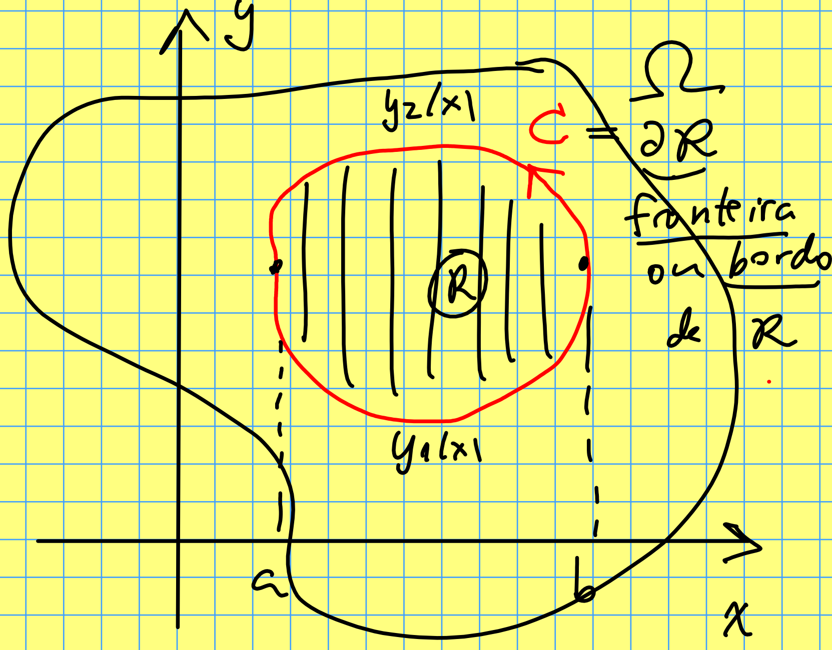
O TEOREMA DE GREEN

28/10/21

(NO PLANO!)



$\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$

campo de vetores

$\vec{F}(x,y) = (P(x,y), Q(x,y))$

P, Q são funções com derivadas parciais contínuas

Seja C um caminho fechado em Ω, orientado no sentido anti-horário. Vamos calcular $\int_C \vec{F} \cdot d\vec{r}$ de uma forma "diferente". Vamos supor que C é um caminho fechado simples*, isto é, que não possui auto-interseções, e que o interior de C está contido em Ω. Vamos supor também que a região interior a C, digamos R, pode ser descrita nas formas:

$$R: \begin{cases} a \leq x \leq b \\ y_1(x) \leq y \leq y_2(x) \end{cases} \quad \text{e} \quad R: \begin{cases} c \leq y \leq d \\ x_1(y) \leq x \leq x_2(y) \end{cases}$$

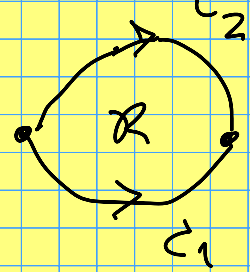
(I)

Calculemos:

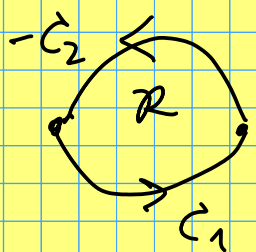
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

$$\int_C P\,dx = \,?$$

$$C = C_1 \cup (-C_2)$$



$$C_1 : \begin{cases} x = t \\ y = y_1(t) \end{cases} \quad a \leq t \leq b$$



$$C_2 : \begin{cases} x = t \\ y = y_2(t) \end{cases} \quad a \leq t \leq b$$

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{-C_2} P\,dx$$

$$= \int_{C_1} P\,dx - \int_{C_2} P\,dx$$

$$= \int_a^b P(t, y_1(t))\,dt - \int_a^b P(t, y_2(t))\,dt$$

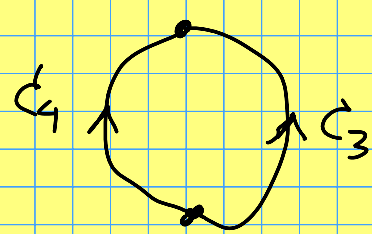
$$= \int_a^b \Big[-P(t,y) \Big]_{y=y_1(t)}^{y=y_2(t)} dt$$

$$= \int_a^b \Big[-P(x,y) \Big]_{y=y_1(x)}^{y=y_2(x)} dx$$

$$= \int_a^b \left(\int_{y_1(x)}^{y_2(x)} -\frac{\partial P}{\partial y}\, dy \right) dx$$

$$= \iint_R -\frac{\partial P}{\partial y}\, dA \qquad \therefore \int_C P\,dx = \iint_R -\frac{\partial P}{\partial x}\, dA \qquad (1)$$

$$\int_C Q\,dy = \,?$$



$$C_3 : \begin{cases} x = x_2(t) \\ y = t \end{cases} \quad c \le t \le d$$

$$C = C_3 \cup (-C_4)$$

$$C_4 : \begin{cases} x = x_1(t) \\ y = t \end{cases} \quad c \le t \le d$$

$$\int_C Q\,dy = \int_{C_3} Q\,dy - \int_{C_4} Q\,dy$$

$$= \int_c^d Q(x_2(t), t)\,dt - \int_c^d Q(x_1(t), t)\,dt$$

$$= \int_c^d \left[Q(x,y) \right]_{x = x_1(t)}^{x = x_2(t)} dt$$

$$= \int_c^d \left(\int_{x_1(t)}^{x_2(t)} \frac{\partial Q}{\partial x}\, dx \right) dy = \iint_R \frac{\partial Q}{\partial x}\, dA$$

$$\therefore \int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA \qquad (2)$$

(1) + (2) :

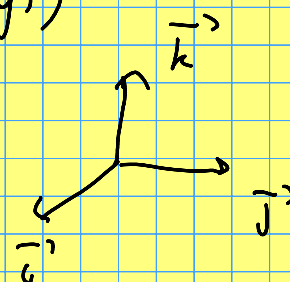
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Teorema de Green

$R = \partial C$

Rotacional de $\vec{F}(x,y) = (P(x,y), Q(x,y))$

$$\vec{F} = P\vec{i} + Q\vec{j}$$



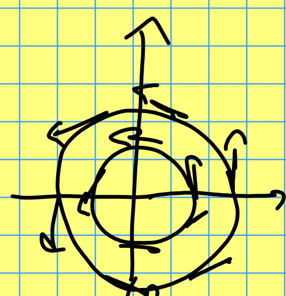
$$\boxed{\operatorname{rot}\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

T. Green

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_R (\operatorname{rot}\vec{F})\cdot\vec{k}\,dA}$$

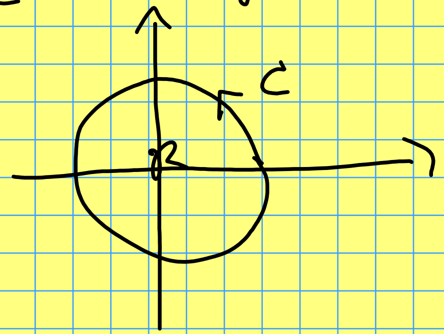
Exemplos

1. $\vec{F}(x,y) = (\overset{P}{-y}, \overset{Q}{x})$



$$\operatorname{rot}\vec{F}(x,y) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \left(1-(-1)\right)\vec{k} = 2\vec{k}$$

$C: x^2+y^2 = R^2 \quad (R>0)$



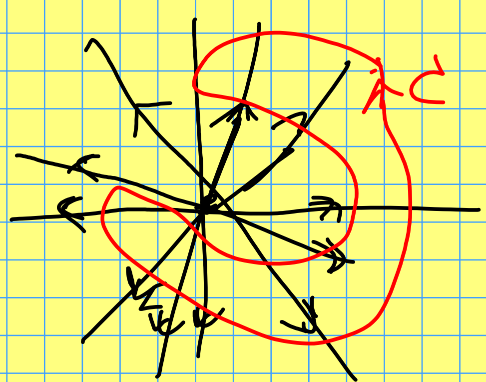
$$\int_C \vec{F}\, d\vec{r} \underset{\text{T. Green}}{=} \iint_R \underbrace{\text{rot}\,\vec{F}}_{=2\vec{k}} \cdot \vec{k}\, dA$$

R: círculo de raio R

$$= 2 \iint_R dA = 2\, \text{Área}(R)$$

$$= 2\pi R^2 \quad //$$

2. $\vec{F}(x,y) = (\overset{P}{x}, \overset{Q}{y})$ campo radial



$$\text{rot}\,\vec{F} = \Big(\underbrace{\frac{\partial Q}{\partial x}}_{=0} - \underbrace{\frac{\partial P}{\partial y}}_{=0}\Big)\vec{k} = \vec{0}$$

$$\oint_C \vec{F}\, d\vec{r} = \iint_R \underbrace{\text{rot}\,\vec{F}\cdot\vec{k}}_{=0}\, dA = 0$$

C qq caminho fechado simples em $\mathbb{R}^2$

Em particular, $\vec{F}$ é conservativo.

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = P = x \\[2mm] \dfrac{\partial \varphi}{\partial y} = Q = y \end{cases}$$

$\varphi(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2} + \text{const}$

é um potencial para $\vec{F}$

$\nabla \varphi = \vec{F}$.

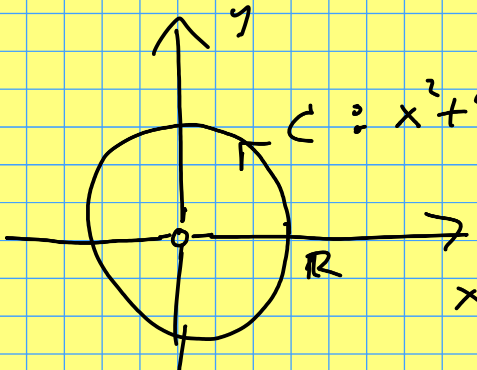
3. $\vec{F}(x,y) = \left(\underbrace{\frac{-y}{x^2+y^2}}_{=P}, \underbrace{\frac{x}{x^2+y^2}}_{=Q} \right)$ $\quad \underbrace{(x,y) \neq (0,0)}_{\Omega}$

$$\frac{\partial P}{\partial y} = \frac{(-1)(x^2+y^2) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1(x^2+y^2) - x\,(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\left.\right] =$$

$$\therefore \ \operatorname{rot} \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} = \vec{0} \ \text{ em } \Omega$$



$C : x^2+y^2 = R^2$

$$C : \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$\int_C \vec{F}\, d\vec{r} = \int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$$= \int_0^{2\pi} \frac{+\cancel{R}\sin t\,(+\cancel{R}\sin t) + \cancel{R}\cos t\ \cancel{R}\cos t}{\cancel{R^2}}\, dt$$

$$= \int_0^{2\pi} dt = \underline{\underline{2\pi}} \neq 0.$$

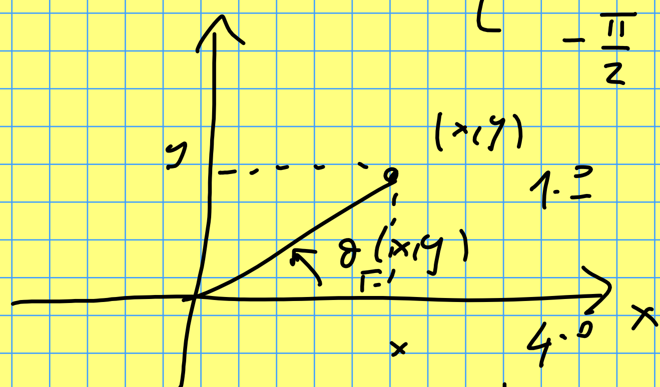
Não vale o T. Green, pois

o interior de C não está inteiramente no domínio de $\vec{F}$ (pois $(0,0) \notin \mathrm{dom}\,\vec{F}$)

$\boxed{\vec{F} \text{ não é conservativo em } \Omega}$

4. Vamos definir a função "ângulo":

$$\theta(x,y) = \begin{cases} \arctan \dfrac{y}{x} & \text{se } x > 0 \\ \dfrac{\pi}{2} - \arctan \dfrac{x}{y} & \text{se } x \leq 0 \text{ e } y > 0 \\ -\dfrac{\pi}{2} - \arctan \dfrac{x}{y} & \text{se } x \leq 0 \text{ e } y < 0 \end{cases}$$

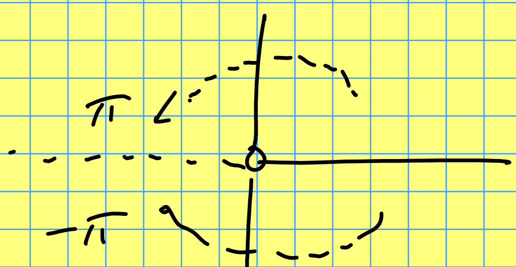


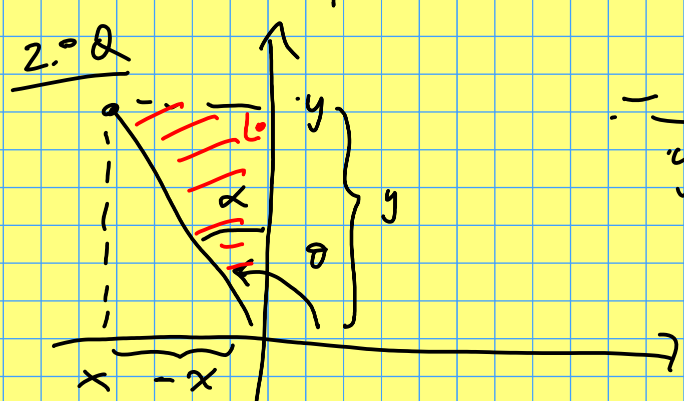
$$\tan \theta(x,y) = \frac{y}{x}$$

$x=0$ e $y>0$: $\theta(x,y) = 0$ $\qquad$ $x=0,\ y<0$: $\theta(x,y) = \pi$

$x = 0 = y$: $\theta(0,0)$ não está definido

$-\pi < \theta(x,y) < \pi$ ← Vamos tomar o ângulo neste intervalo.





$$-\frac{x}{y} = \tan \alpha$$

$$\alpha = \theta - \frac{\pi}{2}$$

$$\theta = \alpha + \frac{\pi}{2}$$

$$\alpha = \arctan -\frac{x}{y} = -\arctan \frac{x}{y}$$

$$\theta = \frac{\pi}{2} - \arctan \frac{x}{y} \qquad \begin{matrix} y > 0 \\ x \leq 0 \end{matrix} \qquad \frac{x}{y} \leq 0$$

$$\arctan \frac{x}{y} \in \left(-\frac{\pi}{2}, 0\right]$$

$$-\arctan \frac{x}{y} \in \left[0, \frac{\pi}{2}\right)$$

Calculemos o gradiente de θ:

$$\nabla \theta = \left(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}\right)$$

$x > 0$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\arctan \frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\arctan \frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \underbrace{\frac{\partial}{\partial y}\left(\frac{y}{x}\right)}_{= 1/x}$$

$$= \frac{x}{x^2+y^2}$$

$$\therefore \nabla \theta = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \vec{F}$$ (campo magnético do ex.3)

$\boxed{x \leq 0 \text{ e } y \neq 0}$

$$\frac{\partial}{\partial x}\left(\pm\frac{\pi}{2} - \arctan\frac{x}{y}\right) = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{1}{y} = \frac{-y}{x^2+y^2}$$

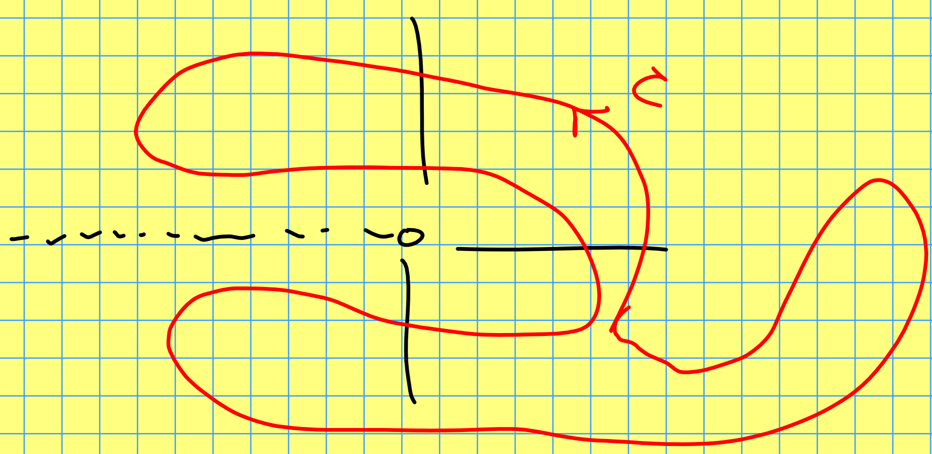
$$\frac{\partial}{\partial y}\left(\pm\frac{\pi}{2} - \arctan\frac{x}{y}\right) = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{-x}{y^2} = \frac{x}{x^2+y^2}$$

<u>Conclusão</u> $\nabla\theta = \vec{F}$ em todo o domínio de θ

$$\operatorname{dom}\theta = \mathbb{R}^2 \setminus \{(x,y) \mid y=0 \text{ e } x<0\}$$
$$\neq$$
$$\operatorname{dom}\vec{F} = \mathbb{R}^2 \setminus \{(0,0)\}$$

$\vec{F}$ é conservativo em $\mathbb{R}^2 \setminus \left\{(x,y) \,\middle|\, \begin{array}{l} y=0 \\ x<0 \end{array}\right\}$

$\vec{F}$ não é conservativo em $\mathbb{R}^2 \setminus \{(0,0)\}$!!



$$\int_C \vec{F}\,d\vec{r} = 0$$

se C não encontra o semi-eixo x negativo.