O TEOREMA DE STOKES, CONT.

29/11/21

EXEMPLO. Calcular o fluxo de $\text{rot}\vec{F}$, onde $\vec{F}(x,y,z) = (x, y, xyz)$ através da superfície

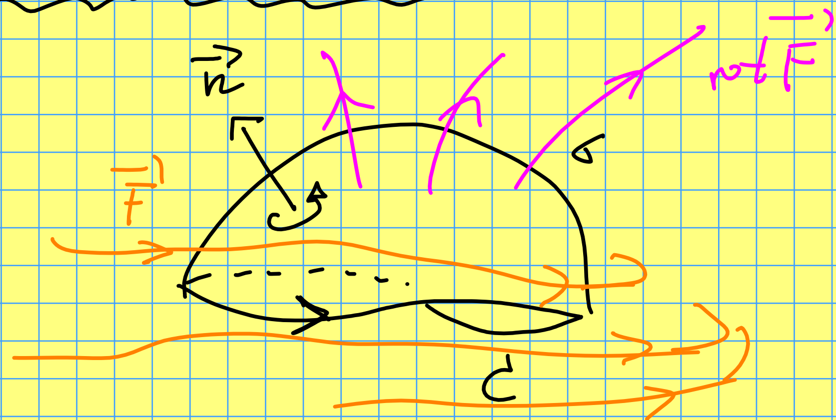
Plano; dado na forma de gráfico

$z = 1 + x + y$ com $x, y \geq 0$ e $x + y \leq 1$,

e a normal apontando para baixo

Resolução. Fórmula de Stokes

Orientações de σ e C compatíveis.

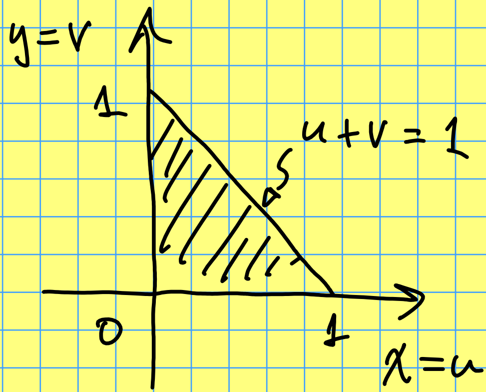


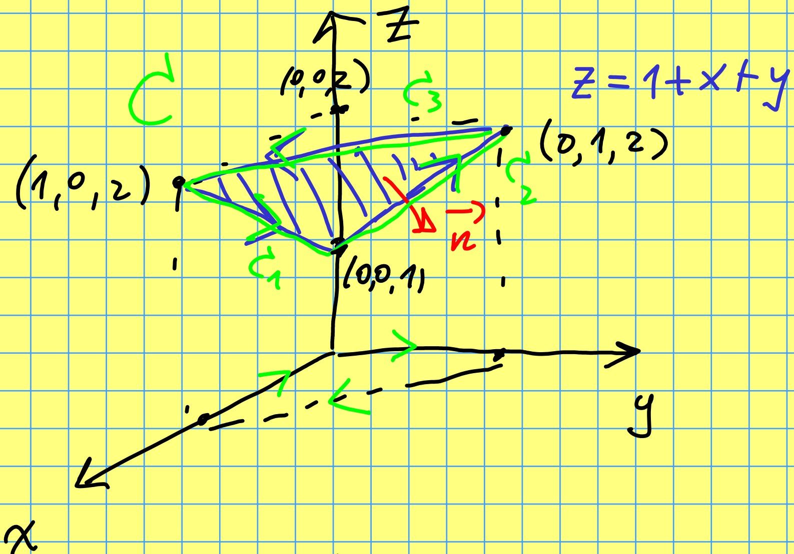
$C = \partial\sigma$ ("C é a fronteira de σ")

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\sigma \text{rot}\vec{F} \cdot \vec{n}\, dS$$

"circulação ao longo de C" = "fluxo através de σ"

Calcularemos. — incógnita do problema





$z = 1 + x + y$

$C = C_1 \cup C_2 \cup C_3$

$$C_1 : \begin{cases} x = 1 - t \\ y = 0 \\ z = 2 - t \end{cases} \qquad 0 \le t \le 1$$

$$C_2 : \begin{cases} x = 0 \\ y = t \\ z = 1 + t \end{cases} \qquad 0 \le t \le 1$$

$$C_3 : \begin{cases} x = t \\ y = 1 - t \\ z = 2 \end{cases} \qquad 0 \le t \le 1$$

$$\int_C \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} + \int_{C_2} \vec{F}\, d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 (1-t, 0, 0) \cdot (-1, 0, -1)\, dt$$

$$+ \int_0^1 (0, t, 0) \cdot (0, 1, 1)\, dt + \int_0^1 (t, 1-t, 2t(1-t)) \cdot (1, -1, 0)\, dt$$

$$= \int_0^1 t - 1 + t + t + t - 1 \; dt = \int_0^1 4t - 2\, dt$$

$$= 2t^2 - 2t\Big|_0^1 = 0 \quad //.$$

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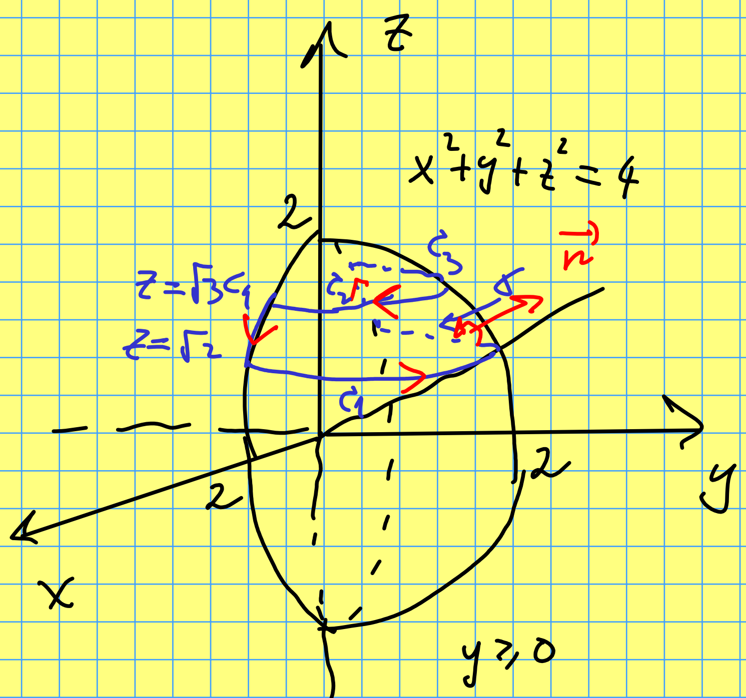
EXEMPLO. Calcular $\iint_\sigma \operatorname{rot}\vec{F} \cdot \vec{n}\, dS$, onde $\vec{F}(x,y,z) = (-y, x, x^2)$ e $\sigma: x^2+y^2+z^2 = 4$, $\sqrt{2} \leq z \leq \sqrt{3}$, $y \geq 0$, com a normal apontando para cima.

Resolução. Identifiquemos a superfície σ e sua fronteira C.



C é a concatenação de quatro arcos de círculo.

$z=\sqrt{2}$: $x^2+y^2 = 4-2 = 2$

$z=\sqrt{3}$: $x^2+y^2 = 4-3 = 1$

$y=0$: $x^2 = 4-z^2$

Stokes: $$\iint_\sigma \operatorname{rot}\vec{F} \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} + \cdots + \int_{C_4} \vec{F} \cdot d\vec{r} = (*)$$

$$C_1 : \begin{cases} x = \sqrt{2}\cos t \\ y = \sqrt{2}\sin t \\ z = \sqrt{2} \end{cases} \qquad 0 \leq t \leq \pi$$

$$C_3 : \begin{cases} x = -\cos t \\ y = \sin t \\ z = \sqrt{3} \end{cases} \qquad 0 \leq t \leq \pi$$

t	x	y
0	-1	0
$\pi/2$	0	1
π	1	0

$$C_2 : \begin{cases} x = -\sqrt{4-t^2} \\ y = 0 \\ z = t \end{cases} \qquad \sqrt{2} \leq t \leq \sqrt{3}$$

$$-C_4 : \begin{cases} x = \sqrt{4-t^2} \\ y = 0 \\ z = t \end{cases} \qquad \sqrt{2} \leq t \leq \sqrt{3}$$

$$(*) = \int_0^{\pi} (-\sqrt{2}\sin t, \sqrt{2}\cos t, 2\cos^2 t) \cdot (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0)\, dt$$

$$+ \int_{\sqrt{2}}^{\sqrt{3}} (0, -\sqrt{4-t^2}, 4-t^2) \cdot (*, 0, 1)\, dt$$

$$+ \int_0^{\pi} (-\sin t, -\cos t, \cos^2 t) \cdot (\sin t, \cos t, 0)\, dt$$

$$- \int_{\sqrt{2}}^{\sqrt{3}} (0, \sqrt{4-t^2}, 4-t^2) \cdot (*, 0, 1)\, dt$$

$$= \int_0^{\pi} 2\,dt + \cancel{\int_{\sqrt{2}}^{\sqrt{3}} 4-t^2\,dt} + \int_0^{\pi} -1\,dt - \cancel{\int_{\sqrt{2}}^{\sqrt{3}} 4-t^2\,dt}$$

$$= \pi \quad //$$

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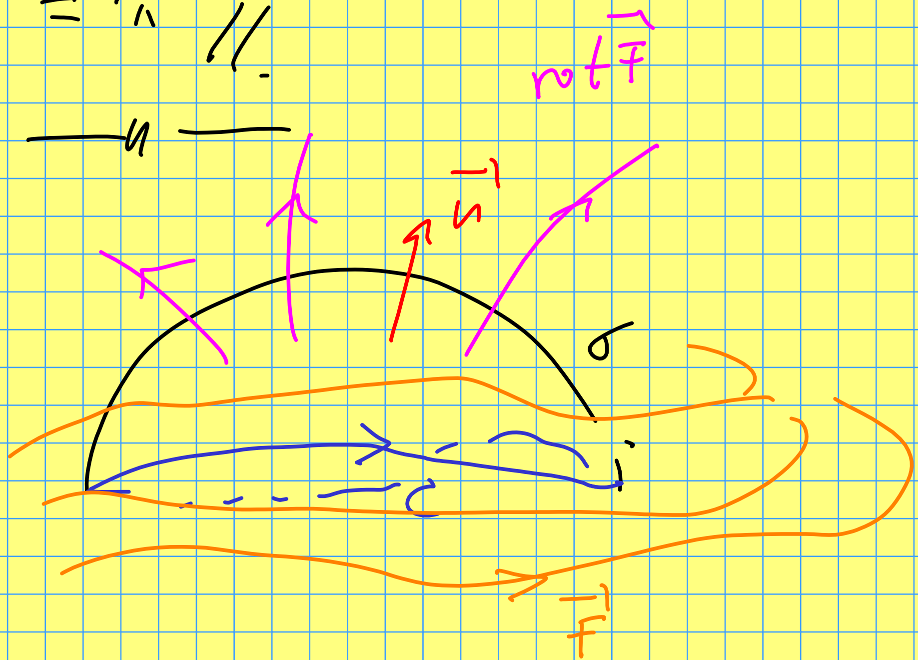

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$$\iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \,?$$

fluxo de $\vec{F}$ através de σ

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$dS = \|\vec{N}\|\,du\,dv$$

$$\vec{n} = \pm\frac{\vec{N}}{\|\vec{N}\|}$$

depende da orientação escolhida.

$$\vec{N} = \frac{\partial\sigma}{\partial u} \times \frac{\partial\sigma}{\partial v}$$

Se σ é escolhida adequadamente, $\vec{n} = +\dfrac{\vec{N}}{\|\vec{N}\|}$.

$$\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3 \qquad \sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in D$$

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(\sigma(u,v))\cdot \frac{\vec{N}(u,v)}{\cancel{\|\vec{N}(u,v)\|}}\cancel{\|\vec{N}(u,v)\|}\,du\,dv$$

$$= \iint_D \Big(P(x(u,v),y(u,v),z(u,v)),\ Q(x(u,v),y(u,v),z(u,v)),\ R(x(u,v),y(u,v),z(u,v))\Big)\cdot \left(\begin{vmatrix} \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{vmatrix}, \begin{vmatrix} \partial z/\partial u & \partial z/\partial v \\ \partial x/\partial u & \partial x/\partial v \end{vmatrix}, \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}\right) du\,dv$$

$$= \iint_D P\,\frac{\partial(y,z)}{\partial(u,v)} + Q\,\frac{\partial(z,x)}{\partial(u,v)} + R\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= \iint_D P\,\underbrace{\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv}_{dy\wedge dz} + \iint_D Q\,\underbrace{\frac{\partial(z,x)}{\partial(u,v)}\,du\,dv}_{dz\wedge dx} + \iint_D R\,\underbrace{\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv}_{dx\wedge dy}$$

$$\overset{\text{Notació}}{=} \iint_\sigma P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$$

(integral de superfície)

Int. de línia

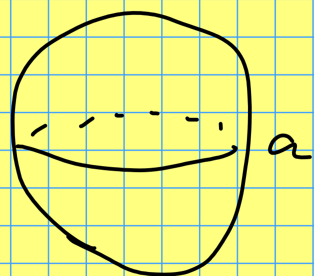
$$\int_C P\,dx + Q\,dy + R\,dz$$

Exemplo. Calcular

$$I = \iint_{\sigma} xz \, dy \wedge dz + yz \, dz \wedge dx + x^2 \, dx \wedge dy$$

onde $\sigma : x^2 + y^2 + z^2 = a^2$ está orientada pela normal exterior.

Resolução



T. Gauss $\qquad \vec{F} = \underline{xz}\,\vec{i} + y\underline{z}\,\vec{j} + x^2\,\vec{k}$

$$\operatorname{div} \vec{F} = z + z + 0 = 2z$$

$$I = \iiint_{B} 2z \, dx\,dy\,dz = 2 \underbrace{z_{CM}}_{=0} \operatorname{vol}(B) = 0 \quad //$$

$B : x^2 + y^2 + z^2 \leq 4$ ("bola")