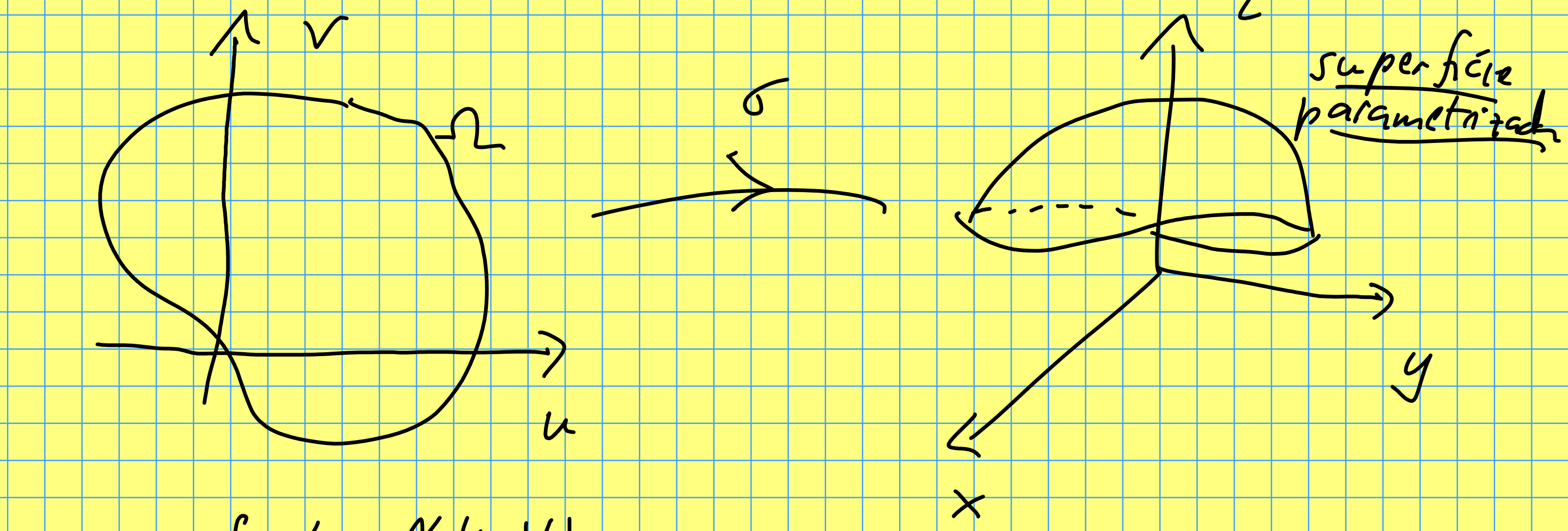
ÁREA DE UMA SUPERFÍCIE

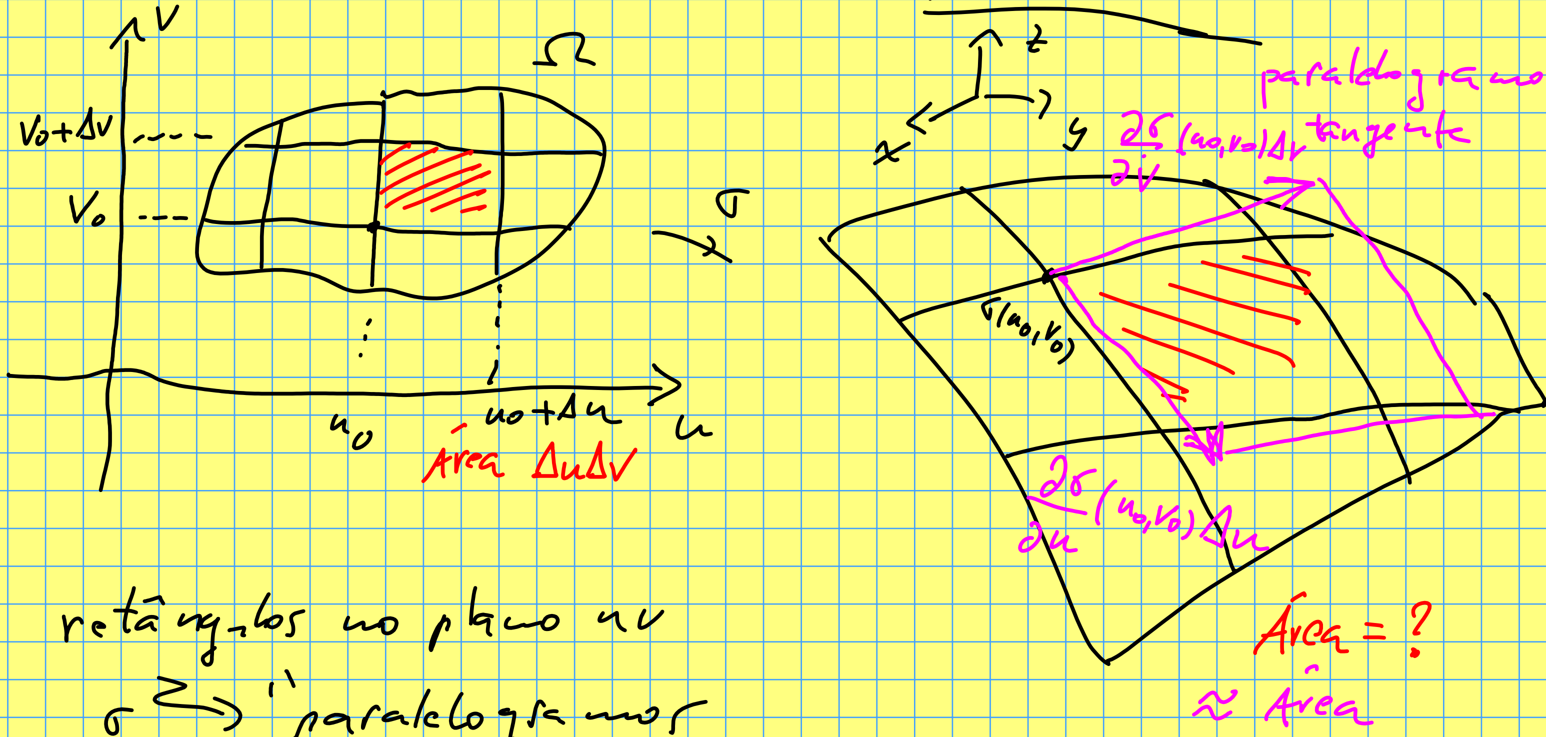
11/11/21



$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$\sigma : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$

parametrização



retângulos no plano uv

$\sigma \Rightarrow$ 'paralelogramos curvilíneos' na superfície

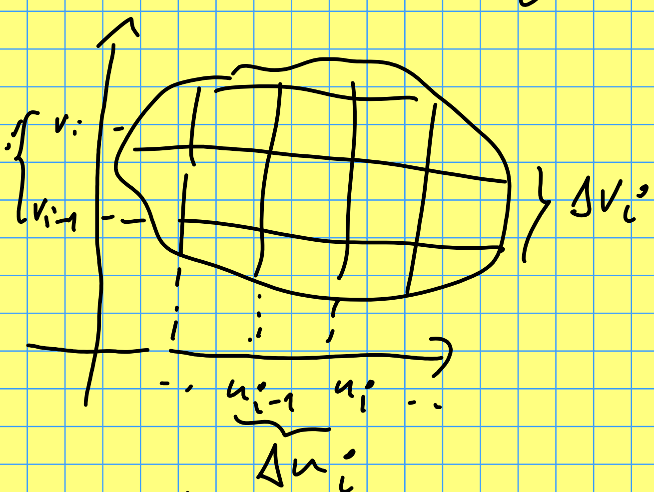
Área do paralelogramo tangente $= \left\| \frac{\partial\sigma}{\partial u}\Delta u \times \frac{\partial\sigma}{\partial v}\Delta v \right\|$

$$= \underbrace{\left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\|}_{= \vec{N}} \underbrace{\Delta u \Delta v}_{\text{área do retângulo no plano } uv}$$

$$\text{Área}(\sigma) = \sum \text{Áreas dos paralelogramos curvilíneos}$$

$$\approx \sum \text{Áreas dos paralelogramos tangentes}$$

$$= \sum_i \left\| \frac{\partial \sigma}{\partial u}(u_i, v_i) \times \frac{\partial \sigma}{\partial v}(u_i, v_i) \right\| \Delta u_i \Delta v_i$$



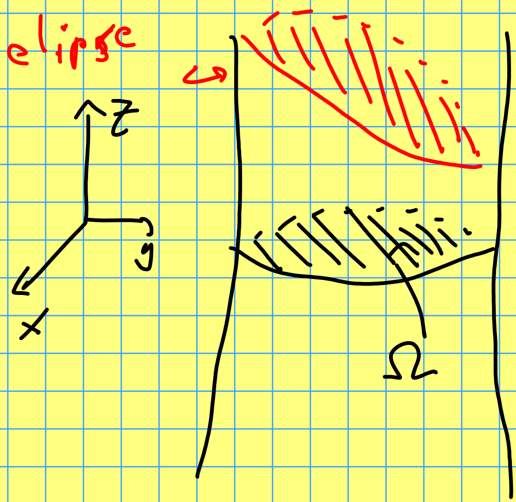
Passando ao limite quando Δu_i e $\Delta v_i \to 0$, obtemos, finalmente:

$$\boxed{\text{Área}(\sigma) = \iint_\Omega \left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| du\, dv}$$

<u>Exemplos.</u>

1. Calcular a área da porção do plano $x + y + z = 1$ que fica no interior do cilindro $x^2 + y^2 = 1$.

<u>Resolução.</u>



$$z = 1 - x - y$$

Gráfico de $f(x,y) = 1 - x - y$

$$\sigma : \begin{cases} x = u \\ y = v \\ z = 1 - u - v \end{cases} \qquad (u,v) \in \Omega$$

$$\Omega : u^2 + v^2 \leq 1$$

$$\text{Área}(\sigma) = \iint_\Omega \left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| du\,dv = \sqrt{3} \iint_\Omega du\,dv = \sqrt{3}\, \text{Área}(\Omega) = \sqrt{3}\pi\ //.$$

$$\left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} = \sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}$$

2. Calcular a área da esfera $x^2 + y^2 + z^2 = R^2$ $(R > 0)$.

<u>Resolução</u>. Na aula passada, usando coords esféricas, obtivemos

$$\vec{N}(\theta, \varphi) = -R \sin\varphi \ \sigma(\theta, \varphi)$$

$$\Rightarrow \|\vec{N}\| = |-R \sin\varphi| \ \|\sigma(\theta,\varphi)\|$$

$$= R \sin\varphi \cdot R$$

$$= R^2 \sin\varphi$$

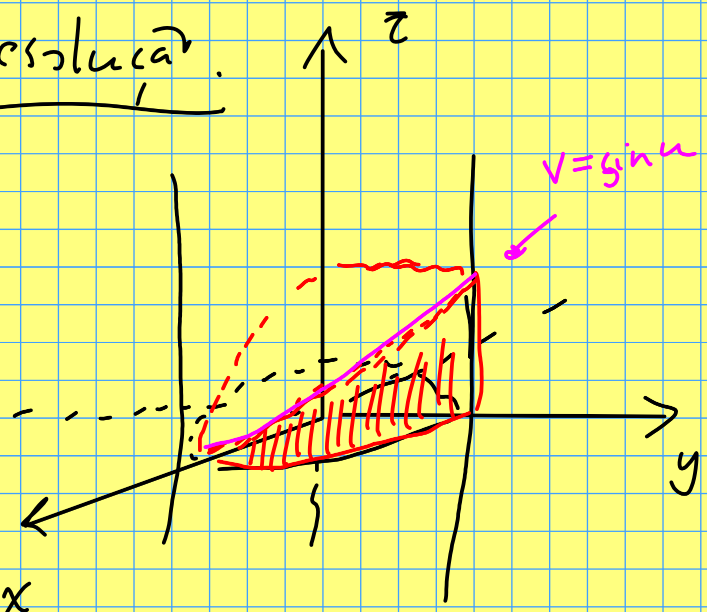
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$

$$\text{Área}(\sigma) = \int_0^{\pi}\int_0^{2\pi} R^2 \sin\varphi \, d\theta \, d\varphi$$

$$= R^2 \cdot 2\pi \cdot \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{=2} = 4\pi R^2 \, //.$$

3. Calcular a área da porção do cilindro $x^2+y^2=1$ delimitada pelo cilindro $x^2+z^2=1$

Resolução:



Coords cilíndricas:

$$\begin{cases} x = \cos u \\ y = \sin u \\ z = v \end{cases} \qquad \|\vec{N}\| = 1$$

1.º octante:

$$\begin{cases} 0 \leq u \leq \frac{\pi}{2} \\ 0 \leq v \leq \sin u \end{cases}$$

Intersecção entre os dois cilindros Ω: no 1.º octante:

$$\begin{cases} x^2+y^2=1 \\ x^2+z^2=1 \end{cases} \Rightarrow \begin{array}{c} y = z \geq 0 \\ \sin u = v \end{array}$$

$$\underset{\text{desejada}}{\text{Área}} = 8 \ \text{Área no 1.º octante}$$

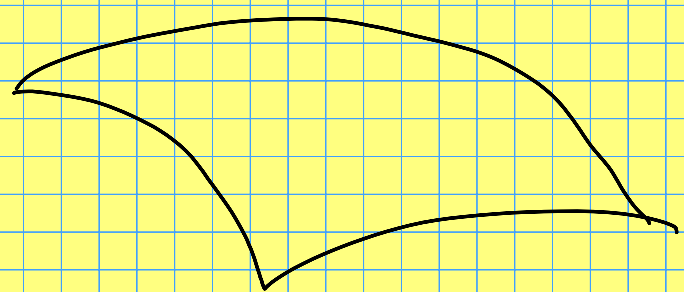
$$= 8 \iint_{\Omega} \|\vec{N}\| \, du\, dv = 8 \int_0^{\pi/2}\int_0^{\sin u} dv \, du$$

$$= 8 \int_0^{\pi/2} \sin u \, du = 8 \left[-\cos u \right]_0^{\pi/2} = -8(0-1) = 8 \; //.$$

— // —

INTEGRAIS DE SUPERFÍCIE

Motivação



Dados: – chapa (fina) no $\mathbb{R}^3$: $\sigma : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$

– densidade superficial : δ (função na superfície)

Queremos calcular a massa da chapa.

Elemento de área é $\|\vec{N}\| \, du dv = dS$

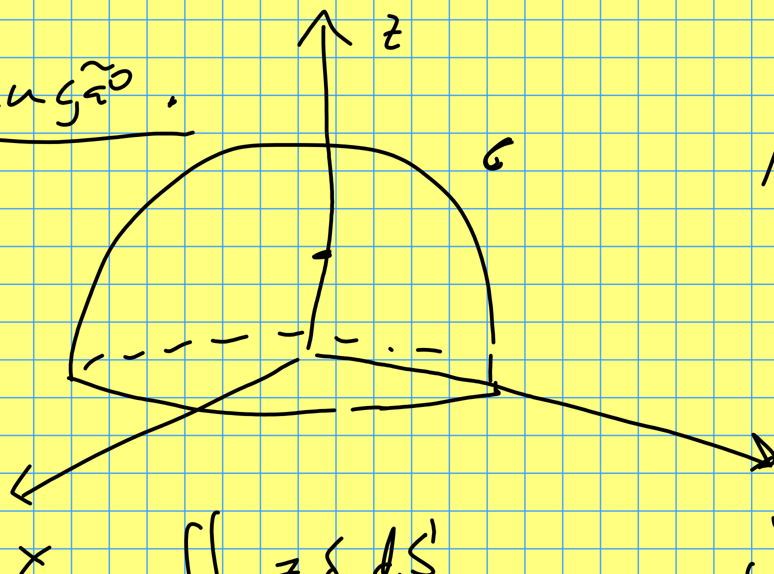
$$\Rightarrow \underbrace{\boxed{\text{Massa} = \iint_\Omega \delta \, \underbrace{\left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| du dv}_{= dS}}}$$

$$\overset{\text{Notação}}{=} \iint_\sigma \delta \, dS \leftarrow \text{Integral de superfície de } \delta \text{ ao longo de } \sigma.$$

$$\text{Área}(\sigma) = \iint_\sigma dS \qquad \text{Massa}(\sigma) = \iint_\sigma \delta \, dS$$

Exemplo. Calcular o centro de massa do hemisfério $x^2+y^2+z^2 = R^2$, $z \geq 0$, supondo a densidade constante

Resolução.



$$\text{Área} = \frac{1}{2} 4\pi R^2 = 2\pi R^2$$

Por simetria do hemisfério e da densidade,

$$x_{CM} = y_{CM} = 0$$

$$z_{CM} = \frac{\iint_\sigma z \delta \, dS}{\iint_\sigma \delta \, dS} = \frac{\cancel{\delta}}{\cancel{\delta}} \frac{\iint_\sigma z \, dS}{\iint_\sigma dS} = \frac{\iint_\sigma z \, dS}{2\pi R^2} \quad (+)$$

$$\iint_\sigma z \, dS = \int_0^{\pi/2} \int_0^{2\pi} R\cos\varphi \, R^2 \sin\varphi \, d\theta \, d\varphi = (*)$$

$$\sigma: \begin{cases} x = R\cos\theta \sin\varphi \\ y = R\sin\theta \sin\varphi \\ z = R\cos\varphi \end{cases} \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{array}$$

$$d_iS' = \|\vec{N}\| \, d\theta \, d\varphi = R^2 \sin\varphi \, d\theta \, d\varphi$$

$$(*) = R^3 \cdot 2\pi \cdot \int_0^{\pi/2} \sin\varphi \underbrace{\cos\varphi \, d\varphi}_{= d(\sin\varphi)} = 2\pi R^3 \left. \frac{\sin^2\varphi}{2} \right|_0^{\pi/2}$$

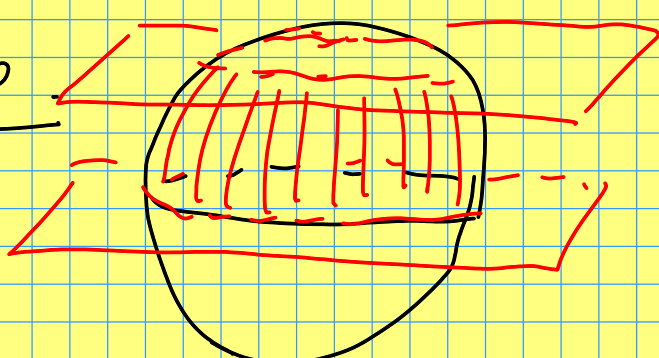
$$= \pi R^3 \quad \therefore \quad z_{cm} \overset{(+)}{=} \frac{\pi R^3}{2\pi R^2} = \frac{R}{2}$$

$$\therefore \; CM = \left(0, 0, \frac{R}{2}\right) \; //.$$

<u>Exemplo.</u> Calcular a área da porção da esfera $x^2 + y^2 + z^2 = R^2$ delimitada pelos planos $z = a$ e $z = a + h$

$(-R \leq a < a+h \leq R)$

<u>Resolução</u>



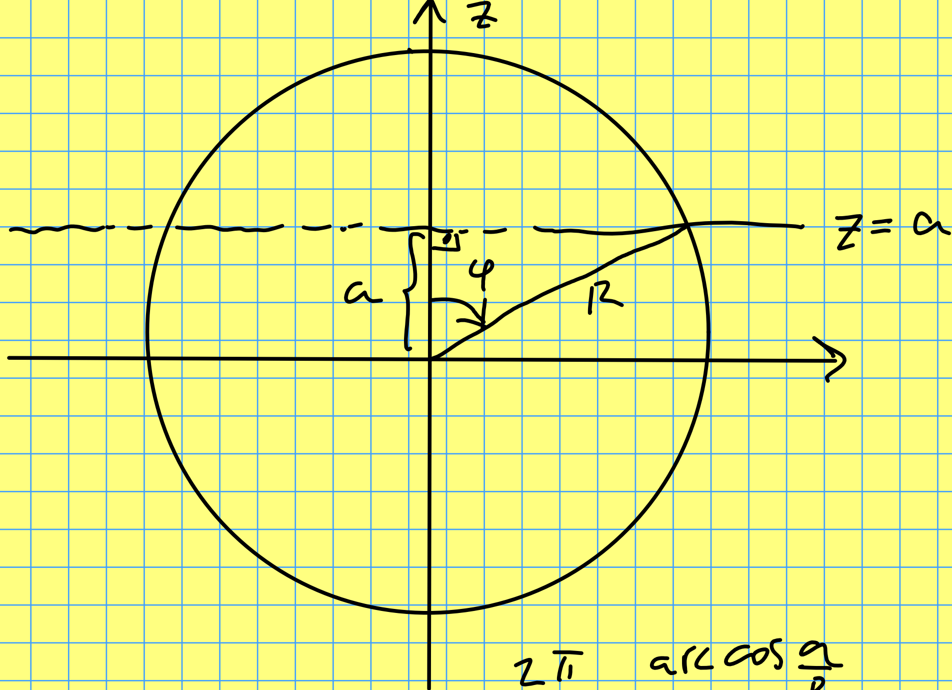
$$\text{Área} = \iint_\sigma d_iS' = \iint_\Omega \|\vec{N}\| \, d\theta \, d\varphi = R^2 \iint_\Omega \sin\varphi \, d\theta \, d\varphi$$

$\Omega = ?$

$b = a + h$

$0 \leq \theta \leq 2\pi$

$\arccos b \leq \varphi \leq \arccos a$



$$\cos\varphi = \frac{a}{R}$$

$$\varphi = \arccos\frac{a}{R}$$

$$\text{Área} = R^2 \int_0^{2\pi} \int_{\arccos\frac{b}{R}}^{\arccos\frac{a}{R}} \sin\varphi \, d\theta \, d\varphi$$

$$= R^2 \cdot 2\pi \cdot \left[-\cos\varphi\right]_{\varphi=\arccos b}^{\varphi=\arccos a}$$

$$= 2\pi R^2 \left(\frac{b}{R} - \frac{a}{R}\right) = 2\pi R h \quad //$$

(Independe de a!)