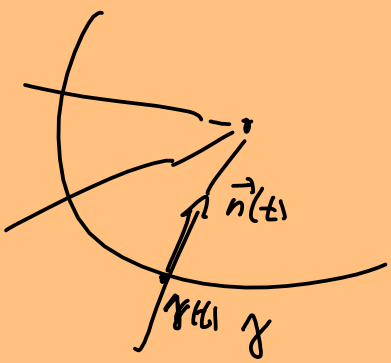
Aula de exercícios

05/09/23

§1.5, Ex. 4 [do Carmo]

Suponhamos que as normais de uma curva parametrizada regular passam por um ponto fixado. Mostrar que a curva é uma parte de um círculo.



Resolução

$\gamma : (a,b) \to \mathbb{R}^3 \quad \|\gamma'\| = 1$

$\kappa(s) \neq 0 \quad \forall s \in (a,b)$

$$\begin{cases} \vec{t} = \gamma' \\ \vec{n} = \dfrac{\gamma''}{\|\gamma''\|} \\ \vec{b} = \vec{t} \times \vec{n} \end{cases}$$

$\kappa = \|\gamma''\|$

$$\gamma(s) + r(s)\vec{n}(s) = p$$

$p \in \mathbb{R}^3$ fixado

$r : (a,b) \to \mathbb{R}$

$$\gamma'(s) + r'(s)\vec{n}(s) + r(s)\vec{n}'(s) = \vec{0}$$

$$\vec{t}(s) + r'(s)\,\vec{n}(s) + r(s)\left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right) = \vec{0}$$

(i) $1 - r(s)\kappa(s) = 0$ (iii) $r(s)\tau(s) = 0$

(ii) $r'(s) = 0$

(ii) $\Rightarrow$ $r(s) = r_0$ (const)

(i) $\Rightarrow$ $r_0 \underbrace{\kappa(s)}_{>0} = 1 \Rightarrow r_0 = 1/\kappa(s) > 0$

$\Downarrow$ $\kappa(s) = 1/r_0$ é const..

(iii) $\Rightarrow$ $\tau \equiv 0 \Rightarrow \gamma$ é uma curva planar.

γ planar, κ const. $\Rightarrow$ γ é um círculo //

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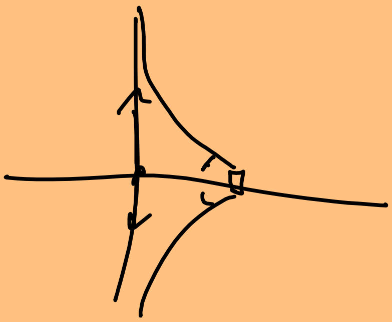

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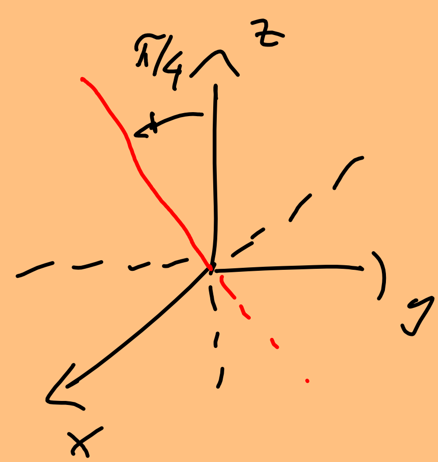
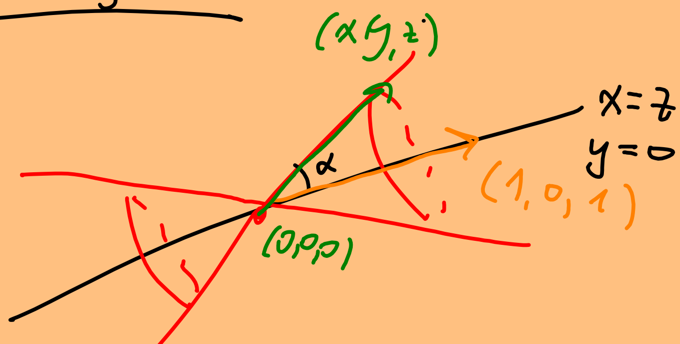
Ex 5, p. 50 [Kühnel]

$\gamma(t) = (3t, 3t^2, 2t^3)$

Mostrar que os vetores tangentes $\gamma'(t)$ a γ estão num cone circular com eixo $\begin{cases} x = z \\ y = 0 \end{cases}$



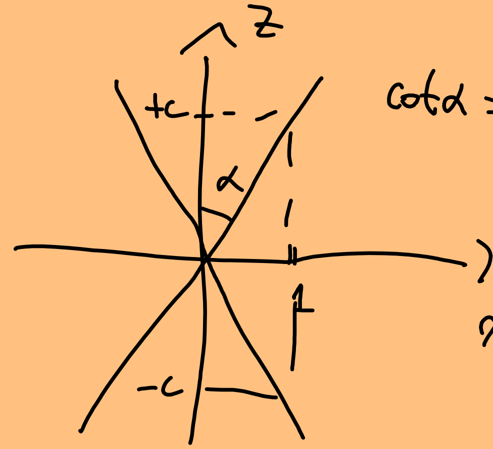
Resolução. $\gamma'(t) = (3, 6t, 6t^2)$



$$z^2 = x^2+y^2 \qquad z = \pm\sqrt{x^2+y^2} \qquad z = \pm c\sqrt{x^2+y^2},\ c \geq 0$$

$$\tan^2\!\alpha\, z = x^2+y^2$$

$$x^2+y^2-\tan^2\!\alpha\, z^2 = 0$$



$\cot\alpha = c$

$$\begin{pmatrix} 1/\sqrt2 & 0 & -1/\sqrt2 \\ 0 & 1 & 0 \\ 1/\sqrt2 & 0 & 1/\sqrt2 \end{pmatrix}$$

$$\cos\alpha = \frac{(x,y,z)\cdot(1,0,1)}{\|(x,y,z)\|\ \|(1,0,1)\|} = \frac{x+z}{\sqrt{x^2+y^2+z^2}\ \sqrt2}$$

$$\gamma'(t) = (3, 6t, 6t^2) \leftarrow \qquad 2.3.6t^2 = (6t)^2 \ \checkmark$$

$$\frac{3+6t^2}{\sqrt{9+36t^2+36t^4}\ \sqrt2} = \frac{\cancel{3+6t^2}}{\sqrt{(\cancel{3+6t^2})^2}\ \sqrt2} = \frac{1}{\sqrt2}$$

$$\alpha = \frac{\pi}{4}$$

$$\frac{1}{\cancel{\sqrt2}} = \frac{x+z}{\sqrt{x^2+y^2+z^2}\ \cancel{\sqrt2}} \qquad \cancel{x^2}+y^2+\cancel{z^2} = \cancel{x^2}+\cancel{z^2}+2xz$$

$$\boxed{y^2 = 2xz}$$

Ex. 15, § 1.5 [do Carmo]

O conhecimento de $\vec{b}(s)$ de uma curva γ com $\tau(s)\neq 0$ $\forall s\in I$ determina $k(s)$ e $|\tau(s)|$, $\forall s \in I$

Resolução.
$$\begin{cases} \vec{t}\,' = \kappa \vec{n} \\ \vec{n}\,' = -\kappa \vec{t} + \tau \vec{b} \\ \vec{b}\,' = -\tau \vec{n} \end{cases}$$

$\Rightarrow \|\vec{b}\,'\| = |\tau|\,\|\vec{n}\| = |\tau|$

Tem $\vec{b} \Rightarrow$ tem $\vec{b}\,' \Rightarrow$ tem $|\tau|$ ✓

$$\vec{n} = -\frac{1}{\tau}\vec{b}\,' = -\operatorname{sgn}(\tau)\frac{1}{|\tau|}\vec{b}\,'$$

$\Rightarrow$ Temos $\vec{n}$ a menos de sinal

$\vec{t} = \vec{n} \times \vec{b} \Rightarrow$ Temos $\vec{t}$ a menos de sinal

$\Rightarrow$ Temos $\vec{t}\,'$ a menos de sinal $\Rightarrow$ Temos $\kappa = \|\vec{t}\,'\|$ //

Ex. 16, §1.5 [do Carmo]

O conhecimento de $\vec{n}(s)$ de uma curva γ com $\tau(s) \neq 0$ $\forall s \in I$ determina $\kappa(s)$ e $\tau(s)$, $\forall s \in I$

Resolução.
$$\vec{t}\,' = \kappa \vec{n}$$
$$\vec{n}\,' = -\kappa \vec{t} + \tau \vec{b}$$
$$\vec{b}\,' = -\tau \vec{n}$$

Tem $\vec{n} \Rightarrow$ Tem $\vec{n}\,' \Rightarrow$ Tem $\|\vec{n}\,'\|^2 = \kappa^2 + \tau^2$

$\kappa\kappa' + \tau\tau'$ ✓

$$\vec{n}' \times \vec{n} = -\kappa \vec{b} - \tau \vec{t}$$

$$\vec{n}'' = -\kappa' \vec{t} - \kappa^2 \vec{n} + \tau' \vec{b} - \tau^2 \vec{n}$$
$$= -\kappa' \vec{t} + \tau' \vec{b} - \|\vec{n}'\|^2 \vec{n}$$

$$\frac{\kappa}{\tau}$$

$$\begin{cases} -\kappa \vec{t} + \tau \vec{b} = \vec{n}' \\ -\kappa' \vec{t} + \tau' \vec{b} = \vec{n}'' + \|\vec{n}'\|^2 \vec{n} \end{cases}$$

$$\to \vec{n}' \times \vec{n} \cdot \vec{n}'' = -\kappa \tau' + \kappa' \tau = \tau^2 \left(\frac{\kappa}{\tau}\right)'$$

$$\frac{-\kappa\tau' + \kappa'\tau}{\kappa^2+\tau^2} = \frac{\tau^2 (\kappa/\tau)'}{\kappa^2+\tau^2} = \frac{(\kappa/\tau)'}{(\kappa/\tau)^2+1} = \mathcal{C}$$

Integrar $\to \kappa/\tau$

$$\Downarrow$$

$$\tau \leftarrow \kappa \leftarrow \kappa^2 \leftarrow \tau^2 \leftarrow (\kappa/\tau)'$$

Ex. 11, §1.5 $\quad \rho = \rho(\theta) \quad \alpha \leq \theta \leq \beta$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \begin{cases} \rho = \rho(t) \\ \theta = \theta(t) \end{cases}$$

(a)
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d\rho}{dt}\cos\theta - \rho\sin\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{d\rho}{dt}\sin\theta + \rho\cos\theta\frac{d\theta}{dt}\right)^2$$
$$= \left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\theta}{dt}\right)^2$$

$t = \theta \in [\alpha, \beta]$ $\quad L(\gamma) = \int_\alpha^\beta \sqrt{\frac{dx}{dt^2} + \frac{dy}{dt^2}}\, dt = \int_\alpha^\beta \sqrt{\rho'^2 + \rho^2}\, d\theta$

(b) $$k = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}$$

$x' = \rho' \cos\theta - \rho \sin\theta \qquad y' = \rho' \sin\theta + \rho \cos\theta$

$x'' = \rho'' \cos\theta - 2\rho' \sin\theta - \rho \cos\theta$

$y'' = \rho'' \sin\theta + 2\rho' \cos\theta - \rho \sin\theta$

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