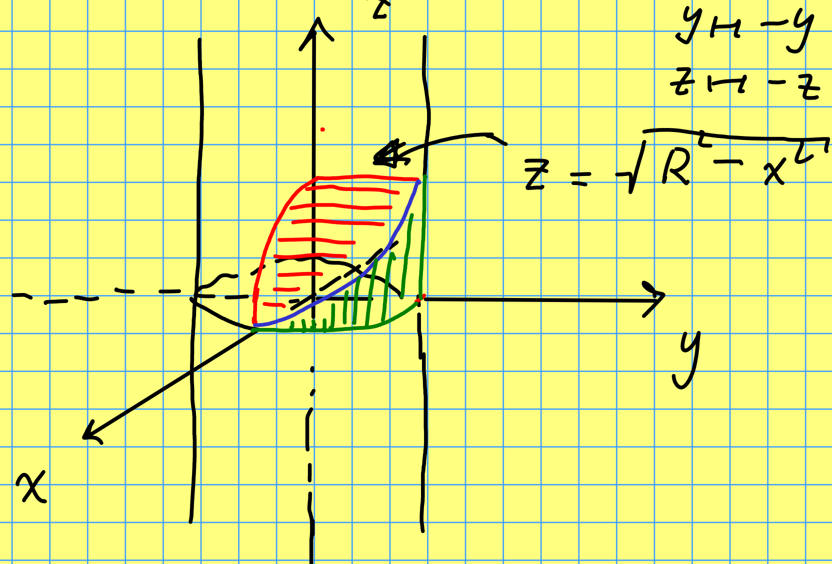
EXERCÍCIOS

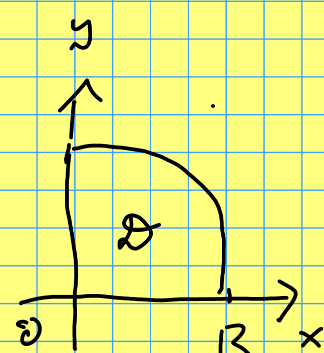
23/09/21

1. Calcular o volume da intersecção de dois cilindros (circulares retos), de mesmo da raio R da base, que se encontram, segundo seus eixos, em um ângulo reto.

Resolução. Sejam $x^2+y^2=R^2$ e $x^2+z^2=R^2$ os cilindros.

$x \mapsto -x$
$y \mapsto -y$
$z \mapsto -z$





Considerando a porção vermelha como gráfico de uma função de x, y, o volume desejado é 8x o volume da região debaixo desse gráfico, no 1.º octante

$$V = 8 \iint_{\mathcal{D}} \sqrt{R^2-x^2}\, dA$$

$\mathcal{D}$ favorece o uso de coords polares:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad dxdy = r\,dr\,d\theta \qquad \begin{array}{l} 0 \le r \le R \\ 0 \le \theta \le \pi/4 \end{array}$$

$$V = 8 \int_0^{\pi/4} \int_0^R \sqrt{R^2 - r^2\cos^2\theta}\; r \, dr \, d\theta$$

$$= 8 \int_0^{\pi/4} \left[\frac{2}{3} \left(R^2 - r^2\cos^2\theta\right)^{3/2} \frac{1}{-2\cos^2\theta} \right]_{r=0}^{r=R} d\theta$$

$$= \frac{8}{3} \int_0^{\pi/4} \frac{1}{-\cos^2\theta} \left(\left(R^2\sin^2\theta\right)^{3/2} - \left(R^2\right)^{3/2} \right) d\theta$$

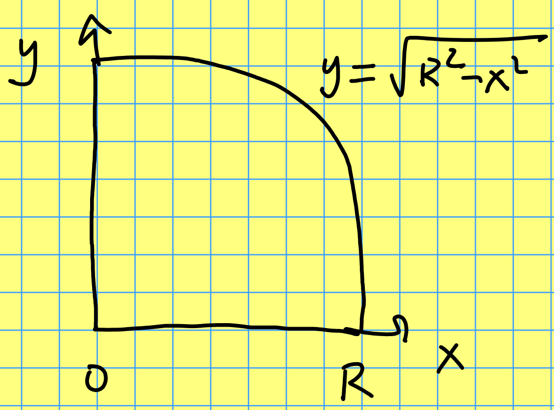
$$= \frac{8R^3}{3} \int_0^{\pi/4} \frac{-\sin^3\theta + 1}{+\cos^2\theta} \, d\theta$$

$$\frac{1 - \sin^3\theta}{\cos^2\theta} = \frac{\cancel{(1-\sin\theta)}\,(1+\sin\theta+\sin^2\theta)}{\cancel{(1-\sin\theta)}\,(1+\sin\theta)}$$

$$= 1 + \frac{\sin^2\theta}{1+\sin\theta}$$

$$\frac{1 - \sin^3\theta}{\cos^2\theta} = \sec^2\theta - \tan^2\theta \, \sin\theta$$

$$\iint_{\mathcal{D}} \sqrt{R^2 - x^2}\, dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2}\, dy\, dx$$

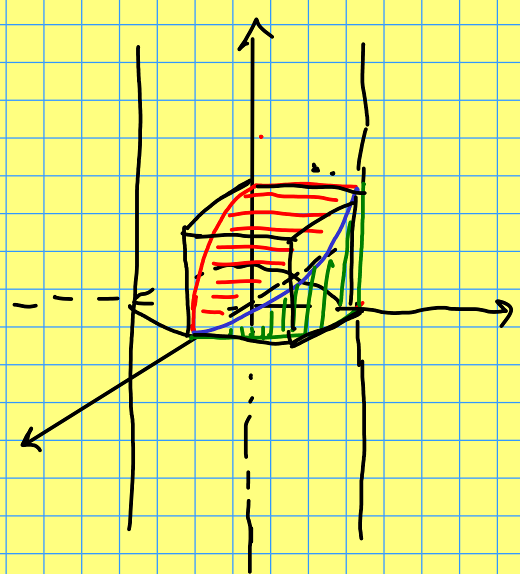


$$= \int_0^R \sqrt{R^2-x^2} \left(\int_0^{\sqrt{R^2-x^2}} dy \right) dx$$

$$= \int_0^R R^2 - x^2\, dx$$

$$= R^2 x - \frac{x^3}{3} \Big|_0^R = R^3 - \frac{R^3}{3} = \frac{2R^3}{3}$$

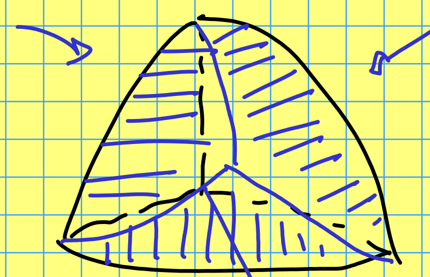
$$V = 8\,\frac{2R^3}{3} = \frac{16}{3} R^3 \quad //$$



Cubo de aresta R ← 1.º oct.
"circunscreve" nossa região,
tem volume R^3

Nossa região no 1.º octante
tem volume calculado $\frac{2}{3}R^3$

Exercício para vocês:



APÊNDICE DO CAP. 20

$$\frac{1}{1-r} = 1 + r^2 + r^3 + \ldots \qquad \text{se} \quad |r| < 1$$

$x, y \in [0,1] \qquad r = xy$

$$\frac{1}{1-xy} = 1 + xy + x^2y^2 + \ldots$$

$$I = \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} = \int_0^1 \left(\int_0^1 1 + xy + x^2y^2 + \ldots \, dx \right) dy$$

$$= \int_0^1 1 + \frac{y}{2} + \frac{y^2}{3} + \ldots \, dy$$

$$= y + \frac{y^2}{2^2} + \frac{y^3}{3^2} + \ldots \Big|_0^1$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Vamos calcular I fazendo uma mudança de variáveis:

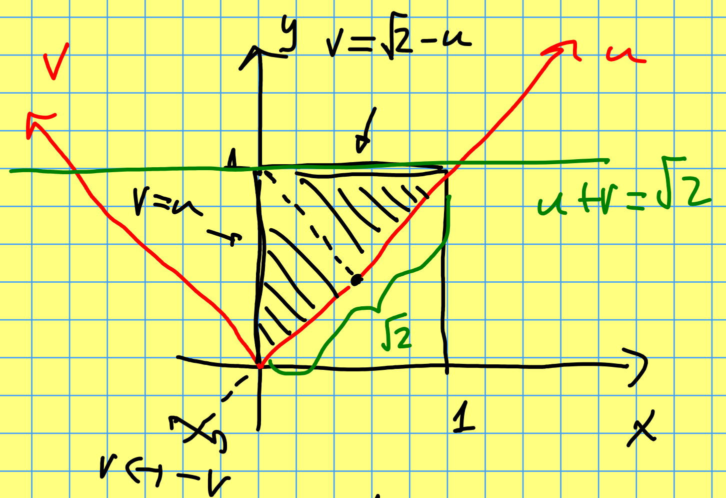
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Rotação de ângulo θ

$\theta = \frac{\pi}{4}$ $\quad \begin{cases} x = \frac{\sqrt{2}}{2}(u-v) \\ y = \frac{\sqrt{2}}{2}(u+v) \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = \underline{\underline{1}}$$

$$xy = \frac{1}{2}(u^2 - v^2) \qquad 1 - xy = \frac{2-u^2+v^2}{2}$$



$0 \leq x \leq 1$
$0 \leq y \leq 1$

$0 \leq u \leq \frac{\sqrt{2}}{2}$ ou $\frac{\sqrt{2}}{2} \leq u \leq \sqrt{2}$

$0 \leq v \leq u \qquad 0 \leq v \leq \sqrt{2} - u$

$$\underline{I} = \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} \qquad dx\,dy = 1 \cdot du\,dv$$

$$= \underbrace{4 \int_0^{\sqrt{2}/2} \int_0^u \frac{dv\,du}{2-u^2+v^2}}_{= I_1} + \underbrace{4 \int_{\sqrt{2}/2}^{\sqrt{2}} \int_0^{\sqrt{2}-u} \frac{dv\,du}{2-u^2+v^2}}_{= I_2}$$

$$\int_0^u \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}} \arctan\left(\frac{v}{\sqrt{2-u^2}}\right)\Bigg|_0^u = \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}}$$

$$\int \frac{dt}{1+t^2} = \arctan t + C \qquad a = \sqrt{2-u^2}$$

$$\int \frac{dt}{a^2+t^2} = \frac{1}{a^2} \int \frac{dt}{1+\left(\frac{t}{a}\right)^2} = \frac{1}{a^2} \int \frac{a\, d\left(\frac{t}{a}\right)}{1+\left(\frac{t}{a}\right)^2}$$

$$= \frac{1}{a} \arctan\left(\frac{t}{a}\right) + C$$

$$I_1 = 4 \int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}}\, du$$

$$u = \sqrt{2} \sin\theta \qquad u = \sqrt{2}/2 : \theta = \pi/6$$

$$du = \sqrt{2}\cos\theta\, d\theta \qquad \sqrt{2-u^2} = \sqrt{2}\cos\theta$$

$$I_1 = 4 \int_0^{\pi/6} \frac{1}{\cancel{\sqrt{2}\cos\theta}} \underbrace{\left(\arctan \frac{\cancel{\sqrt{2}}\sin\theta}{\cancel{\sqrt{2}}\cos\theta}\right)}_{=\theta} \cancel{\sqrt{2}\cos\theta}\, d\theta$$

$$= 4 \int_0^{\pi/6} \theta\, d\theta = 2\theta^2 \Big|_0^{\pi/6} = 2\frac{\pi^2}{36} = \frac{\pi^2}{18}$$

$$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \left[\frac{1}{\sqrt{2-u^2}} \arctan\left(\frac{v}{\sqrt{2-u^2}}\right)\right]_{v=0}^{v=\sqrt{2}-u} du$$

$$= 4 \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}\, du$$

$$\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}} = \arctan\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}} \qquad (u=\sqrt{2}\sin\theta)$$

$$=\arctan\left(\frac{1-\sin\theta}{\cos\theta}\right) = t \overset{!}{=} \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)$$

$$\tan t = \frac{1-\sin\theta}{\cos\theta} = \sec\theta - \tan\theta$$

$$I_2 = 4\int_{\pi/6}^{\pi/2} \frac{1}{\cancel{\sqrt{2}\cos\theta}}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cancel{\sqrt{2}\cos\theta}\, d\theta$$

$$= \cancel{4}\left[\frac{\pi}{\cancel{4}}\theta - \frac{\theta^2}{\cancel{4}}\right]_{\pi/6}^{\pi/2} = \pi\underbrace{\left(\frac{\pi}{2}-\frac{\pi}{6}\right)}_{=\pi/3} - \left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right)$$

$$= \frac{\pi^2}{3} - \left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right) = \frac{\pi^2}{3} - \frac{\cancel{8}\,2\pi^2}{\cancel{36}\,9} = \frac{\pi^2}{9}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = I = I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6} \quad //$$

(Euler, séc. XVIII)