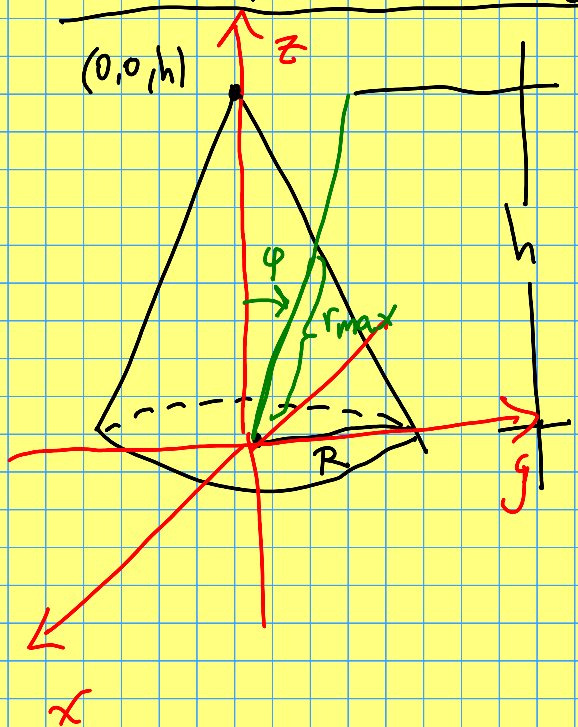
EXERCÍCIOS

30/09/21

Ex.9, p.511 [Português]



Calcular o volume do cone indicado.

1.º método Usando coords esféricas:

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

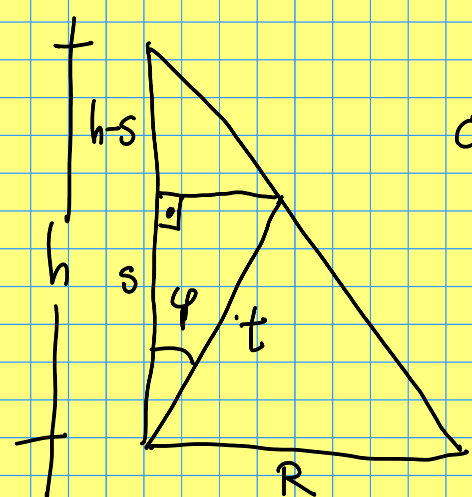
$0 \leq \theta \leq 2\pi$

↑

sólido de rotação

$0 \leq \varphi \leq \dfrac{\pi}{2}$

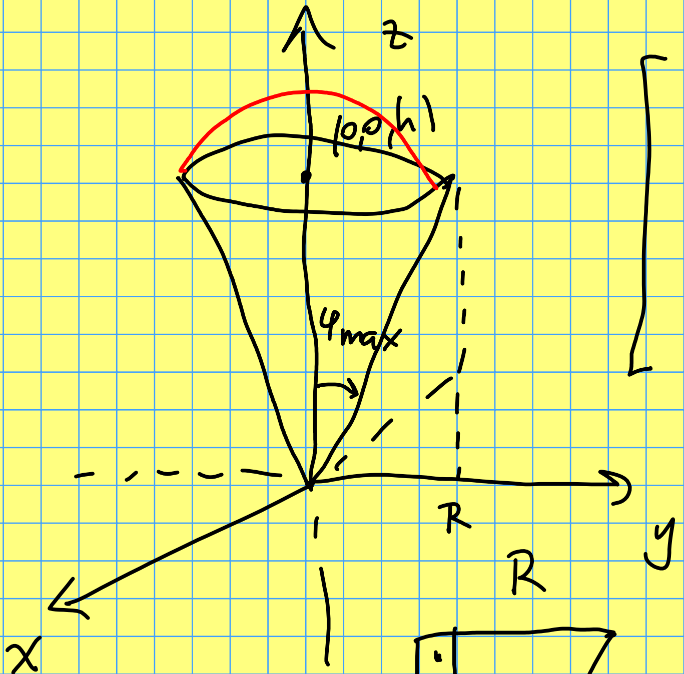
$0 \leq r \leq$



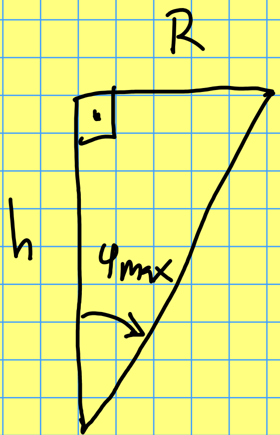
$\cos\varphi = \dfrac{s}{t}$

⋮

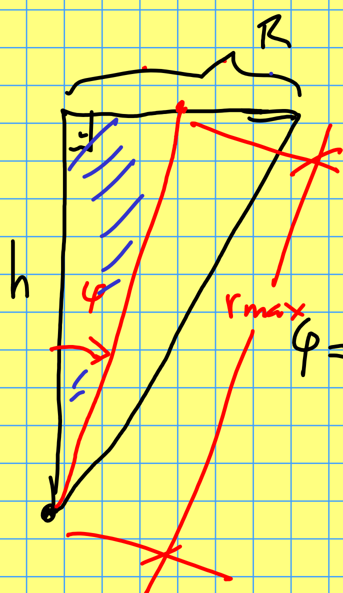
2.º método: coords esféricas, cone invertido



$$\left[\begin{array}{l} \left.\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \arctan \frac{R}{h} \end{array}\right\} \text{extremos constantes} \\ 0 \le r \le h \sec\varphi \end{array}\right.$$



$$\tan \varphi_{max} = \frac{R}{h}$$



$$\cos\varphi = \frac{h}{r_{max}}$$

$$\varphi = \varphi_{max} \Rightarrow r = h \sec \varphi_{max}$$

$$= h \sec\left(\arctan \frac{R}{h}\right) = h \sqrt{1 + \frac{R^2}{h^2}} = \sqrt{h^2 + R^2}$$

$$\sec \underbrace{\arctan x}_{=y} = \sqrt{1 + x^2}$$

$$\tan y = x \qquad \sec^2 y = 1 + \tan^2 y$$

$$\mathrm{Vol} = \iiint_R d\bar{V} = \int_0^{\varphi_{max}} \int_0^{2\pi} \int_0^{r_{max}(\varphi)} \underline{r^2 \sin\varphi}\, dr\, d\theta\, d\varphi$$

$$= 2\pi \int_0^{\varphi_{max}} \frac{r_{max}(\varphi)^3}{3} \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\arctan\frac{R}{h}} h^3 \sec^3\varphi \, \sin\varphi \, d\varphi$$

$$= \frac{2\pi h^3}{3}\int_0^{\arctan\frac{R}{h}} \frac{\sin\varphi \, d\varphi}{\cos^3\varphi}$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

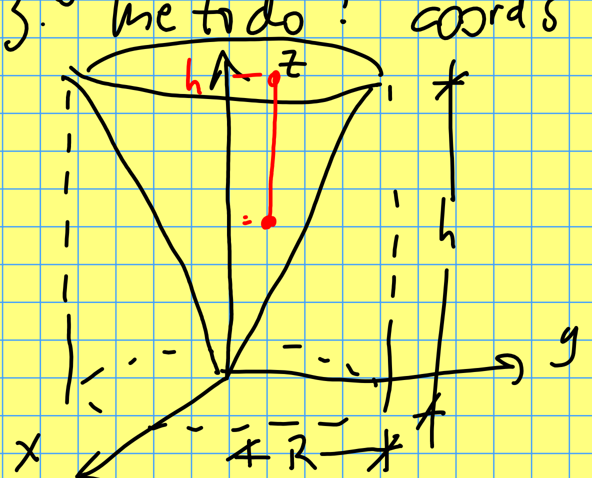
$$= \frac{\cancel{2}\pi h^3}{3} \cdot \frac{1}{\cancel{2}} \left[\sec^2\varphi\right]_0^{\arctan\frac{R}{h}} = \frac{\pi h^3}{3}\left(\cancel{1} + \frac{R^2}{h^2} - \cancel{1}\right)$$

$$= \frac{\pi h R^2}{3} \quad //$$

$$\int \frac{\sin\varphi \, d\varphi}{\cos^3\varphi} = \int \frac{-du}{u^3} = -\int u^{-3}\, du = -\frac{u^{-2}}{-2} + C$$

$$= +\frac{1}{2\cos^2\varphi} + C = \frac{\sec^2\varphi}{2} + C$$

3.º método: coords cilíndricas

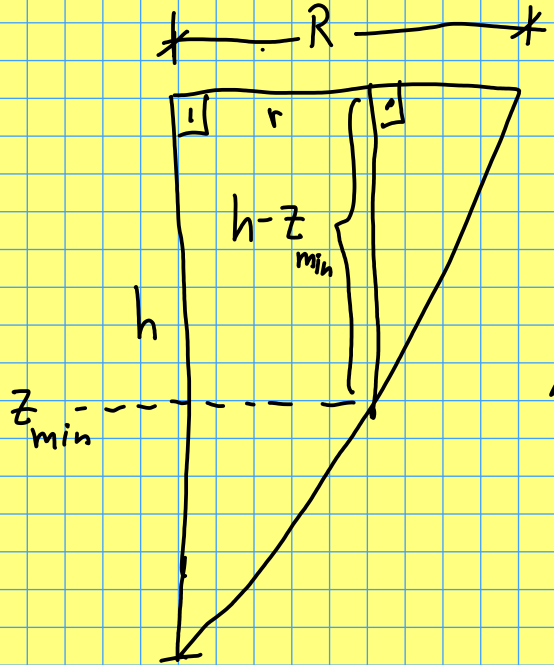


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq R$

$\frac{h}{R} r \leq z \leq h$



$$\frac{R}{h} = \frac{R-r}{h-z_{min}}$$

$$\cancel{Rh} - R z_{min} = \cancel{Rh} - hr$$

$$z_{min} = \frac{hr}{R}$$

$$Vol = \iiint_R dV = \int_0^R \int_{\frac{h}{R}r}^{h} \int_0^{2\pi} r \, d\theta \, dz \, dr$$

$$= 2\pi \int_0^R r \, [z]_{z=\frac{h}{R}r}^{z=h} \, dr$$

$$= 2\pi \int_0^R r\left(h - \frac{h}{R} r\right) dr = 2\pi h \int_0^R r - \frac{r^2}{R} \, dr$$

$$= 2\pi h \left[\frac{r^2}{2} - \frac{r^3}{3R}\right]_0^R = 2\pi h \left(\frac{R^2}{2} - \frac{R^2}{3}\right) = 2\pi h \frac{R^2}{6}$$

$$= \pi h R^2/3 \quad //$$

Ex. 11, p. 518

Calcular a área da superfície de sela $az = x^2 - y^2$ cortada pelo cilindro $x^2 + y^2 = a^2$. $(a>0)$.

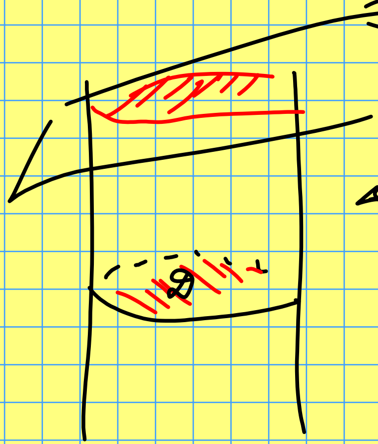
Resolução $az = x^2 - y^2$ ← gráfico de f

$f(x,y) = \frac{1}{a}(x^2 - y^2)$ ← gráfico de $f(x,y)$

$f_x = \frac{1}{a} 2x$

$f_y = \frac{-2y}{a}$

$f_y^2 = \frac{4y^2}{a^2}$



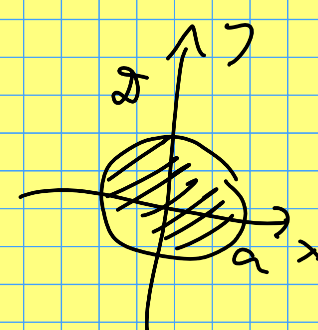
← cilindro é vertical

$x^2 + y^2 = a^2$

$\mathcal{D} : x^2 + y^2 \leq a^2$

$$\text{Área} = \iint_{\mathcal{D}} \sqrt{1 + f_x^2 + f_y^2} \, dxdy$$

$$= \iint_{\mathcal{D}} \sqrt{1 + \frac{1}{a^2} 4x^2 + \frac{1}{a^2} 4y^2} \, dxdy$$



$$= \frac{1}{a} \iint_{\mathcal{D}} \sqrt{a^2 + 4(x^2+y^2)} \, dxdy$$

Coords polares : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$dxdy = r\,dr\,d\theta$

$0 \leq r \leq a$

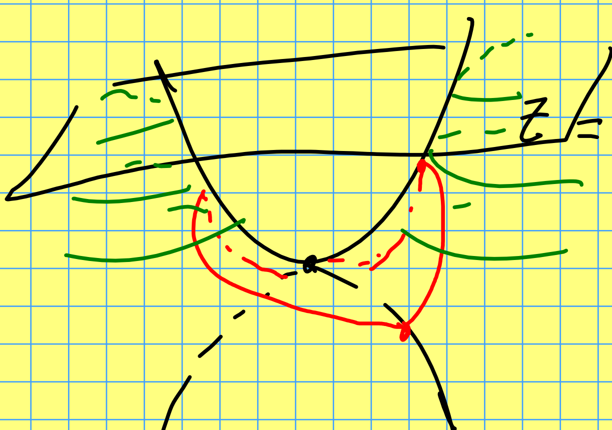
$0 \leq \theta \leq 2\pi$

$$= \frac{1}{a} \int_0^{2\pi} \int_0^a \sqrt{a^2 + 4r^2}\, r \, dr \, d\theta$$

$$= \frac{2\pi}{a} \int_0^a r \left(a^2 + 4r^2\right)^{1/2} dr$$

$$= \frac{2\pi}{a} \, \frac{2}{3} \left(a^2 + 4r^2\right)^{3/2} \frac{1}{8} \Bigg|_0^a$$

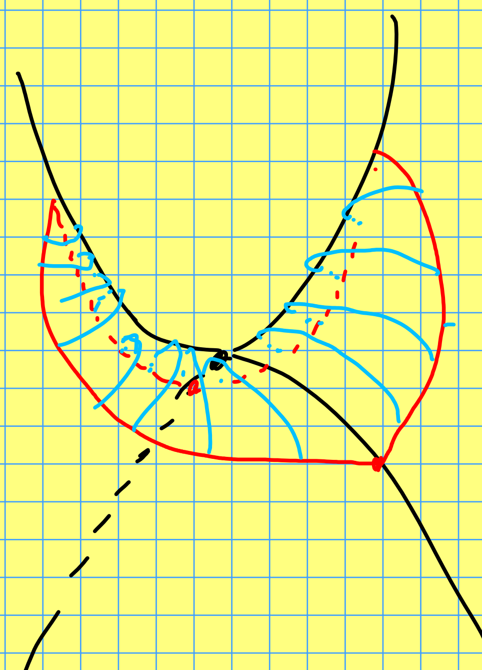
$$= \frac{\pi}{6a} \left[\left(5a^2\right)^{3/2} - \left(a^2\right)^{3/2} \right] = \frac{\pi a^2}{6} \left(5\sqrt{5} - 1\right) //$$



$$z = \frac{1}{a}\left(x^2 - y^2\right)$$

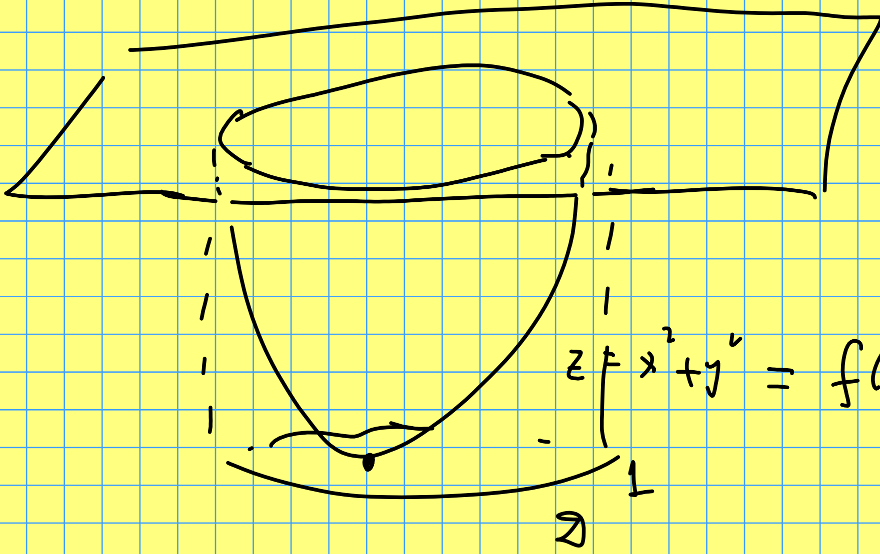
$$\frac{5\sqrt{5} - 1}{6} \approx 1.696$$

parabolóide hiperbólico



Ex. 9

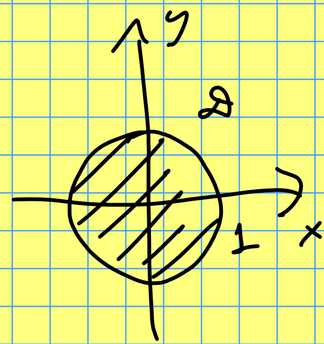
Área do parabolóide $z = x^2 + y^2$ cortada pelo plano $z = 1$.



$z = 1$ parabolóide de revolução

$z = x^2 + y^2 = f(x,y)$

$$A = \iint_{D} \sqrt{1 + f_x^2 + f_y^2}\; dx\,dy = \iint_{D} \sqrt{1 + 4x^2 + 4y^2}\; dx\,dy$$



$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right) //$$