SUPERFÍCIES EM $\mathbb{R}^3$

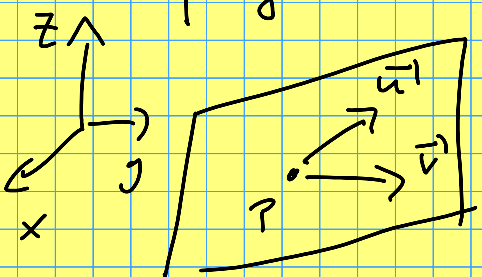
08/11/21

(& SUAS PARAMETRIZAÇÕES)

Exemplos

1. Plano em $\mathbb{R}^3$

Eq. geral $\quad ax+by+cz=d$



Eqs. parametricas

$$\sigma: \begin{cases} x = x_0 + \lambda a_1 + \mu a_2 \\ y = y_0 + \lambda b_1 + \mu b_2 \\ z = z_0 + \lambda c_1 + \mu c_2 \end{cases} \qquad \lambda, \mu \in \mathbb{R} \text{ parâmetros}$$

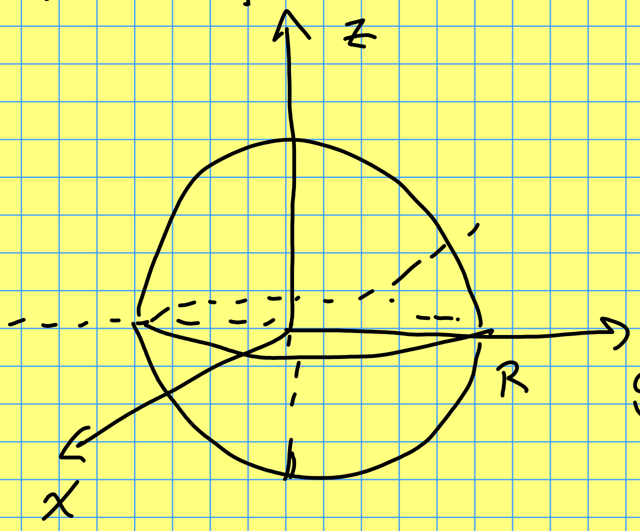
$P = (x_0, y_0, z_0) \leftarrow$ um ponto do plano

$\left.\begin{array}{l} \vec{u} = (a_1, b_1, c_1) \\ \vec{v} = (a_2, b_2, c_2) \end{array}\right\}$ vetores diretores do plano

$$\sigma(\underset{u}{\lambda}, \underset{v}{\mu}) = (x_0 + \lambda a_1 + \mu a_2,\ y_0 + \lambda b_1 + \mu b_2,\ z_0 + \lambda c_1 + \mu c_2)$$

$\sigma: \mathbb{R}^2 \to \mathbb{R}^3$ parametrização do plano

2. Esfera em $\mathbb{R}^3$



$$\sigma : \begin{cases} x = R\cos\theta \sin\varphi \\ y = R\sin\theta \sin\varphi \\ z = R\cos\varphi \end{cases}$$

(coords esféricas)

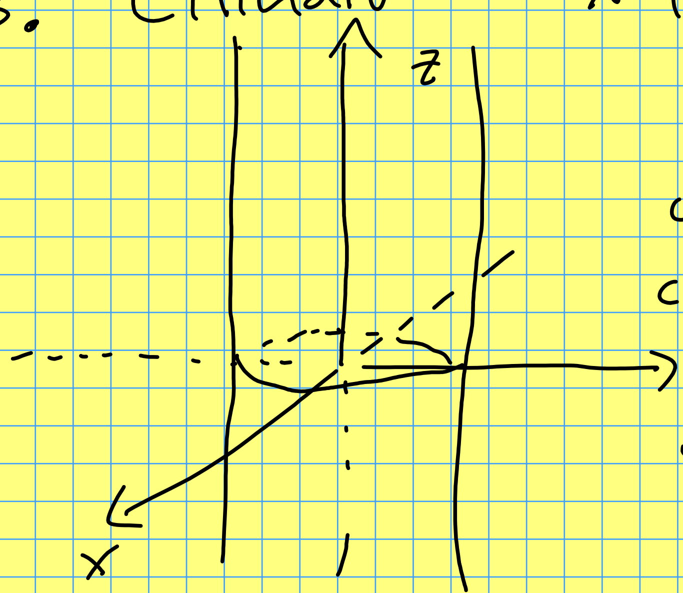
θ, φ : parâmetros

θ: longitude $\quad \theta \in [0, 2\pi]$

φ: co-latitude $\quad \varphi \in [0, \pi]$

$$\sigma : \underbrace{[0, 2\pi] \times [0, \pi]}_{\text{retângulo em } \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

3. Cilindro $\quad x^2 + y^2 = R^2$

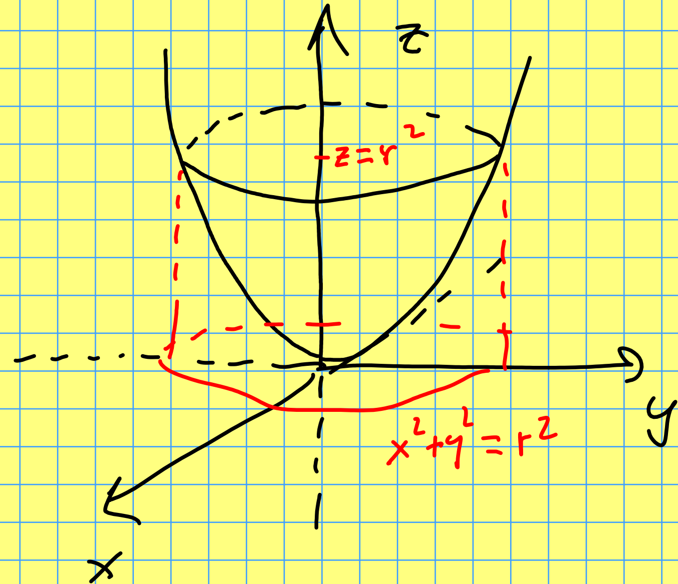


cilindro vertical
circular reto

$$\sigma : \begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = h \end{cases} \qquad h \in \mathbb{R} \qquad 0 \leq \theta \leq 2\pi$$

(coords. cilíndricas)

4. Parabolóide de revolução $z = \underbrace{x^2+y^2}_{\text{(dist ao eixo } z)^2}$.



$$\sigma : \begin{cases} x = u \\ y = v \\ z = u^2+v^2 \end{cases}$$

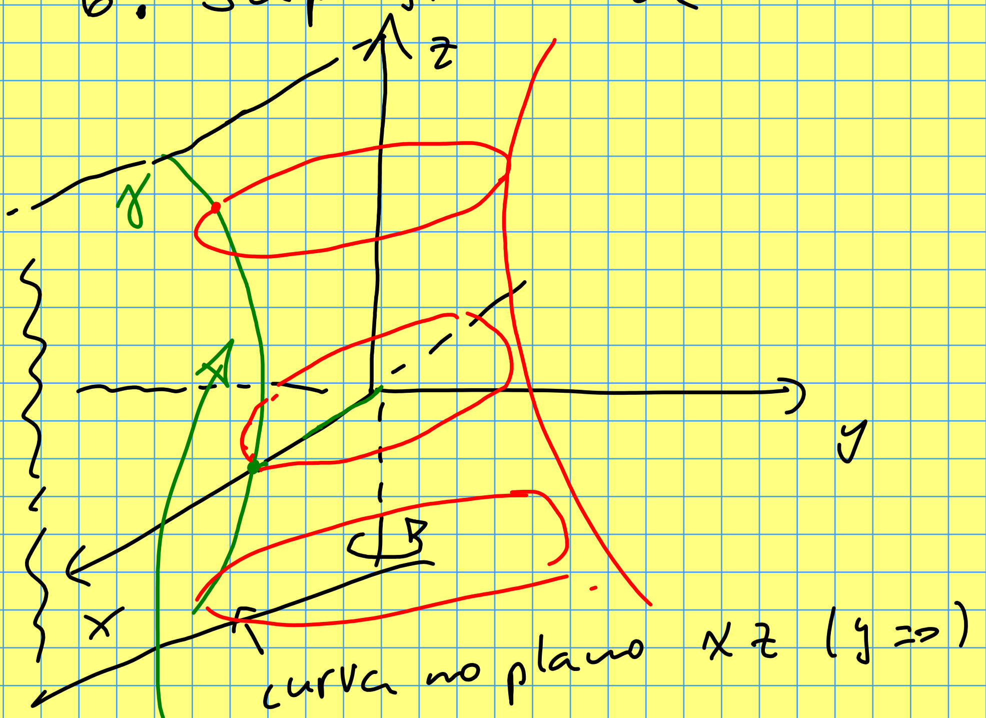
$u, v \in \mathbb{R}$

5. Gráficos $z = f(x,y)$

$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

$$\sigma : \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases} \qquad (u,v) \in \Omega$$

6. Superfícies de revolução



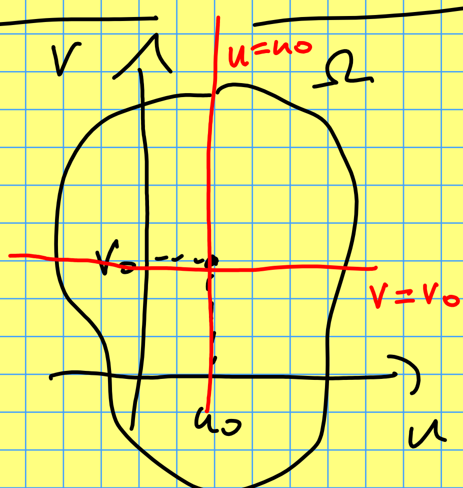
$\gamma(u) = (f(u), 0, g(u)) \qquad u \in I$

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix} = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{pmatrix}$$

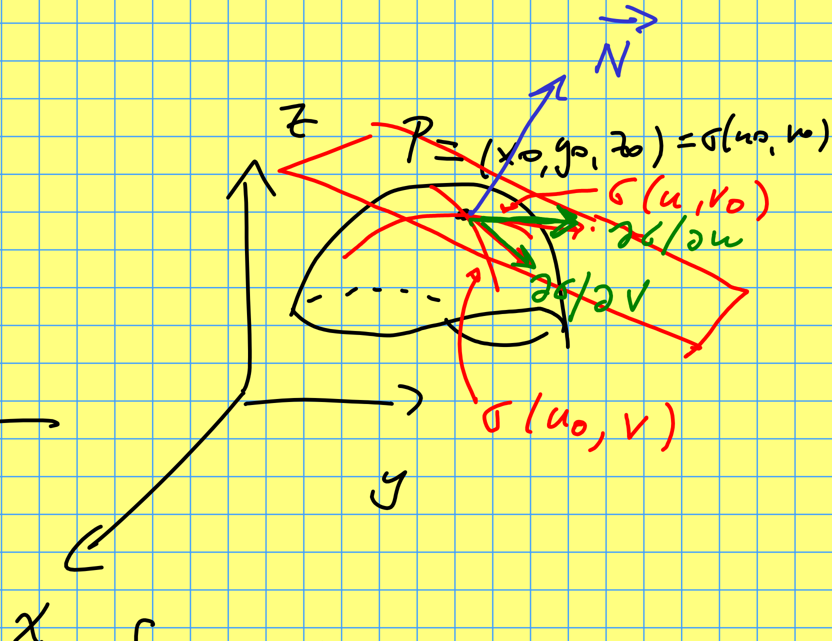
$$\sigma : \begin{cases} x = f(u)\cos v \\ y = f(u)\sin v \\ z = g(u) \end{cases} \qquad \begin{array}{l} u \in I \\ v \in [0, 2\pi] \end{array}$$

— n —

PLANO TANGENTE



$\xrightarrow{\sigma}$



$$\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad (u,v) \in \Omega$$

Eq. vetorial do plano tangente à superfície no ponto P:

$$\boxed{(x,y,z) = (x_0, y_0, z_0) + \lambda \frac{\partial\sigma}{\partial u}(u_0, v_0) + \mu \frac{\partial\sigma}{\partial v}(u_0, v_0)}$$

$\lambda, \mu \in \mathbb{R}$

$\frac{\partial \sigma}{\partial u}(u_0, v_0)$, $\frac{\partial \sigma}{\partial v}(u_0, v_0)$: vetores diretores do plano tangente

$$\vec{N} = \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \quad \text{(produto vetorial)}$$

é um vetor normal à superfície

[Obs $\vec{N} \neq \vec{0} \iff \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial v}$ são L.I.

Neste caso, $\vec{n} = \frac{\vec{N}}{\|\vec{N}\|}$ é um vetor normal unitário.]

Se $\sigma : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in \Omega$

então $\frac{\partial \sigma}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$ e

$$\frac{\partial \sigma}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

$$\vec{N} = \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)$$

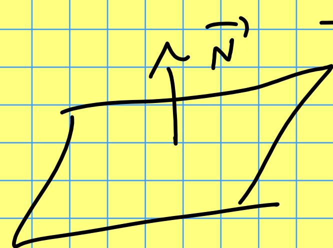
$$= \left(\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}, \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}, \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right)$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)$$

Cálculo de $\vec{N}$ nos exemplos:

1. $\frac{\partial \sigma}{\partial \lambda} = (a_1, b_1, c_1) = \vec{u}$ $\qquad \vec{N} = \vec{u} \times \vec{v}$ constante

$\frac{\partial \sigma}{\partial \mu} = (a_2, b_2, c_2) = \vec{v}$



2.

$$\sigma: \begin{cases} x = R \cos\theta \sin\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\varphi \end{cases}$$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin\theta \sin\varphi & R\cos\theta \sin\varphi & 0 \\ R\cos\theta \cos\varphi & R\sin\theta \cos\varphi & -R\sin\varphi \end{vmatrix}$$

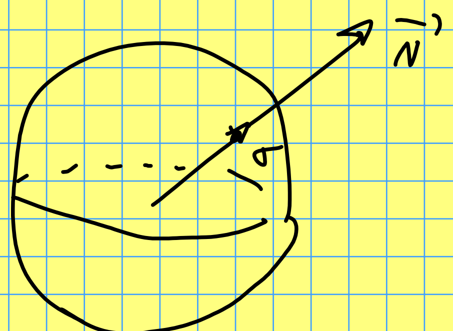
$$= R^2 \sin\varphi \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix}$$

$$= R^2 \sin\varphi \left(-\cos\theta \sin\varphi,\ -\sin\theta\sin\varphi,\ -\cos\varphi\right)$$

$$= -R\sin\varphi \underbrace{\left(R\cos\theta\sin\varphi,\ R\sin\theta\sin\varphi,\ R\cos\varphi\right)}_{\sigma(\theta,\varphi)}$$

$$\vec{N}(\theta,\varphi) = -R\sin\varphi \cdot \sigma(\theta,\varphi)$$

"A normal é um múltiplo do vetor posição"



$$\|\vec{N}\| = |R\sin\varphi|\ \|\sigma\| = R\sin\varphi \cdot R = R^2\sin\varphi$$

$$0 \le \varphi \le \pi$$

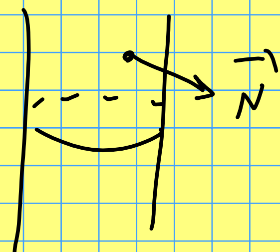
3.

$$\sigma: \begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = h \end{cases}$$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ - & \partial\sigma/\partial\theta & - \\ - & \partial\sigma/\partial h & - \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (R\cos\theta,\ R\sin\theta,\ 0)$$

$\vec{N}$ é um vetor horizontal



5. Gráfico $z = f(x,y)$ $(x,y) \in \Omega$

$$\sigma : \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases} \quad (u,v) \in \Omega$$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial f/\partial u \\ 0 & 1 & \partial f/\partial v \end{vmatrix} = \left(-\frac{\partial f}{\partial u},\ -\frac{\partial f}{\partial v},\ 1\right)$$

$$\|\vec{N}\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}$$

— n —

LISTA 7

11. $\displaystyle\int_{(-2,-1)}^{(1,5)} 2y\,dx + 2x\,dy$

<u>Resolução</u>: Cálculo da função potencial

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = 2y \implies \displaystyle\int \frac{\partial \varphi}{\partial x}\,dx = \int 2y\,dx \\ \dfrac{\partial \varphi}{\partial y} = 2x \end{cases}$$

$$\varphi(x,y) = 2xy + h(y)$$

$$2x = \frac{\partial \varphi}{\partial y} = 2x + h'(y)$$

$$0 = h'(y) \Rightarrow h(y) = \mathbb{C}$$

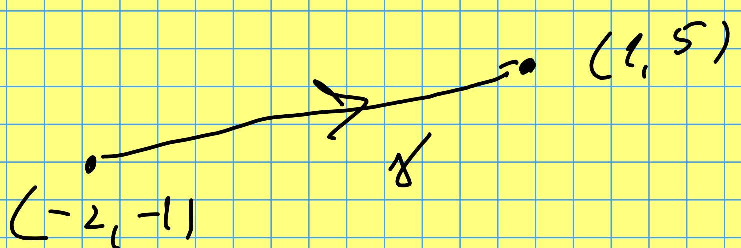
$$\therefore \varphi(x,y) = 2xy + \mathbb{C}$$

é uma função potencial para $\vec{F} = 2y\vec{i} + 2x\vec{j}$.

De fato: $\nabla\varphi = \left(\dfrac{\partial \varphi}{\partial x}, \dfrac{\partial \varphi}{\partial y}\right) = (2y, 2x) = \vec{F}(x,y)$

$\vec{F}$ é um campo gradiente $\therefore$ $\vec{F}$ é conservativo

$\therefore \displaystyle\int_{(-2,-1)}^{(1,5)} \vec{F}\,d\vec{r}$ independe do caminho.



$$\gamma(t) = (-2,-1) + t\big((1,5) - (-2,-1)\big)$$

$$= (-2,-1) + t(3,6)$$

$$\gamma: \begin{cases} x = -2 + 3t \\ y = -1 + 6t \end{cases} \quad 0 \leq t \leq 1$$

$$\int_{(-2,-1)}^{(1,5)} 2y\,dx + 2x\,dy = \int_{\gamma} 2y\,dx + 2x\,dy$$

$$= \int_0^1 2(-1+6t)\,3 + 2(-2+3t)\,6\;dt$$

$$= \int_0^1 -6 + 36t - 24 + 36t\;dt$$

$$= \int_0^1 -30 + 72t\;dt = 6\int_0^1 -5 + 12t\;dt$$

$$= 6 \quad -5t + 6t^2 \Big|_0^1 = 6$$

Alternativamente:

$$\int_{(-2,-1)}^{(1,5)} 2y\,dx + 2x\,dy = \varphi(1,5) - \varphi(-2,-1)$$

$$= 2.1.5 - 2(-2)(-1)$$

$\varphi(x,y) = 2xy$

$$= 10 - 4$$

$$= 6 \;//$$