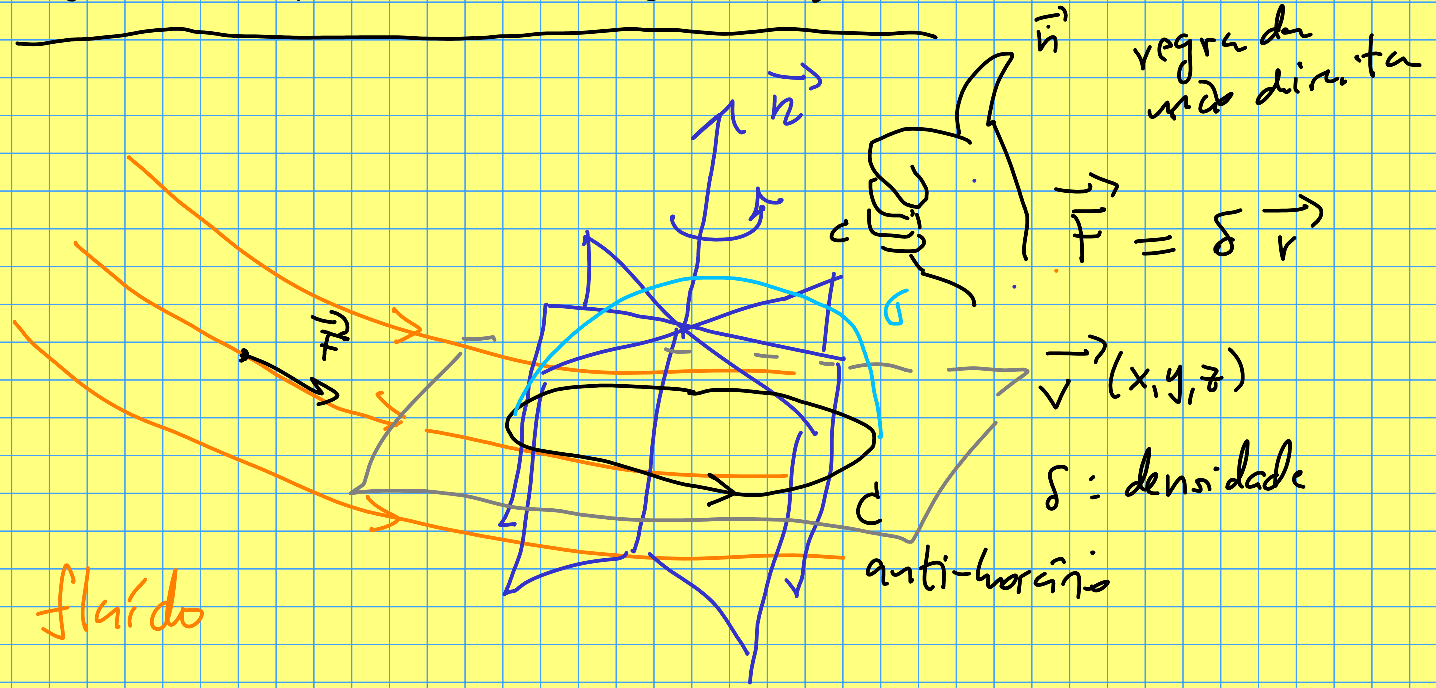
O TEOREMA DE STOKES

25/11/21



$$\int_C \vec{F} \cdot \underline{d\vec{r}} := \text{circulação do fluído ao longo de } C \text{ por unidade de tempo}$$

integral de linha

Teorema de Stokes

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{CIRCULAÇÃO DE } \vec{F} \text{ AO LONGO DE } C} = \iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, dS$$

FLUXO DE $\operatorname{rot}\vec{F}$ ATRAVÉS DE σ

integral de superfície

$C = \partial\sigma$

onde, sendo $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, temos:

- $$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

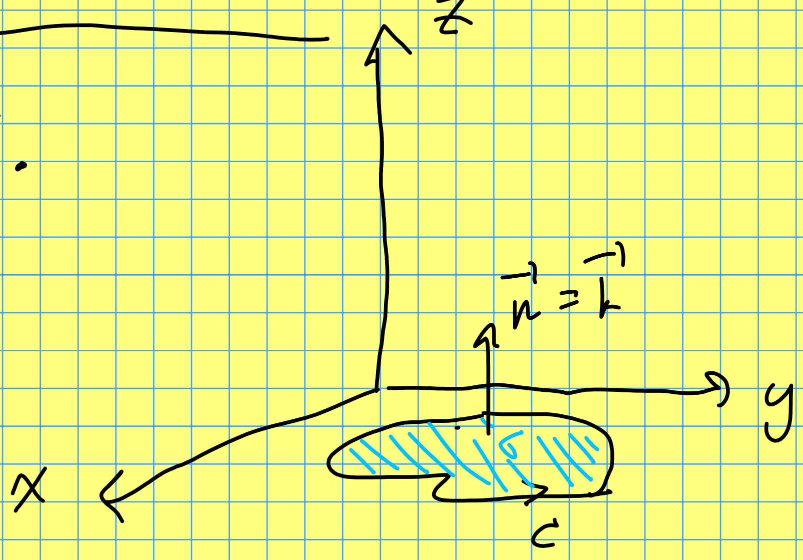
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \underline{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

$\text{rot}\,\vec{F} \cdot \vec{n}$ = mede o quanto $\vec{F}$ roda em torno de $\vec{n}$

- σ é qualquer superfície com fronteira C.
- Orientações de σ e C são compatíveis: regra da mão direita.

EXEMPLOS

1.



C: curva plana (fechada simples) no plano xy

σ: região delimitada por C

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\underline{\text{rot}\,\vec{F} \cdot \vec{n}} = \text{rot}\,\vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$
= \begin{cases} \displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + \cancel{R\,dz} \\[2em] \displaystyle\iint_\sigma \operatorname{rot}\vec{F}\cdot\vec{n}\,dS = \iint_\sigma \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dS \end{cases}
$$

$z = 0$ ao longo de C $\Rightarrow dz = 0$

dS = elemento de área de σ = elemento de área do plano xy $= dx\,dy$

$$
= \iint_\sigma \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy
$$

integral dupla.

$$
\therefore \int_C P\,dx + Q\,dy = \iint_\sigma \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy
$$

Neste caso, o T. de Stokes se reduz ao T. de Green.

2. Calcular a integral de linha

$$
I = \oint_C y^3 z^2\,dx + 3xy^2z^2\,dy + 2xy^3z\,dz
$$

ao longo da curva fechada C: $\begin{cases} x = \cos^5 t \\ y = \sin^{15} t \\ z = \cos^7 t \end{cases}$

$0 \le t \le 2\pi$.

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

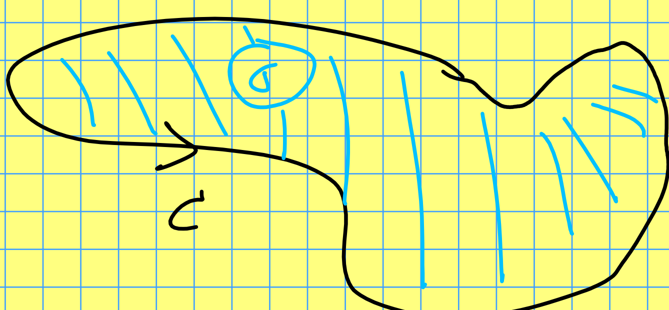
$$= \left(\frac{\partial(2xy^3z)}{\partial y} - \frac{\partial(3xy^2z^2)}{\partial z} \right)\vec{i} + \left(\frac{\partial(y^3z^2)}{\partial z} - \frac{\partial(2xy^3z)}{\partial x} \right)\vec{j}$$

$$+ \left(\frac{\partial(3xy^2z^2)}{\partial x} - \frac{\partial(y^3z^2)}{\partial y} \right)\vec{k}$$

$$= (6xy^2z - 6xy^2z)\vec{i} + (2y^3z - 2y^3z)\vec{j}$$

$$+ (3y^2z^2 - 3y^2z^2)\vec{k} = \vec{0}$$

Pela fórmula de Stokes $I = 0$ ∴



Deve existir σ bordeada por C e onde $\vec{F}$ está definido

$$\vec{F} = \frac{1}{x^2+y^2}(x,y,z)$$

$$(x,y) \neq (0,0)$$

$$\operatorname{dom} \vec{F} = \mathbb{R}^3 \setminus \text{eixo } z$$

Não podemos aplicar o T. Stokes!

σ contém um ponto (pelo menos) do eixo z.

3. Calcular $\displaystyle\iint_{\sigma} \text{rot}\,\vec{F}\cdot\vec{n}\,dS$ onde

$\vec{F}(x,y,z) = y\vec{i} + (x+y)\vec{k}$,

$$\sigma : \begin{cases} x = u \\ y = v \\ z = 2-u^2-v^2 \end{cases} \quad \text{com } \underbrace{u^2+v^2 \le 1}_{D},$$

e $\vec{n}$ aponta para cima.

Resolução. Cálculo direto

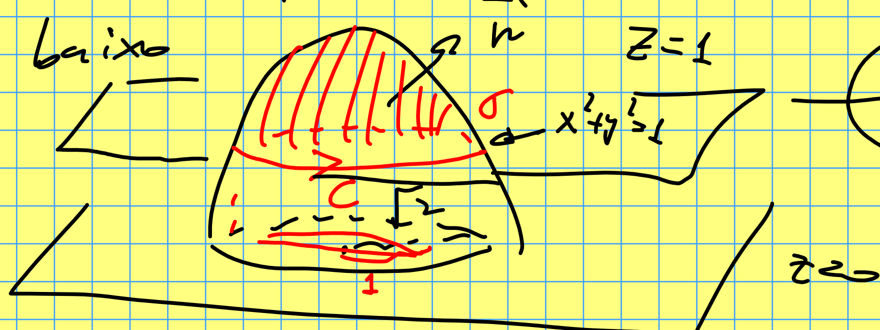
$$\text{rot}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

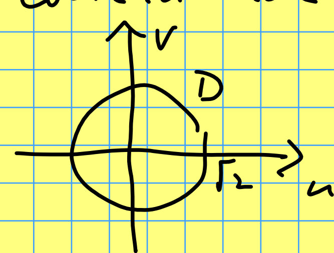
$$= \left(\frac{\partial(x+y)}{\partial y} - \cancel{\frac{\partial(0)}{\partial z}}\right)\vec{i} + \left(\cancel{\frac{\partial(y)}{\partial z}} - \frac{\partial(x+y)}{\partial x}\right)\vec{j}$$

$$+ \left(\cancel{\frac{\partial(0)}{\partial x}} - \frac{\partial(y)}{\partial y}\right)\vec{k} = \vec{i} - \vec{j} - \vec{k}$$

σ: $z = 2 - x^2 - y^2$ parabolóide com concavidade para baixo

gráfico de $f(x,y) = 2 - x^2 - y^2$





$$\vec{N} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

$$= (+2u, +2v, 1)$$

$$\vec{n} = \frac{\vec{N}}{\|\vec{N}\|} \qquad dS = \|\vec{N}\| \, du\, dv$$

$$\iint_{\sigma} \operatorname{rot}\vec{F} \cdot \vec{n} \, dS = \iint_{D} \operatorname{rot}\vec{F} \, \frac{\vec{N}}{\cancel{\|\vec{N}\|}} \cancel{\|\vec{N}\|} \, du\, dv$$

$$= \iint_{D} (1, -1, -1) \cdot (2u, 2v, 1) \, du\, dv$$

$$= \iint_{D} 2u - 2v - 1 \, du\, dv$$

Coord. polares:

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$du\,dv = r\,dr\,d\theta$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$= \int_0^{2\pi} \int_0^{1} (2r\cos\theta - 2r\sin\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^1 2r^2 dr \underbrace{\cancel{\int_0^{2\pi} \cos\theta \, d\theta}}_{0} - \int_0^1 2r^2 dr \underbrace{\cancel{\int_0^{2\pi} \sin\theta \, d\theta}}_{0}$$

$$- \int_0^1 r\,dr \int_0^{2\pi} d\theta = -\frac{r^2}{\cancel{2}}\Big|_0^1 \cdot \cancel{2}\pi = -\pi$$

Usando o T. de Stokes

$$\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + (x+y)\underbrace{dz}_{=0} = \int_C y\,dx$$

$(\because z=1)$

C orientada no sentido anti-horário (olhando de cima)

$$C:\begin{cases} x = \cos t \\ y = \sin t \qquad 0 \leq t \leq 2\pi \\ z = 1 \end{cases}$$

$$\vec{F}(x,y,z) = (y,0,x+y)$$

$$= \int_0^{2\pi} \sin t(-\sin t)\,dt$$

$$= -\int_0^{2\pi} \sin^2 t\,dt$$

$$= -\int_0^{2\pi} \frac{1-\cancel{\cos 2t}}{2}\,dt$$

$$= -\frac{1}{2}\,2\pi = -\pi \quad //$$

4. Seja $\vec{F} = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$.

Então

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \underbrace{\left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)}_{=}\vec{i} + \underbrace{\left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}\right)}_{=}\vec{j}$$

$$+\left(\frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial^2\varphi}{\partial y\partial x}\right)\vec{k}=\vec{0}$$

$$\therefore \boxed{\operatorname{rot}\nabla\varphi=\vec{0}\quad\forall\varphi}$$

5. Seja $\vec{G}=\operatorname{rot}\vec{F}$. Então

$$\operatorname{div}\vec{G}=?$$

Sendo $\vec{F}=(P,Q,R)$, temos

$$\vec{G}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) \text{ e}$$

$$\operatorname{div}\vec{G}=\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$$=\cancel{\frac{\partial^2 R}{\partial x\partial y}}-\cancel{\frac{\partial^2 Q}{\partial x\partial z}}+\cancel{\frac{\partial^2 P}{\partial y\partial z}}-\cancel{\frac{\partial^2 R}{\partial y\partial x}}+\cancel{\frac{\partial^2 Q}{\partial z\partial x}}-\cancel{\frac{\partial^2 P}{\partial z\partial y}}$$

$= 0$

$\therefore \boxed{\operatorname{div} \operatorname{rot} \vec{F} = 0 \quad \forall \vec{F}}$