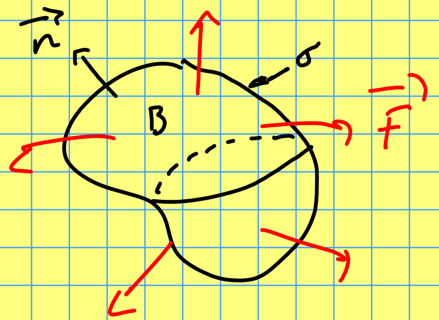
O TEOREMA DA DIVERGÊNCIA, CONT.

22/11/21



$$\underbrace{\iint_\sigma \vec{F}\cdot\vec{n}\,dS}_{\text{fluxo}} = \iiint_B \operatorname{div}\vec{F}\;dx\,dy\,dz$$

Exemplos.

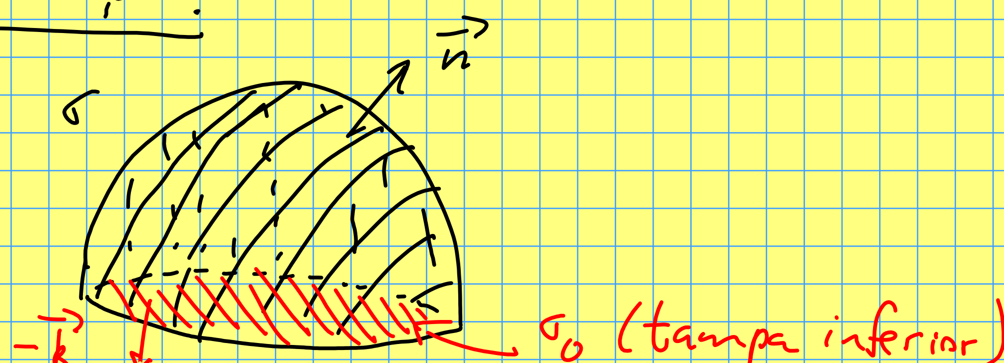
1. Dados: $\vec{F}(x,y,z) = (x^2y,\ xy^2,\ 5-4xyz)$

$\sigma: x^2+y^2+z^2 = 4,\ z \geq 0$

$\vec{n}$: normal unitária com componente z positiva

Calcular o fluxo de $\vec{F}$ através de σ.

Resolução.



(σ não inclui a "tampa inferior")

Agora $\sigma \cup \sigma_0$ é uma superfície fechada, que delimita uma região B, à qual portanto podemos aplicar o T. de Gauss:

$$\underbrace{\iint_{\sigma\cup\sigma_0} \vec{F}\cdot\vec{n}\, dS}_{} = \iiint_B \operatorname{div}\vec{F}\, dx\,dy\,dz = 0$$

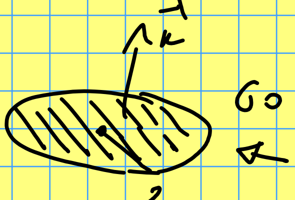
$$= \iint_{\sigma} \vec{F}\cdot\vec{n}\, dS + \iint_{\sigma_0} \vec{F}\cdot(-\vec{k})\, dS$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(5-4xyz)$$

$$= 2xy + 2xy - 4xy = 0$$

$$\Rightarrow \iint_{\sigma} \vec{F}\cdot\vec{n}\, dS = \iint_{\sigma_0} \vec{F}\cdot\vec{k}\, dS$$

Fluxo desejado



$$= \iint_{\sigma_0} 5 - \cancel{4xyz}^{\,0}\, dS$$

$$= 5\,\text{Área}(\sigma_0)$$

$$= 5\cdot\pi 2^2 = 20\pi$$

2. (Campo elétrico)

$$\vec{E} = \frac{q}{r^3\varepsilon_0}\vec{r}$$

$q \neq 0$: carga elétrica

$\vec{r}$: vetor posição

$\varepsilon_0 \approx 8.854\times 10^{-12}\,\dfrac{C^2}{N\,m^2}$

$r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

$\vec{E} = P\vec{i} + Q\vec{j} + R\vec{k}$ $\quad\varepsilon_0$: permissividade do vácuo

$$\vec{E}(x,y,z) = \frac{q}{\varepsilon_0 (x^2+y^2+z^2)^{3/2}} (x,y,z) \qquad (x,y,z) \neq (0,0,0)$$

(a) Calcular $\operatorname{div} \vec{E}$.

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\frac{q x}{\varepsilon_0 (x^2+y^2+z^2)^{3/2}} \right) = \frac{q (x^2+y^2+z^2)^{3/2} - q x \frac{3}{\cancel{2}} (x^2+y^2+z^2)^{1/2} \cancel{2} x}{\varepsilon_0 (x^2+y^2+z^2)^3}$$

$$= \frac{q (x^2+y^2+z^2)^{1/2}}{\varepsilon_0 (x^2+y^2+z^2)^3} \left(x^2+y^2+z^2 - 3x^2 \right)$$

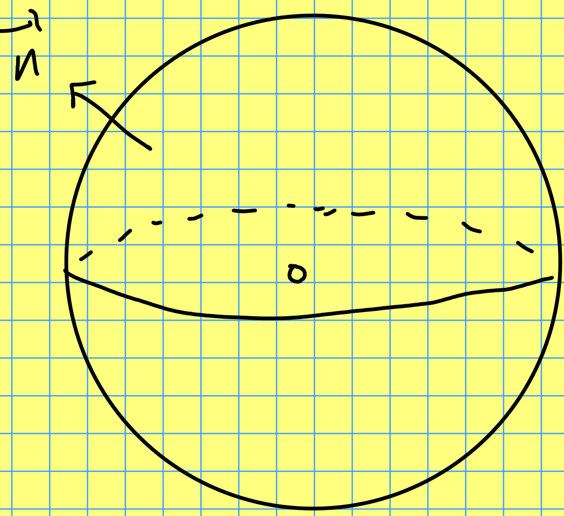
$$= \frac{q}{\varepsilon_0 (x^2+y^2+z^2)^{5/2}} (y^2+z^2-2x^2)$$

$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}\left(\frac{q y}{\varepsilon_0 (x^2+y^2+z^2)^{3/2}} \right) = \frac{q}{\varepsilon_0 (x^2+y^2+z^2)^{5/2}} (x^2+z^2-2y^2)$$

$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}\left(\frac{q z}{\varepsilon_0 (x^2+y^2+z^2)^{3/2}} \right) = \frac{q}{\varepsilon_0 (x^2+y^2+z^2)^{3/2}} (x^2+y^2-2z^2)$$

$$\therefore \operatorname{div} \vec{E} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

(b) Calcular o fluxo de $\vec{E}$ através da superfície esférica $\sigma: x^2+y^2+z^2 = R^2$ $(R>0)$, orientada pela normal exterior.



$\vec{E}$ não está definido em todo o interior de σ!

$(0,0,0) \notin \text{dom}\,\vec{E}$

Assim, não é permitido aplicar o T. Gauss.

$$\text{Fluxo} = \iint_\sigma \vec{E} \cdot \vec{n}\, dS = \iint_\sigma \frac{q}{\varepsilon_0 r^3}\, \vec{r} \cdot \frac{\vec{r}}{R}\, dS = (*)$$

Em se tratando de uma superfície esférica, $\vec{n}$ é múltiplo positivo de $\vec{r}$: $\vec{n} = \dfrac{\vec{r}}{\|\vec{r}\|} = \dfrac{\vec{r}}{r} = \dfrac{\vec{r}}{R}$ $\quad r = R$ sobre σ

$$(*) = \iint_\sigma \frac{q}{\varepsilon_0 R^3}\, \frac{1}{R} \cdot (\vec{r} \cdot \vec{r})\, dS$$

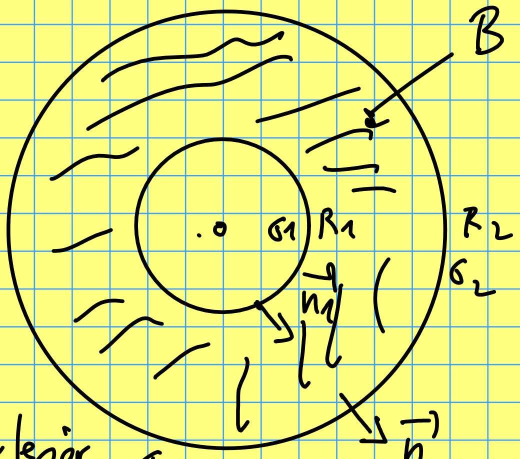
$\vec{r} \cdot \vec{r} = \|\vec{r}\|^2 = r^2 = R^2$

$$= \iint_\sigma \frac{q}{\varepsilon_0 R^4}\, R^2\, dS$$

$$= \frac{q}{\varepsilon_0 R^2} \iint_\sigma dS = \frac{q}{R^2 \varepsilon_0}\, \text{Área}(\sigma)$$

$$= \frac{q}{\cancel{R^2}\, \varepsilon_0} \cdot 4\pi \cancel{R^2} = \underline{\frac{4\pi q}{\varepsilon_0}}.$$

$$\text{Fluxo} = \frac{4\pi q}{\varepsilon_0} \text{ é independente de } R.$$



$\vec{n}_1$ exterior a σ_1
e interior a B

$\partial B = \sigma_1 \cup \sigma_2$

$$\boxed{\operatorname{div}\vec{E} = 0 \text{ em } B}$$

B: região intermediária entre a esfera de raio R_1 e a esfera de raio R_2.

Singularidade de $\vec{E}$ não pertence a B!
$\underset{(0,0,0)}{\uparrow}$

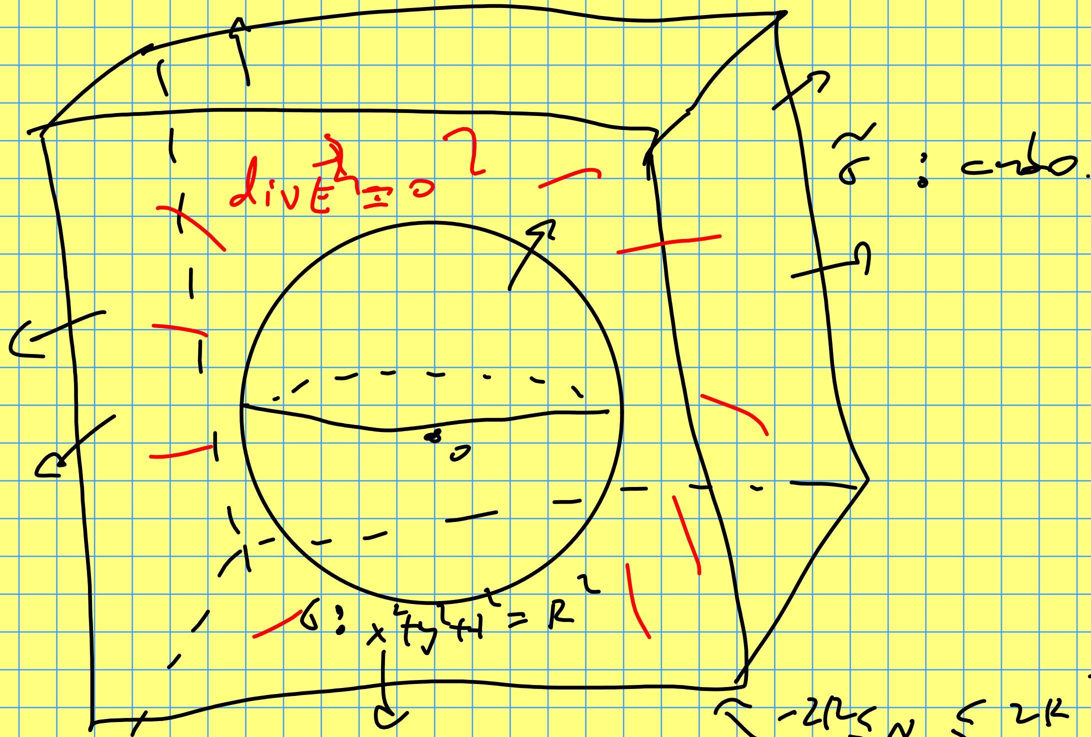
<u>T. Gauss</u>:

$$\underbrace{\iint_{\sigma_1 \cup \sigma_2} \vec{E}\cdot\vec{n}\, dS}_{} = \iiint_B \underbrace{\operatorname{div}\vec{E}}_{=0}\, dx\,dy\,dz = 0$$

$$= \iint_{\sigma_1} \vec{E}\cdot(-\vec{n}_1)\, dS + \iint_{\sigma_2} \vec{E}\cdot\vec{n}_2\, dS$$

$$\Rightarrow \iint_{\sigma_2} \vec{E}\cdot\vec{n}_2\, dS = \iint_{\sigma_1} \vec{E}\cdot\vec{n}_1\, dS$$

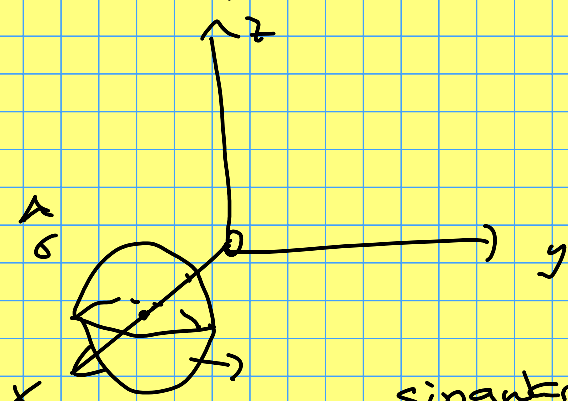
Fluxo de $\vec{E}$ através de σ_2 = Fluxo de $\vec{E}$ através de σ_1



$$\left.\begin{array}{l} -2R \le x \le 2R \\ -2R \le y \le 2R \\ -2R \le z \le 2R \end{array}\right\} \tilde{\sigma}$$

$$\iint_{\tilde{\sigma}} \vec{E}\cdot\vec{n}\, dS = \iint_{\sigma} \vec{E}\cdot\vec{n}\, dS = \frac{4\pi q}{\varepsilon_0}$$

$$\hat{\sigma} : (x-2)^2 + y^2 + z^2 = 1$$



$$\iint_{\hat{\sigma}} \vec{E}\cdot\vec{n} = 0$$

pois $\hat{\sigma}$ não envolve a singularidade de $\vec{E}$ e $\operatorname{div}\vec{E} = 0$

no interior de $\hat{J}$.