underset{\pi-\beta_2}{\alpha_2} + \underset{\pi-\beta_3}{\alpha_3} = 2\pi$$

$$\therefore \boxed{\int_\Delta K\,dA = \beta_1 + \beta_2 + \beta_3 - \pi}$$

Cor. A soma dos ângulos internos de um triângulo geodésico em S é $\begin{cases} > \pi & \text{se } K > 0 \\ = \pi & \text{se } K = 0 \\ < \pi & \text{se } K < 0. \end{cases}$

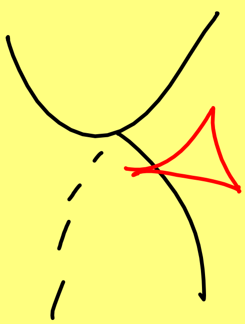
área $= \frac{1}{8} 4\pi = \frac{\pi}{2}$



$S^2(1)$

$K = +1 \qquad \beta_1 + \beta_2 + \beta_3 - \pi = \text{Área}(\Delta)$

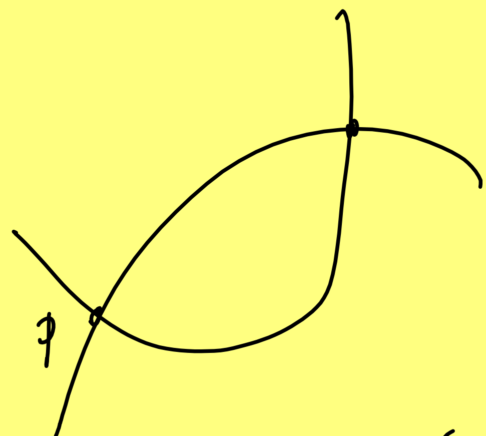
Lambert



$K < 0 \longleftarrow$ $K = -1$

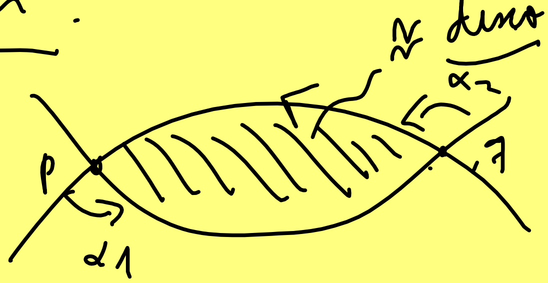
Aplicação

Prop. S $K \leq 0 \Rightarrow$ simples/e convexa.



$\nexists$ 2-ágono geodésico

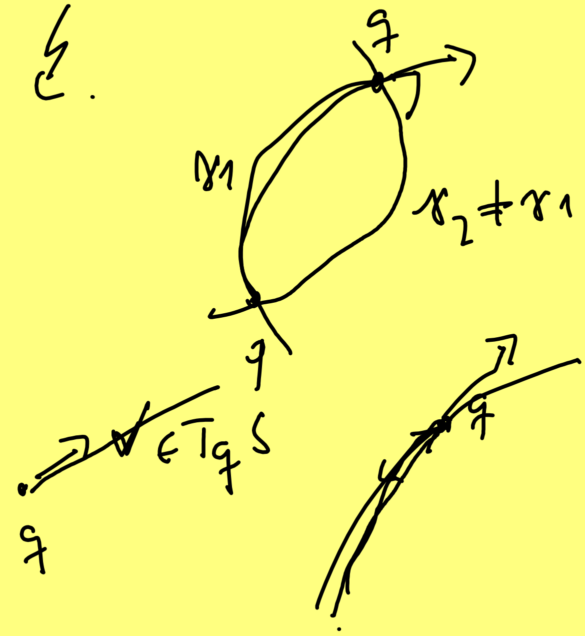
Dem.



$$\int_R K\,dA + \alpha_1 + \alpha_2 = 2\pi$$

$\alpha_1, \alpha_2 < \pi$

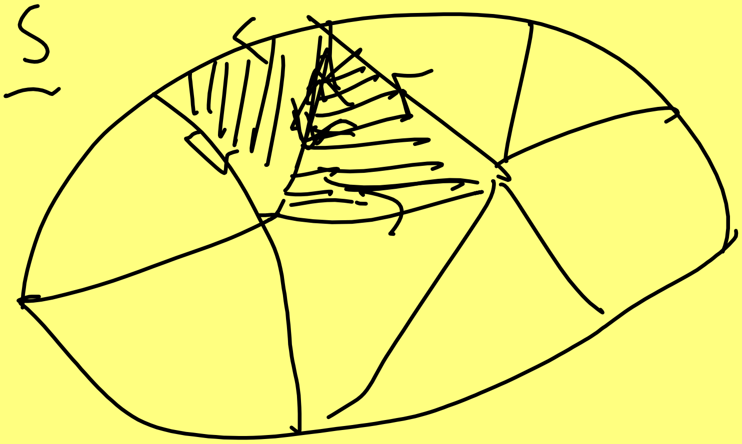
$$0 \geq \int_R \underbrace{K}_{\leq 0} \, dA = \underbrace{2\pi - \alpha_1 - \alpha_2}_{> 0} \quad \text{↯}$$



Teor. (versão global)

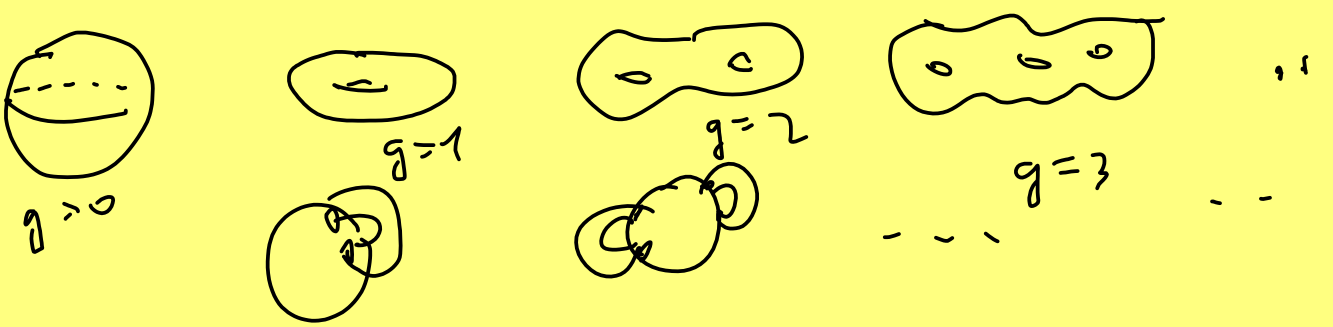
opta

S



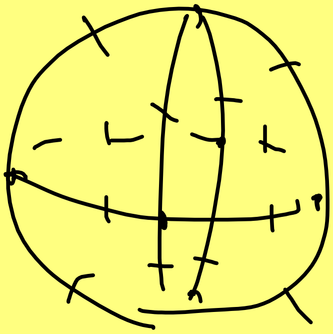
$$\int_S K \, dA = 2\pi \underbrace{\chi(S)}_{\text{característica de Euler}}$$

$$\chi(S) = V - E + F$$



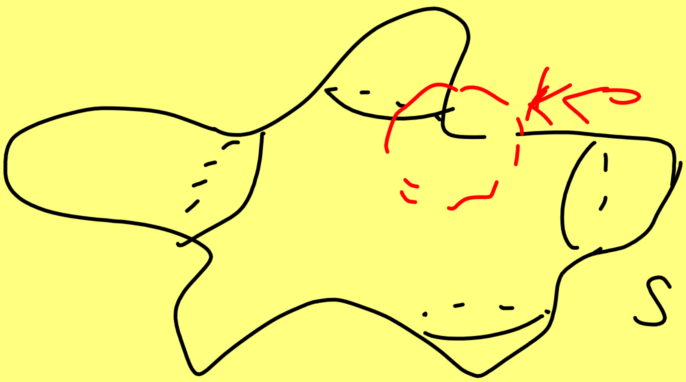
g = número de alças (gênero)

$$\chi(S) = 2 - 2g$$



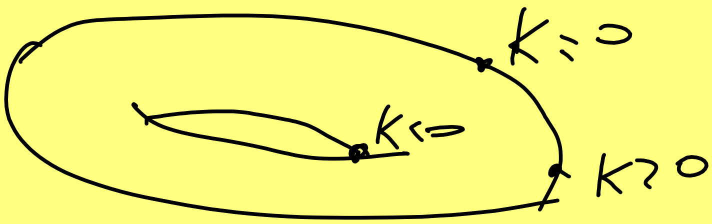
$F = 8$
$A = 12$
$V = 6$

$\chi(S^2) = 2$

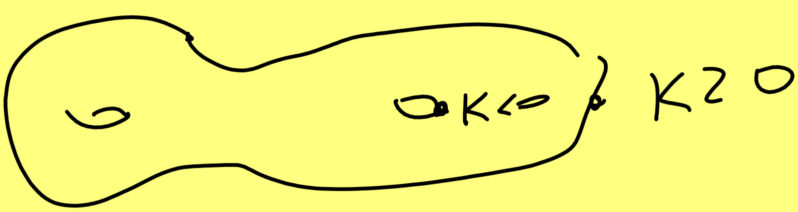


$S \simeq S^2$

$$\int_S K\,dA = \chi(S) = \chi(S^2) = 2$$



$$\int_{T^2} K\,dA = 0$$



$$\int_S K\,dA = -2$$

$$\nabla_X(fY) = df(X)Y + f\nabla_X Y$$

$b^j : \varphi(U) \to \mathbb{R}$

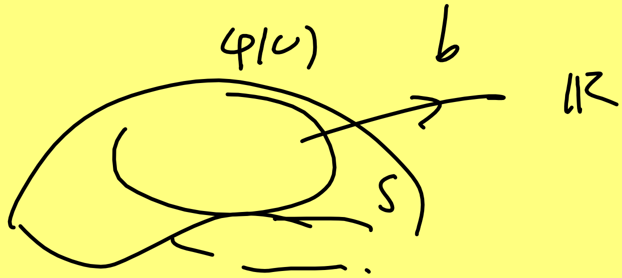
$$\nabla_{\frac{\partial\varphi}{\partial u^i}}\left(b^j \frac{\partial\varphi}{\partial u^j}\right) = db^j\Big(\underbrace{\frac{\partial\varphi}{\partial u^i}}_{= \frac{\partial\varphi}{\partial u^i}\circ\varphi^{-1} : \varphi(U) \to \mathbb{R}^3}\Big)\frac{\partial\varphi}{\partial u^j} + \dots$$

$$\varphi : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3 \qquad \frac{\partial \varphi}{\partial u^i} : U \to \mathbb{R}^3$$

$$b^j \circ \varphi : U \to \mathbb{R}$$

$$d(b^j \circ \varphi)\left(\frac{\partial \varphi}{\partial u^i}\right)$$

$$=$$



$$db\left(\frac{\partial \varphi}{\partial u}(u,v)\right) = db\left(\frac{d}{dt}\Big|_{t=0} \varphi(u+t,v)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} b(\varphi(u+t,v))$$

$$= \frac{\partial (b \circ \varphi)}{\partial u}(u,v)$$