UM CURSO DE CÁLCULO, VOL. 3 02/12/21

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Cap 11 Stokes

Cap 10 Gauss

Cap 9 Int. de superfície

Cap. 8 Green

Cap. 7 Campos conservativos

Cap 6 Int. de linha

Ex. 15, p.784 Simons [inglês]

σ metade superior do elipsóide

→ $x^2+y^2+\frac{z^2}{9}=1$, orientada de modo que $\vec{n}$ aponte para cima. Sendo

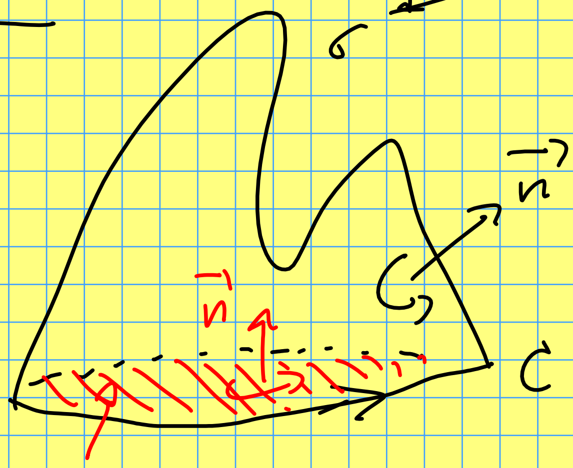
$\vec{F}(x,y,z)=(x^3, y^4, z^3 \sin xy)$, calcular

$$\iint_\sigma \operatorname{rot}\vec{F}\cdot\vec{n}\, dS,$$ (curl = rot)

substituindo σ por uma superfície mais simples, com o mesmo bordo.

Resolução.

"complicada"

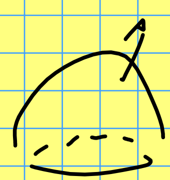


$\tilde{\sigma}$ ← "mais simples"

Stokes duas vezes!

$$\iint_{\sigma} \operatorname{rot}\vec{F}\cdot\vec{n}\, dS = \int_C \vec{F}\cdot d\vec{r} = \iint_{\tilde{\sigma}} \operatorname{rot}\vec{F}\cdot\vec{n}\, dS$$

?



—//—

$$\vec{N} = \left(-\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, 1\right) \qquad \sigma = \text{gráfico de } f$$

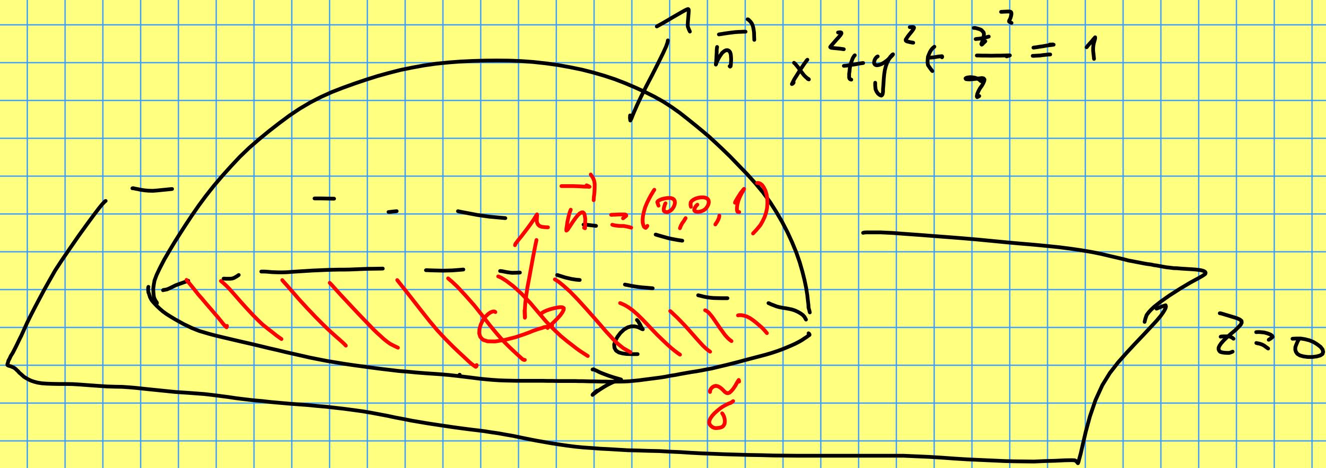
$$\|\vec{N}\|$$

$$\vec{n} = \vec{N}/\|\vec{N}\|$$

$$\vec{N} = \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \quad \leftarrow \text{mais geral}$$

σ parametrizada

—//—



$$\tilde{\sigma}: \begin{cases} z=0 \\ x^2+y^2\leq 1 \end{cases}$$

$$\iint_{\sigma} \operatorname{rot}\vec{F}\cdot\vec{n}\,dS = \iint_{\tilde{\sigma}} \operatorname{rot}\vec{F}\cdot\underbrace{\vec{n}}_{=(0,0,1)}\,dS' = \underbrace{-12\iint_{\tilde{\sigma}} x^2y^3\,dS'}_{(*)}$$

$$\left[\begin{aligned} \operatorname{rot}\vec{F} &= \left(\frac{\partial(R,Q)}{\partial(y,z)}, \frac{\partial(P,R)}{\partial(z,x)}, \frac{\partial(Q,P)}{\partial(x,y)}\right) \\ &= \left(*,*,\begin{vmatrix} \frac{\partial}{\partial x}(y^4) & \frac{\partial}{\partial y}(y^4) \\ \frac{\partial}{\partial x}(x^3) & \frac{\partial}{\partial y}(x^3) \end{vmatrix}\right) \\ &= (*,*,-3x^2\cdot 4y^3) = (*,*,-12x^2y^3) \end{aligned}\right.$$

$$(*) = -12\iint_{D} x^2y^3\,dx\,dy \quad \text{(int. duplo)}$$

$D: x^2+y^2\leq 1$

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \qquad \begin{array}{l} 0\leq r\leq 1 \\ 0\leq\theta\leq 2\pi \end{array}$$

$dx\,dy = r\,dr\,d\theta$

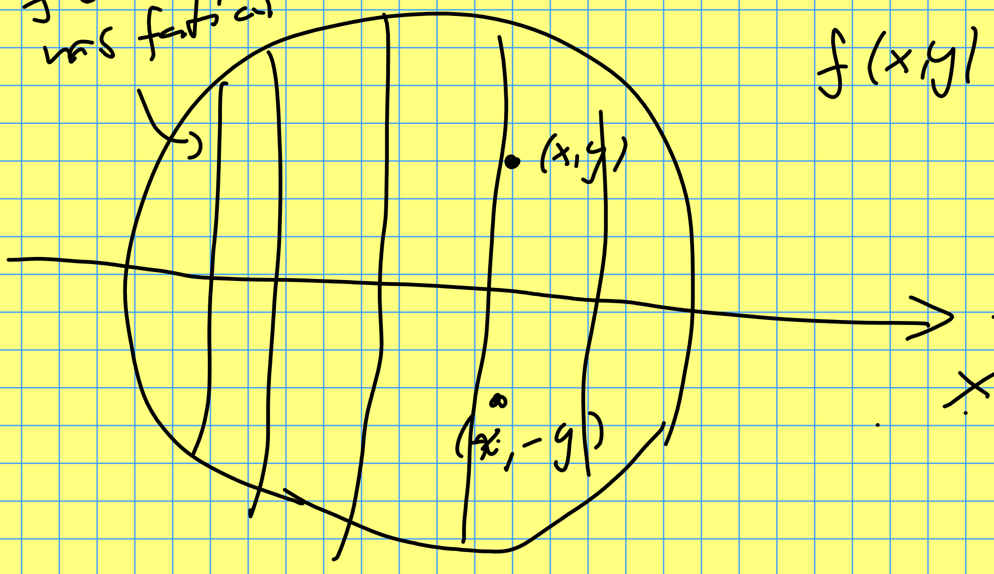
$$= -12 \int_0^{2\pi}\int_0^1 r^2\cos^2\theta \, r^3 \sin^3\theta \, r\,dr\,d\theta$$

$$= -12 \int_0^1 r^6\,dr \int_0^{2\pi} \cos^2\theta \sin^3\theta \, d\theta = 0$$

$$\int_0^{2\pi} \cos^2\theta \underbrace{\sin^3\theta}_{= \sin\theta(1-\cos^2\theta)} d\theta = \int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta - \int_0^{2\pi} \cos^4\theta \sin\theta \, d\theta$$

$$= -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\Big|_0^{2\pi} = 0$$

f é impar nas fatias verticais



$$f(x,y) = x^2y^3$$
$$= -f(x,-y)$$

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p 770 Ex. 23-28 [inglês]

Verificar que $\vec{F}$ é conservativo e calcular um potencial para $\vec{F}$.

23. $\vec{F}(x,y) = (y^3, 3xy^2)$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = y^3 \\[2mm] \dfrac{\partial \varphi}{\partial y} = 3xy^2 \end{cases} \Rightarrow \varphi(x,y) = \int \frac{\partial \varphi}{\partial x}\, dx = \int y^3 dx = xy^3 + h(y)$$

$$3xy^2 = \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left(xy^3 + h(y)\right) = 3xy^2 + h'(y)$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = \text{const}$$

$$\Rightarrow \varphi(x,y) = xy^3 + \text{const}$$

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (y^3; 3xy^2) = \vec{F}(x,y) \checkmark$$

De fato, $\varphi(x,y) = xy^3 + \text{const}$ é um potencial para $\vec{F}$.

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§7.1 Ex. 1 [Hamilton]

É o campo dado conservativo?

(a) $\vec{F}(x,y,z) = (x-y,\ x+y+z,\ z^2)$, $\operatorname{dom} \vec{F} = \mathbb{R}^3$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x+y+z & z^2 \end{vmatrix}$$

$$= (-1, *, *) \neq (0,0,0)$$

$\therefore$ $\vec{F}$ não é conservativo

(d) $$\vec{F}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^2}(x,y,z)$$

$$\text{dom}\,\vec{F} = \mathbb{R}^3 \setminus \{(0,0,0)\}$$

$$P = \frac{x}{(x^2+y^2+z^2)^2} \qquad Q = \frac{y}{(x^2+y^2+z^2)^2} \qquad R = \frac{z}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{0.(x^2+y^2+z^2)^2 - y\,2(x^2+y^2+z^2)\,2x}{(x^2+y^2+z^2)^4} = \frac{-4xy}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial P}{\partial y} = \frac{0 \cdot \cdots - x\,2(x^2+y^2+z^2)\,2y}{(x^2+y^2+z^2)^4}$$

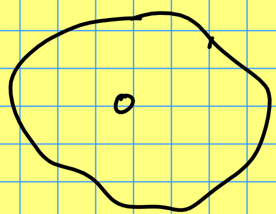
$$\frac{\partial R}{\partial y} = \frac{-4yz}{(x^2+y^2+z^2)^3} \qquad \text{rot}\,\vec{F} = \vec{0}.$$

$$\frac{\partial Q}{\partial z} = \qquad \text{etc}$$

$\mathbb{R}^3 \setminus \{(0,0,0)\}$ é simplesmente conexo

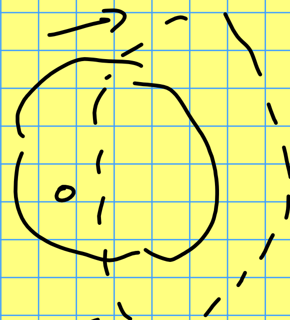
$\Rightarrow$ $\vec{F}$ é conservativo.

$\mathbb{R}^2 \setminus$ ponto



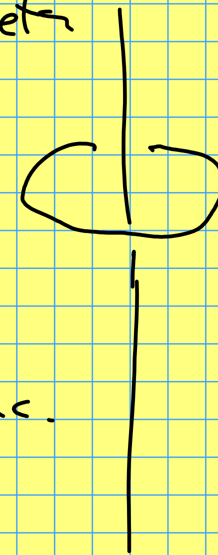
não é s.c.

$\mathbb{R}^3 \setminus$ ponto



é s.c.

$\mathbb{R}^3 \setminus$ reta



não é s.c.

Calculemos um potencial para $\vec{F}$:

$$
\begin{cases}
\dfrac{\partial \varphi}{\partial x} = \dfrac{x}{(x^2+y^2+z^2)^2} & (1) \\[2ex]
\dfrac{\partial \varphi}{\partial y} = \dfrac{y}{(x^2+y^2+z^2)^2} & (2) \\[2ex]
\dfrac{\partial \varphi}{\partial z} = \dfrac{z}{(x^2+y^2+z^2)^2} & (3)
\end{cases}
$$

$$\int \frac{x\,dx}{(x^2+a^2)^2} = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$$

$u = x^2 + a^2$

$du = 2x\,dx$

$$= \frac{1}{2}\cdot\frac{u^{-1}}{-1} = -\frac{1}{2u} + C$$

$$= -\frac{1}{2(x^2+a^2)} + C$$

(1) $\varphi(x,y,z) = \int \frac{\partial \varphi}{\partial x}\, dx$

$= -\frac{1}{2(x^2+y^2+z^2)} + h_1(y,z)$

(2) $\varphi(x,y,z) = \int \frac{\partial \varphi}{\partial y}\, dy = -\frac{1}{2(y^2+x^2+z^2)} + h_2(x,z)$

(3) $\varphi = -\frac{1}{2(z^2+x^2+y^2)} + h_3(x,y)$

$\therefore\ \varphi(x,y,z) = -\frac{1}{2(x^2+y^2+z^2)} + \text{const}$

é um potencial de $\vec{F}$ //.

<u>Ex</u>: $\vec{F}(x,y,z) = \frac{1}{x^2+y^2+z^2}(x,y,z)$.

$$\begin{cases} \frac{\partial \varphi}{\partial x} = \frac{x}{x^2+y^2+z^2} \\ \frac{\partial \varphi}{\partial y} = \frac{y}{x^2+y^2+z^2} \\ \frac{\partial \varphi}{\partial z} = \frac{z}{x^2+y^2+z^2} \end{cases}$$

$\Rightarrow \varphi = \frac{1}{2}\int \frac{2x\,dx}{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2) + h(y,z)$

$\therefore\ \varphi(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2) + \text{const}$

é um potencial para $\vec{F}$

$\therefore\ \vec{F}$ é conservativo.

$\underline{\text{Ex}}$. $\vec{F}(x,y,z) = (\overset{P}{x+z}, \overset{Q}{-(y+z)}, \overset{R}{x-y})$

$$\left.\begin{array}{l} \dfrac{\partial R}{\partial y} = -1 \\ \dfrac{\partial Q}{\partial z} = -1 \end{array}\right] = \quad \left.\begin{array}{l} \dfrac{\partial P}{\partial z} = 1 \\ \dfrac{\partial R}{\partial x} = 1 \end{array}\right] \checkmark \quad \left.\begin{array}{l} \dfrac{\partial Q}{\partial x} = 0 \\ \dfrac{\partial P}{\partial y} = 0 \end{array}\right] \checkmark$$

$$\left.\begin{array}{l} \text{rot } \vec{F} = \vec{0} \\ \text{dom } \vec{F} = \mathbb{R}^3 \text{ é simplesle connexo} \end{array}\right\} \Rightarrow \vec{F} \text{ é cons}$$

$$\begin{cases} \partial\varphi/\partial x = x+z \Rightarrow \varphi = \int x+z\,dx = \dfrac{x^2}{2} + xz + h(y,z) \\ \partial\varphi/\partial y = -y-z \\ \partial\varphi/\partial z = x-y \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -y-z = \dfrac{\partial\varphi}{\partial y} = 0 + \dfrac{\partial h}{\partial y} \\ \cancel{x}-y = \dfrac{\partial\varphi}{\partial z} = \cancel{x} + \dfrac{\partial h}{\partial z} \end{cases}$$

$$\begin{cases} \dfrac{\partial h}{\partial y} = -y-z \Rightarrow h(y,z) = \int -y-z\,dy = -\dfrac{y^2}{2} - yz + k(z) \\ \dfrac{\partial h}{\partial z} = -y \end{cases}$$

$$\Downarrow$$

$$-y = \frac{\partial h}{\partial z} = -y + k'(z)$$

$$\Downarrow$$

$$k'(z) = 0 \Rightarrow k(z) = \text{const}$$

$$h(y, z) = -\frac{y^2}{2} - yz + \text{const}$$

$$\varphi(x, y, z) = \frac{x^2}{2} + xz - \frac{y^2}{2} - yz + \text{const}$$

$$\frac{\partial \varphi}{\partial x} = x + z = P \checkmark \qquad \frac{\partial \varphi}{\partial y} = -y - z = Q \checkmark$$

$$\frac{\partial \varphi}{\partial z} = x - y = R \checkmark$$

$$\therefore \varphi(x, y, z) = \frac{x^2 - y^2}{2} + (x - y)z + \text{const}$$

é um potencial para $\vec{F}$ //