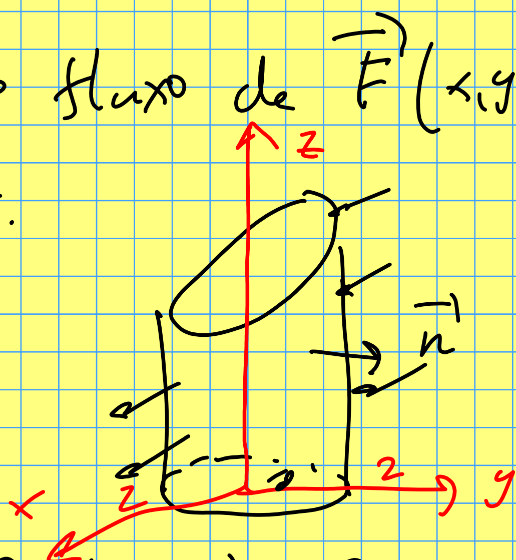
§21.4 [Simons, em inglês] 09/12/21

Gauss

19. σ: superfície do sólido B delimitado pelo cilindro $x^2+y^2=4$, o plano $x+z=4$ e o plano $z=0$

Calcular o fluxo de $\vec{F}(x,y,z) = (x^2+e^y, xy-\tan z, \sin x)$ através de σ.



$z = 4-x > 0$

$|x| \leq 2$

$x<0 \Rightarrow z>4$

Resolução.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2+e^y) + \frac{\partial}{\partial y}(xy-\tan z) + \frac{\partial}{\partial z}(\sin x)$$

$$= 2x + x + 0 = 3x$$

$$\text{Fluxo desejado} = \iint_\sigma \vec{F}\cdot\vec{n}\, dS \underset{\text{T. Gauss}}{=} \iiint_B \operatorname{div}\vec{F}\, dx\,dy\,dz$$

$$= \iiint_B 3x\, dx\,dy\,dz \leftarrow$$

Coords cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = h \end{cases} \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \\ 0 \leq h \leq 4 - r\cos\theta \end{array}$$

$x + z = 4$

$z = 4 - x$

$dx\,dy\,dz = r\,dr\,d\theta\,dh$

$z = 0$

$$\text{fluxo} = \int_0^2 \int_0^{2\pi} \int_0^{4-r\cos\theta} 3r^2\cos\theta \; dh\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} \underbrace{3r^2\cos\theta\,(4 - r\cos\theta)}_{= 12r^2\cos\theta \, - \, 3r^3\cos^2\theta} \, d\theta\, dh$$

$$= 12 \int_0^2 r^2\,dr \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0} - 3 \int_0^2 r^3\,dr \int_0^{2\pi} \underbrace{\cos^2\theta}_{= \frac{1+\cos 2\theta}{2}}\, d\theta$$

$$= -3 \left. \frac{r^4}{4} \right|_0^2 \cdot \left. \frac{\theta}{2} + \cancel{\frac{\sin 2\theta}{4}} \right|_0^{2\pi}$$

$$= -3 \cdot 16 \cdot \pi = -48\pi \; //$$

Complemento: Cálculo do x_{CM} de B.

$$\iiint_B x\,dx\,dy\,dz = -16\pi$$

$$Vol(B) = \iiint_B dx\,dy\,dz = \int_0^2\int_0^{2\pi}\int_0^{4-r\cos\theta} r\,dh\,d\theta\,dr$$

$$= \int_0^2\int_0^{2\pi} r(4-r\cos\theta)\,d\theta\,dr$$

$$= 4\int_0^2 r\,dr\int_0^{2\pi} d\theta - \int_0^2 r^2\,dr \cancel{\int_0^{2\pi}\cos\theta\,d\theta}^{\,0}$$

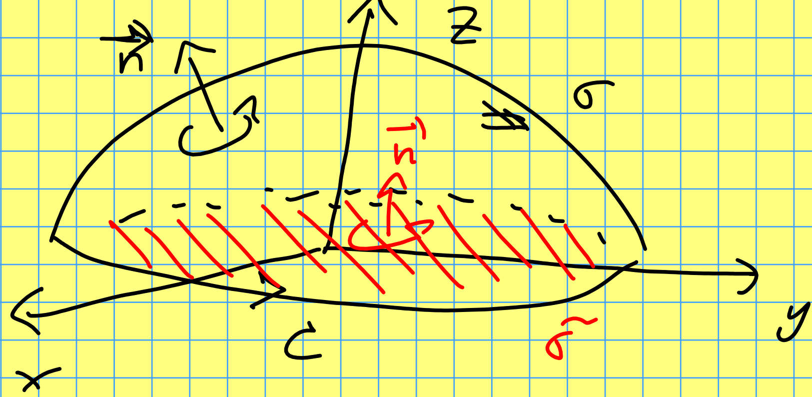
$$= 4\left.\frac{r^2}{2}\right|_0^2 2\pi = 16\pi$$

$$x_{CM} = \frac{-16\pi}{16\pi} = -1$$

— // —

16. σ: $\begin{cases} x^2+y^2+4z^2 = 1 \\ z \geq 0 \end{cases}$

orientado pela normal que aponta para cima.



$C = \partial\sigma$

$$\vec{F}(x,y,z) = (xz^2,\ x^3,\ \cos xz)$$

Calcular o fluxo de rot $\vec{F}$ atraves de σ trocando σ por uma superfície mais simples com a mesma fronteira.

$$\tilde{\sigma}: \begin{cases} z = 0 \\ x^2 + y^2 \cancel{+ z^2} \leq 1 \end{cases}$$

(superfície planar) círculo

$$\iint_\sigma \text{rot}\,\vec{F}\cdot\vec{n}\, dS \underset{\text{stokes}}{=} \oint_C \vec{F}\cdot d\vec{r} \underset{\text{stokes}}{=} \iint_{\tilde{\sigma}} \text{rot}\,\vec{F}\cdot\vec{n}\, dS = (*)$$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^2 & x^3 & \cos xz \end{vmatrix}$$

$$= (0,\ +z\sin xz + 2zx,\ 3x^2) \overset{z=0 \text{ em } \tilde{\sigma}}{=} (0,\ 0,\ 3x^2)$$

$\vec{n} = \vec{k}$ em $\tilde{\sigma}$

$$(*) = \iint_D 3x^2 \, dx\,dy = \int_0^{2\pi}\int_0^1 3 r^3 \cos^2\theta \, dr\,d\theta$$

$D: x^2+y^2 \leq 1$ $\quad \begin{cases} x = r\cos\theta & 0 \leq r \leq 1 \\ y = r\sin\theta & 0 \leq \theta \leq 2\pi \end{cases}$

$$dx\,dy = r\,dr\,d\theta$$

$$= 3\int_0^1 r^3 dr \int_0^{2\pi} \underbrace{\cos^2\theta}_{= \frac{1+\cos 2\theta}{2}} \, d\theta$$

$$= 3 \cdot \frac{1}{4} \cdot \pi = \frac{3\pi}{4} \quad //$$

— // —

Complemento ao ex. 16

Cálculo de $\oint_C \vec{F} \cdot d\vec{r}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \cancel{x z^2 dx} + x^3 dy + \cancel{\cos xz \, \underbrace{dz}_{=0}}$$

$C: x^2+y^2 = 1, \; z = 0$ $\quad \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \leq t \leq 2\pi$

$$= \oint_C x^3\,dy = \int_0^{2\pi} \cos^4 t\,dt = \int_0^{2\pi} \frac{3}{8}\,dt$$

$$= \frac{3}{8} 2\pi = \frac{3\pi}{4}$$

$$\cos^4 t = (\cos^2 t)^2 = \left(\frac{1+\cos 2t}{2}\right)^2$$

$$= \frac{1 + 2\cos 2t + \cos^2 2t}{4}$$

$$= \frac{1}{4} + \frac{\cos 2t}{2} + \frac{1}{4}\left(\frac{1+\cos 4t}{2}\right)$$

$$= \frac{3}{8} + \underbrace{\frac{\cos 2t}{2} + \frac{\cos 4t}{8}}_{\leftarrow \int_0^{2\pi} = 0}$$