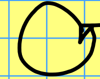
# O TEOREMA DE GREEN (CONT.)

04/11/21

**Exemplo 1.** Calcular $I = \oint_C \underbrace{(3x-y)}_{P}\,dx + \underbrace{(x+5y)}_{=Q}\,dy$, onde

$C: x=\cos t,\ y=\sin t,\ 0\le t\le 2\pi.$

orientado no sentido anti-horário!



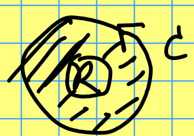
**Resolução.**

T. Green: $$I = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$R$: interior de $C$

$= x^2+y^2\le 1$

$$= \iint_R 1-(-1)\,dx\,dy$$



$$= 2\iint_R dx\,dy = 2\,\text{Área}(R) = 2\pi$$

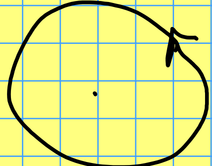
**Exemplo 2.** Calcular

$$I = \oint_C \underbrace{(2y+\sqrt{1+x^5})}_{=P}\,dx + \underbrace{(5x-e^{y^2})}_{=Q}\,dy$$

ao longo do círculo $x^2+y^2=4$ orientado no sentido anti-horário.

**Resolução.**

$$C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0\le t\le 2\pi$$



**T. Green**

$$I = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R 5-2 \; dx\,dy = 3 \text{ Área}(R) = 3 \cdot \pi \cdot 2^2 = 12\pi.$$

**Exemplo.** Se $C$ é uma curva fechada simples e delimita uma região $R$ do plano, então:

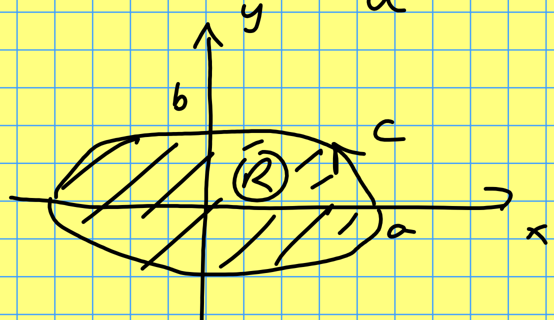
$$\text{Área}(R) = \iint_R dA = \iint_R \underbrace{1}_{= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dA \overset{\text{T.Green}}{=} \oint_C P\,dx + Q\,dy$$

$$= \oint_C x\,dy \qquad Q = x, \; P = 0$$

$$= -\oint_C y\,dx \qquad Q = 0, \; P = -y$$

$$= \frac{1}{2}\oint_C x\,dy - y\,dx \qquad Q = \frac{x}{2}, \; P = -\frac{y}{2}$$

**Exemplo.** Calcular a área da região $R$ delimitada pela elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a, b > 0)$.




$$\text{Área}(R) = \oint_C x\,dy$$

$$= \int_0^{2\pi} \overbrace{a\cos t}^{=x} \overbrace{(b\cos t)\,dt}^{=dy}$$

$$C: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$= ab\int_0^{2\pi} \underbrace{\cos^2 t}_{=\frac{1+\cos 2t}{2}}\, dt$$

$$= ab\left[\frac{t}{2} + \cancel{\frac{\sin 2t}{4}}\right]_0^{2\pi}$$

$$= \frac{ab}{2}\, 2\pi = \pi ab$$

—//—

**Def** $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ é <u>conservativo</u> se vale qualquer uma das condições equivalentes:

- $\vec{F} = \nabla\varphi$, $\varphi: \Omega \to \mathbb{R}$ ($\varphi$: função potencial do $\vec{F}$)
- $\int_C \vec{F}\cdot d\vec{r}$ é independente do caminho usado para unir dois pontos em $\Omega$
- $\oint_C \vec{F}\, d\vec{r} = 0$ para todo caminho fechado em $\Omega$

<u>Condição necessária</u> para $\vec{F}$ ser conservativo

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

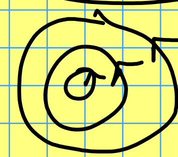
Se $\vec{F}$ é cons., $\vec{F} = \nabla\varphi$, então $P = \frac{\partial\varphi}{\partial x}$, $Q = \frac{\partial\varphi}{\partial y}$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial P}{\partial y}$$

(sempre supondo que nossas funções são suficientem/e deriváveis).

$$\boxed{\vec F = P\vec i + Q\vec j \ \text{cons.} \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$$

<u>Exemplo</u> $\vec F(x,y) = (\underbrace{-y}_{=P}, \underbrace{x}_{=Q})$



$$\left.\begin{array}{l}\dfrac{\partial Q}{\partial x}=1\\[2ex] \dfrac{\partial P}{\partial y}=-1\end{array}\right] \neq \quad \therefore \text{ Não é cons.}$$

—"—

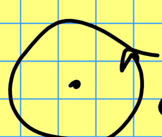
Vale a recíproca?

Em geral, não: $\vec F(x,y) = \left(\overbrace{\dfrac{-y}{x^2+y^2}}^{P}, \overbrace{\dfrac{x}{x^2+y^2}}^{Q}\right)$

Aqui vale

$\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, mas $\vec F$ não é cons., $(x,y)\neq(0,0)$

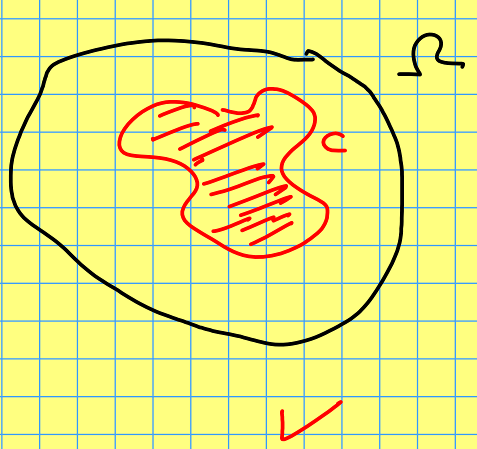
pois $\displaystyle\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi \neq 0.$

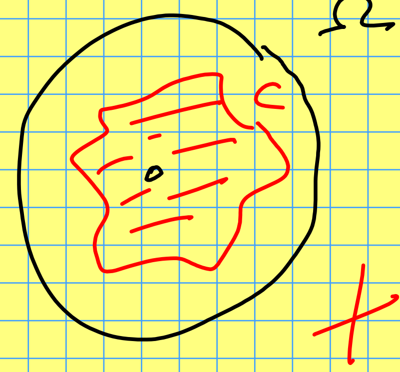

$C: x^2+y^2=1$

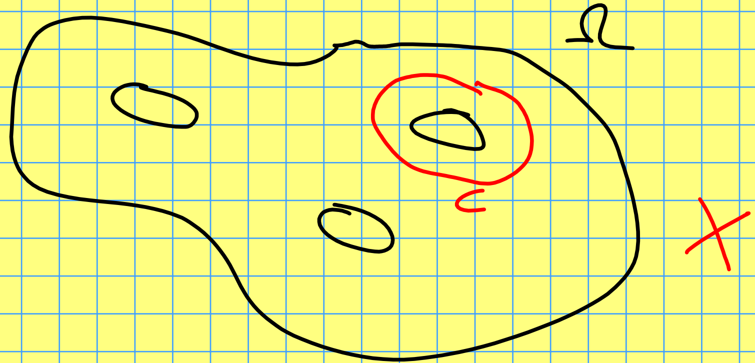
—"—

Mas a recíproca vale se $\Omega = dom(\vec F)$ for <u>simplesmente conexo</u>; isso quer dizer que para toda curva fechada simples $C$ em $\Omega$,

o interior do $C$ também está contido em $\Omega$
($\Omega$ não tem "buracos")

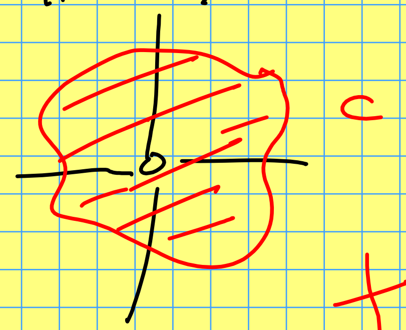



Plano perfurado

$\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$

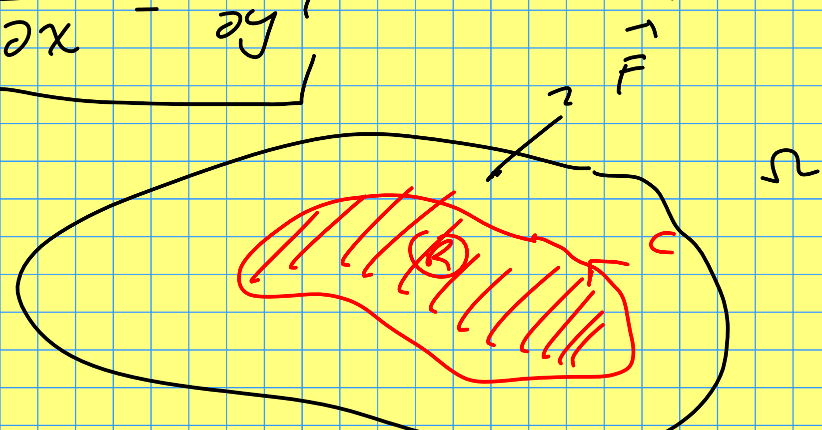



Se $\Omega$ é <u>simplesmente</u> <u>conexa</u>

e $\vec{F} = P\vec{i} + Q\vec{j}$ é um campo em $\Omega$

com $\underbrace{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$, então $\vec{F}$ é conservativo.

<u>Dem.</u>




$R = \text{int}(C) \subset \Omega$

$\Omega$ é simpl. conexa.

Para todo caminho fechado simples $C$ em $\Omega$:

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

$$= \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dA$$

(pelo T. Green, pois $R \subset \Omega$)

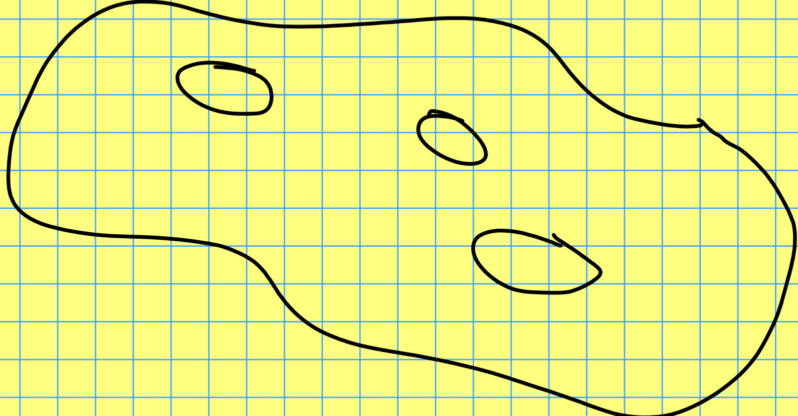
$$= 0$$

Segue que $\vec{F}$ é conservativo //

$$\operatorname{rot} \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\boxed{\vec{F} \text{ é cons} \iff \operatorname{rot}\vec{F} = \vec{0} \text{ no caso de } \Omega \text{ simplesmente conexo}}$$

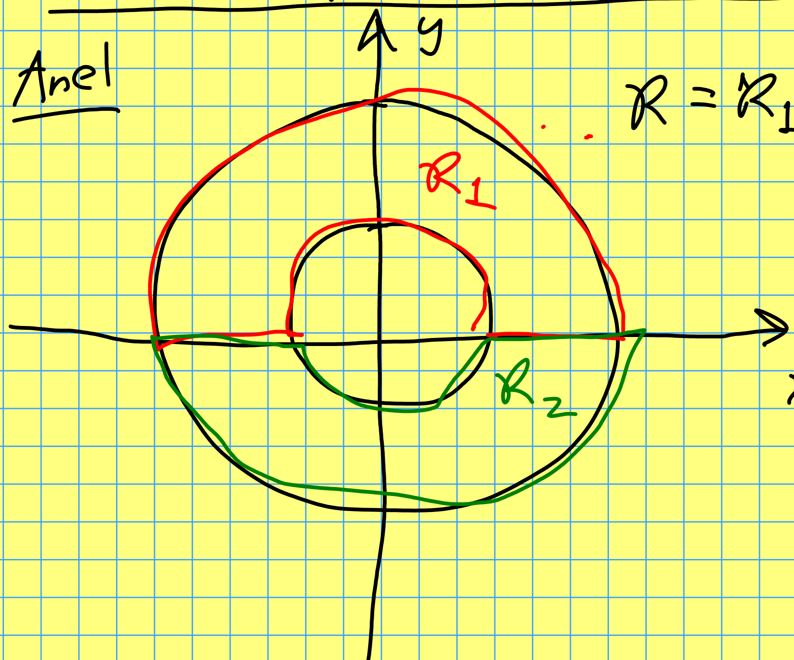
—//—



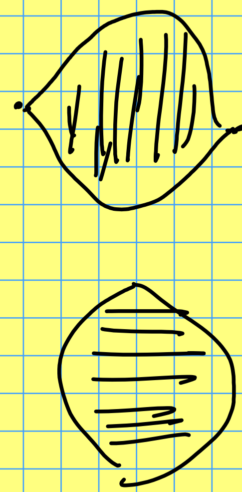
Região multiplamente conexa

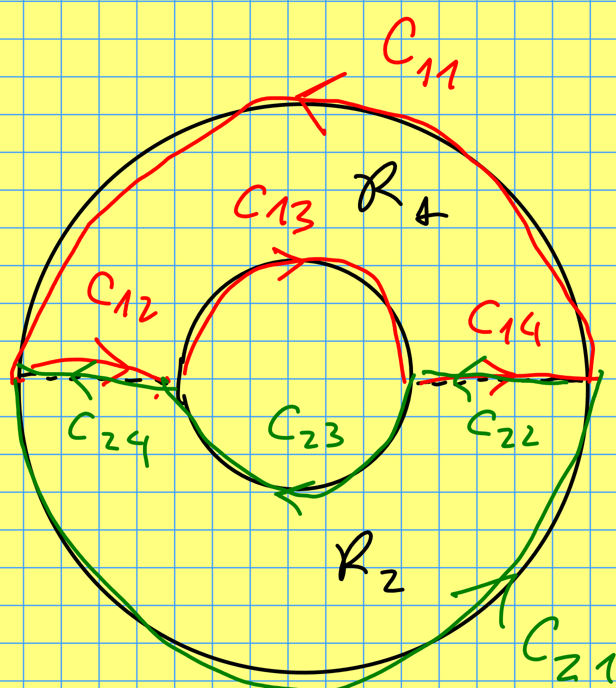
# T. Green para regiões multiplamente conexas

Anel




$R = R_1 \cup R_2$






$R = R_1 \cup R_2$

$\partial R = C$

$C_1 = \partial R_1 = C_{11} \cup C_{12} \cup C_{13} \cup C_{14}$

$C_2 = \partial R_2 = C_{21} \cup C_{22} \cup C_{23} \cup C_{24}$

$R_1$ e $R_2$ são duas regiões do tipo que nós sabemos vale o T.Green:

$$\oint_{C_1} \vec{F}\, d\vec{r} = \iint_{R_1} \operatorname{rot}\vec{F} \cdot \vec{k}\, dA$$

$+$

$$\oint_{C_2} \vec{F}\, d\vec{r} = \iint_{R_2} \text{rot}\,\vec{F}\cdot \vec{k}\, dA$$

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$$\iint_R \text{rot}\,\vec{F}\, dA = \oint_{C_1} \vec{F}\, d\vec{r} + \oint_{C_2} \vec{F}\, d\vec{r}$$

$$= \oint_{C_{11}} + \cancel{\oint_{C_{12}}} + \oint_{C_{13}} + \cancel{\oint_{C_{14}}} + \oint_{C_{21}} + \cancel{\oint_{C_{22}}} + \oint_{C_{23}} + \cancel{\oint_{C_{24}}}$$

$C_{14} = -C_{22}$

$C_{12} = -C_{24}$

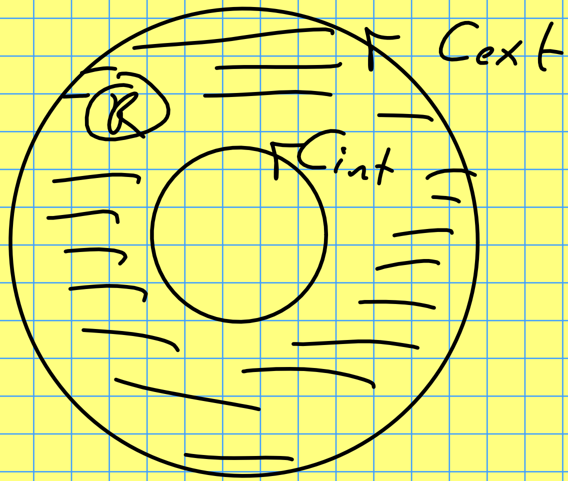
$$= \oint_{C_{11}\cup C_{21}} + \oint_{C_{13}\cup C_{23}}$$

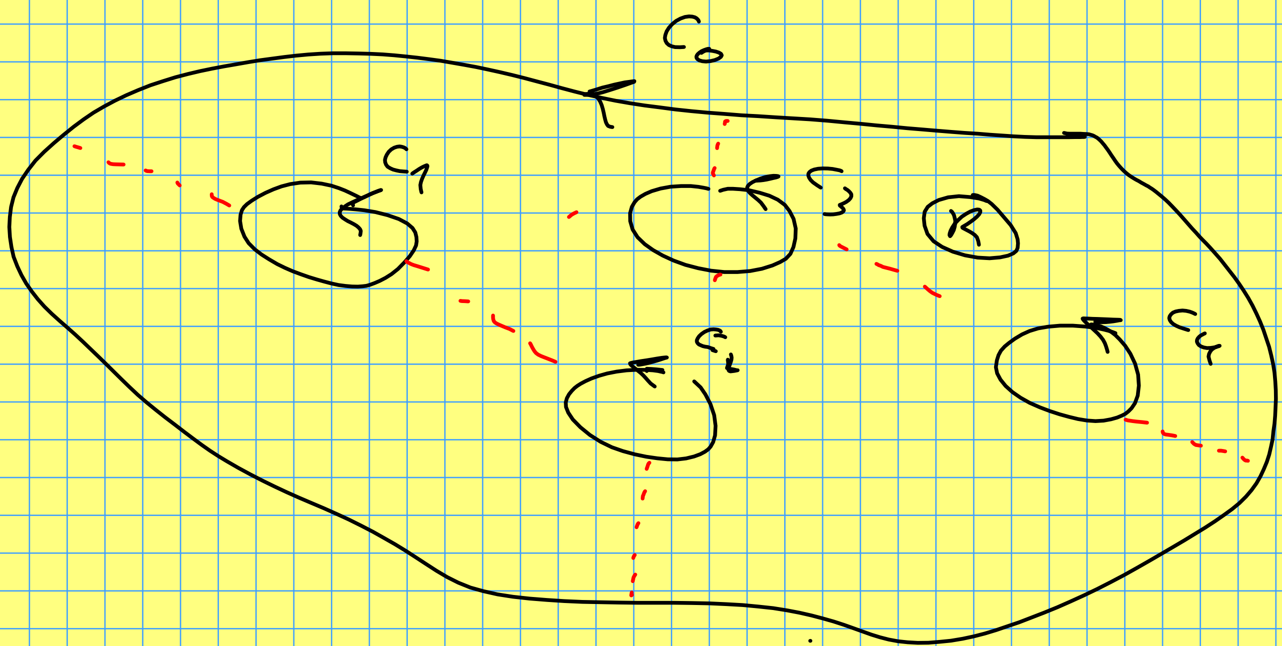
$C_{11}\cup C_{21} = C_{ext}$

$C_{13}\cup C_{23} = -C_{int}$

onde $C_{ext}$ e $C_{int}$ são os círculos exterior e interior que delimitam R, orientados no sentido anti-horário.

$$\therefore \boxed{\iint_R \text{rot}\,\vec{F}\, dA = \oint_{C_{ext}} \vec{F}\, d\vec{r} - \int_{C_{int}} \vec{F}\, d\vec{r}}$$



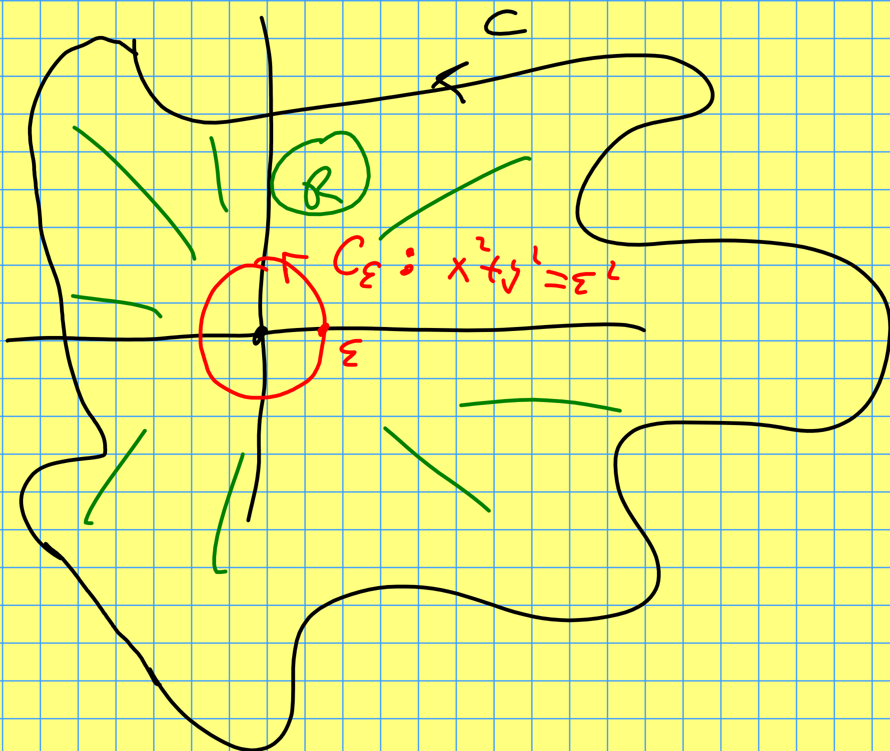




$$\iint_R \operatorname{rot}\vec{F}\,dA = \oint_{C_0} \vec{F}\,d\vec{r} - \oint_{C_1} \vec{F}\,d\vec{r} - \oint_{C_2} \vec{F}\,d\vec{r}$$

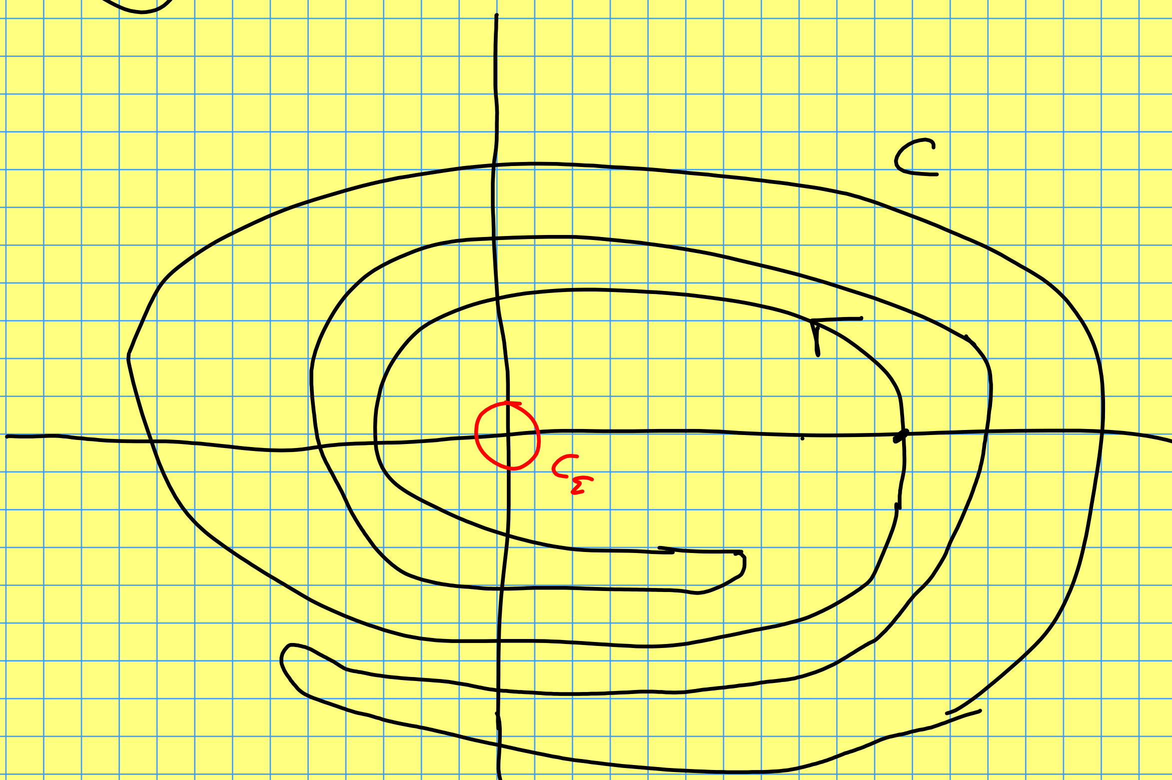
$$- \oint_{C_3} \vec{F}\,d\vec{r} - \oint_{C_4} \vec{F}\,d\vec{r}$$

<u>Exemplo</u>.

Seja C qualquer curva fechada simples que contém a origem em seu interior e dá uma volta em torno dela, no sentido anti-horário, como na figura:




$$\int_{C_\varepsilon} \vec{F}\, d\vec{r} = 2\pi$$




Calcular

$$I = \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}.$$

Resolução.  • $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ tem

$\text{rot}\,\vec{F} = \vec{0}$.  $(x,y) \neq (0,0)$

• $\oint_{C_R} \vec{F}\,d\vec{r} = 2\pi$.

$C_R : x^2+y^2 = R^2$ , orient. sentido anti-horário

T. Green p/ região R (multiplamente conexa):

$$\underbrace{\iint_R \underbrace{\text{rot}\,\vec{F}}_{=0} \cdot \vec{k}\,dA}_{=0} = \oint_C \vec{F}\,d\vec{r} - \underbrace{\oint_{C_\varepsilon} \vec{F}\,d\vec{r}}_{2\pi}$$

$$\therefore \oint_C \vec{F}\,dr = 2\pi \;//$$