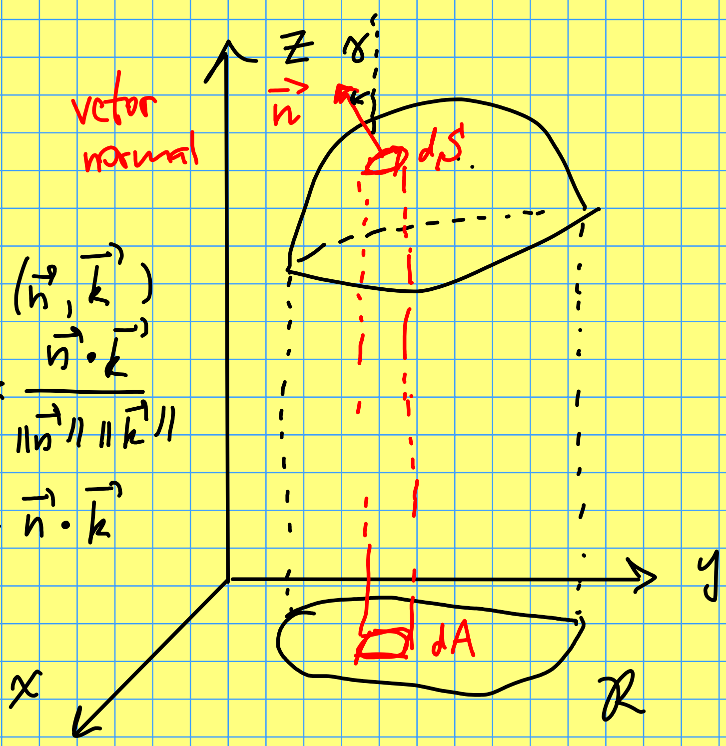
ÁREA DE UM GRÁFICO

20/09/21



$$\gamma = \measuredangle(\vec{n}, \vec{k})$$

$$\cos\gamma = \frac{\vec{n}\cdot\vec{k}}{\|\vec{n}\|\,\|\vec{k}\|} = \vec{n}\cdot\vec{k}$$

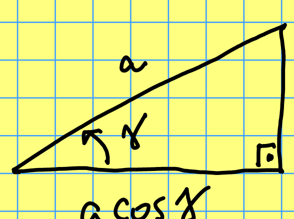
$f : R \subset \mathbb{R}^2 \to \mathbb{R}$

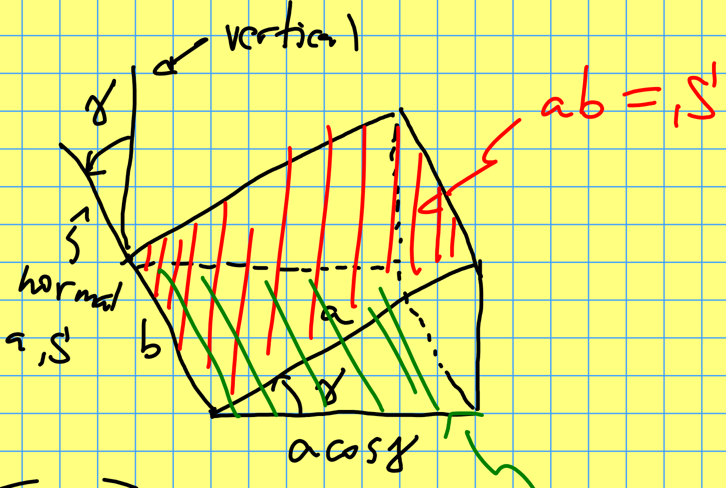
Gráfico: ← É uma superfície!

$S : \; z = f(x,y)$

dS : elemento de superfície

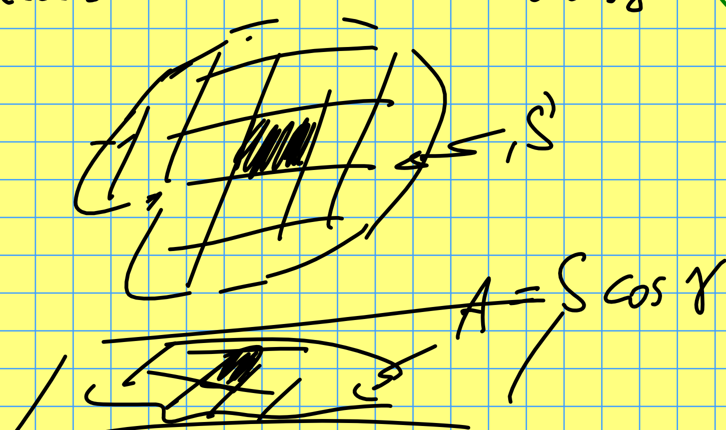
$$\text{Área}(S) = \iint_R dS \qquad dA = dx\,dy$$





$ab\cos\gamma = A$

$S\cos\gamma = A$

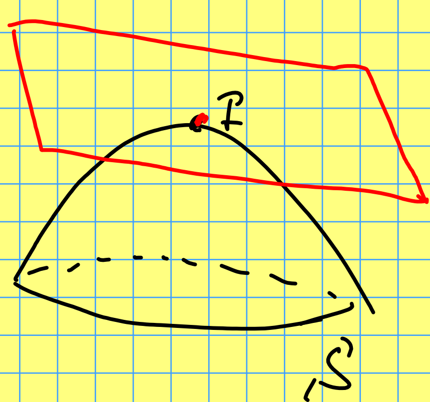


$$S = \frac{A}{\cos\gamma} \qquad d\mathcal{S} = \frac{dA}{\cos\gamma} \simeq \frac{dx\,dy}{\cos\gamma}$$

$$\text{Área}(\mathcal{S}) = \iint_R \frac{dx\,dy}{\cos\gamma}$$

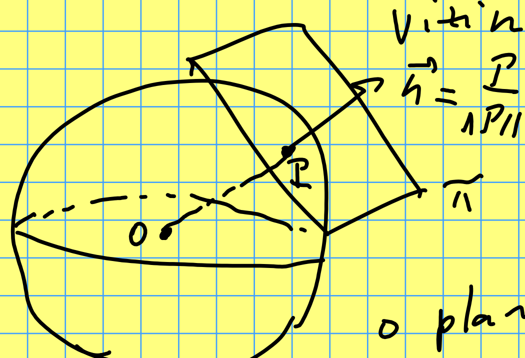
γ é o ângulo formado entre o plano horizontal (plano xy) e o plano tangente à nossa superfície: ele varia de ponto para ponto.

PLANO TANGENTE E VETOR NORMAL A UM GRÁFICO



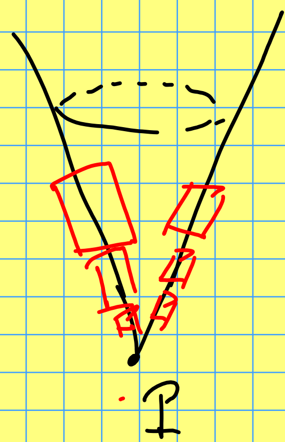
Plano tangente à superfície $\mathcal{S}$ em um ponto P de $\mathcal{S}$: Plano contendo P que "melhor aproxima" $\mathcal{S}$ numa vizinhança de P.

Esfera



Plano tangente à esfera em P é o plano por P perpendicular à reta OP.

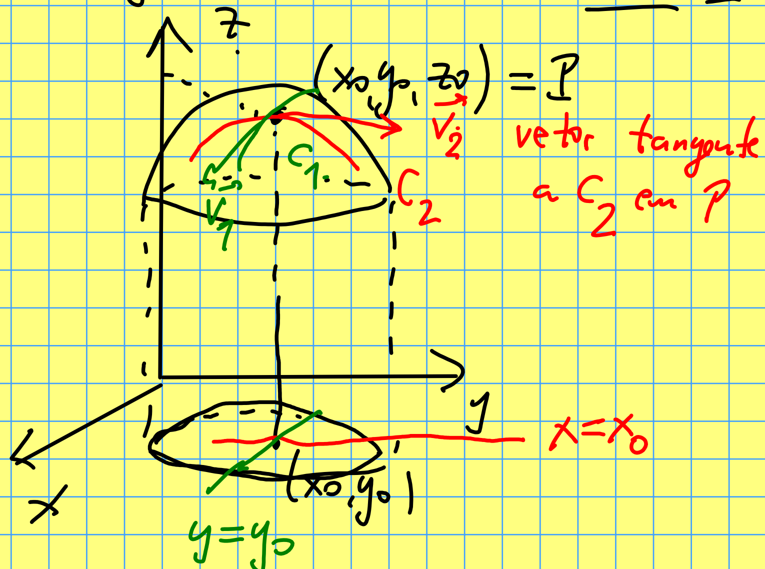
Cone



Não existe um plano tangente ao cone em P

Como obter uma equação para o plano tangente?

Caso de um gráfico $z = f(x,y)$



$C_2 : y \mapsto (x_0, y, \underbrace{f(x_0, y)}_{=z})$ curva contida no gráfico, passando por P no instante $y = y_0$

$\vec{v}_2$ vetor tangente à curva C_2 em P é dado por $\vec{v}_2 = \frac{\partial}{\partial y}\Big|_{y=y_0} (x_0, y, f(x_0, y)) = \left(0, 1, \frac{\partial f}{\partial y}(x_0, y_0)\right)$

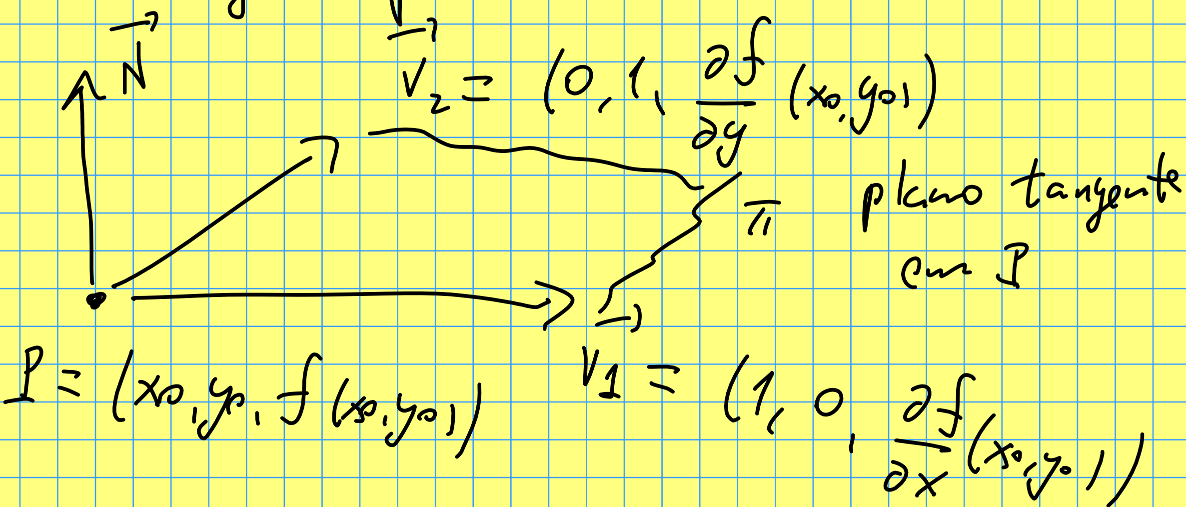
$\vec{v}_2$ também é tangente ao gráfico em P.

$C_1 : x \mapsto f(x, y_0)$

$$\vec{v}_1 = \frac{\partial}{\partial x}\Big|_{x=x_0} (x, y_0, f(x, y_0)) = \left(1, 0, \frac{\partial f}{\partial x}(x_0, y_0)\right)$$

$\vec{v}_1, \vec{v}_2$ dois vetores tangentes ao gráfico em P

Se eles forem L. I., então determinam o plano tangente por P.



$\vec{N} = \vec{v}_1 \times \vec{v}_2$ vetor normal ao gráfico em P

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0) \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(x_0, y_0)\,\vec{i} - \frac{\partial f}{\partial y}(x_0, y_0)\,\vec{j} + \vec{k}$$

$\vec{n} = \dfrac{\vec{N}}{\|\vec{N}\|}$ vetor normal unitário ao gráfico em P.

$$\boxed{\vec{n} = \frac{\left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right)}{\sqrt{\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2 + 1}}}$$

$\pi : ax + by + cz = d$

onde (a,b,c) é um vetor normal

$$\pi : -\frac{\partial f}{\partial x}(x_0,y_0)\, x - \frac{\partial f}{\partial y}(x_0,y_0)\, y + 1.z = d$$

$(x_0, y_0, f(x_0,y_0)) \in \pi$

$$\Rightarrow -\frac{\partial f}{\partial x}(x_0,y_0)\, x_0 - \frac{\partial f}{\partial y}(x_0,y_0)\, y_0 + f(x_0,y_0) = d$$

$$-\frac{\partial f}{\partial x}(x_0,y_0)\, x - \frac{\partial f}{\partial y}(x_0,y_0)\, y + z = -\frac{\partial f}{\partial x}(x_0,y_0)\, x_0 - \frac{\partial f}{\partial y}(x_0,y_0)\, y_0 + f(x_0,y_0)$$

$$\therefore \boxed{z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\,(y - y_0)}$$

Plano tangente ao gráfico $z = f(x,y)$ em $(x_0, y_0, f(x_0,y_0))$.

Exemplo

Esfera $\quad x^2+y^2+z^2=R^2 \quad (R>0)$

$$z=\sqrt{R^2-x^2-y^2}=f(x,y)$$

$$\frac{\partial f}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2-y^2}}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

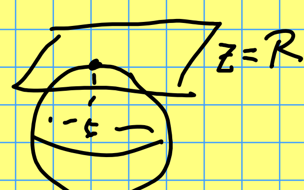
$$\frac{\partial f}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}}$$

$$z=\sqrt{R^2-x_0^2-y_0^2}-\frac{x_0}{\sqrt{R^2-x_0^2-y_0^2}}(x-x_0)-\frac{y_0}{\sqrt{R^2-x_0^2-y_0^2}}(y-y_0)$$

$$=\frac{1}{\sqrt{R^2-x_0^2-y_0^2}}\left[R^2-\cancel{x_0^2}-\cancel{y_0^2}-x_0x+\cancel{x_0^2}-y_0y+\cancel{y_0^2}\right]$$

$$\boxed{z=\frac{1}{\sqrt{R^2-x_0^2-y_0^2}}\left(R^2-x_0x-y_0y\right)}$$

$x_0=y_0=0$: $\quad z=\frac{R^2}{\sqrt{R^2}}=R$



$$\vec{n} = \frac{\left(\dfrac{x_0}{\sqrt{R^2-x_0^2-y_0^2}} , \dfrac{y_0}{\sqrt{R^2-x_0^2-y_0^2}} , 1 \right)}{\sqrt{\dfrac{x_0^2}{R^2-x_0^2-y_0^2} + \dfrac{y_0^2}{R^2-x_0^2-y_0^2} + 1}}$$

$$= \frac{\dfrac{1}{\cancel{\sqrt{R^2-x_0^2-y_0^2}}} \left(x_0 , y_0 , \sqrt{R^2-x_0^2-y_0^2} \right)}{\sqrt{\dfrac{\cancel{x_0^2} + \cancel{y_0^2} + R^2 - \cancel{x_0^2} - \cancel{y_0^2}}{\cancel{R^2-x_0^2-y_0^2}}}}$$

$$= \frac{1}{R} \left(x_0 , y_0 , \sqrt{R^2-x_0^2-y_0^2} \right)$$

$$= \frac{1}{R} \left(x_0 , y_0 , f(x_0,y_0) \right) = \frac{1}{R} \cdot P$$

$\vec{n} = \frac{1}{R} P$ $\quad$ $\vec{n}$ e P são paralelos

$= \dfrac{P}{\|P\|}$ $\quad$ $R = \|P\|$

<u>Exemplo</u>. Problema 2 § 19.3 [Inglês]

$z = 4xy$ $\quad$ Plano tangente em $\left(4, \frac{1}{4}, \underset{f(x_0,y_0)=z_0}{\underset{\shortparallel}{4}}\right)$

Resolução $f(x,y) = 4xy \quad \frac{\partial f}{\partial x} = 4y \quad \frac{\partial f}{\partial y} = 4x$

$$z = z_0 + 4y_0(x-x_0) + 4x_0(y-y_0)$$

$$= 4 + 1\cdot(x-4) + 16\left(y - \frac{1}{4}\right)$$

$$= \cancel{4} + x - \cancel{4} + 16y - 4$$

$$= x + 16y - 4$$

ou $\boxed{x + 16y - z = 4}$

— u —

$$\vec{n} = \frac{\left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right)}{\sqrt{\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2 + 1}}$$

$$\cos\gamma = \vec{n}\cdot\vec{k} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2}}$$

$$\therefore \boxed{\text{Área}(S) = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy}$$

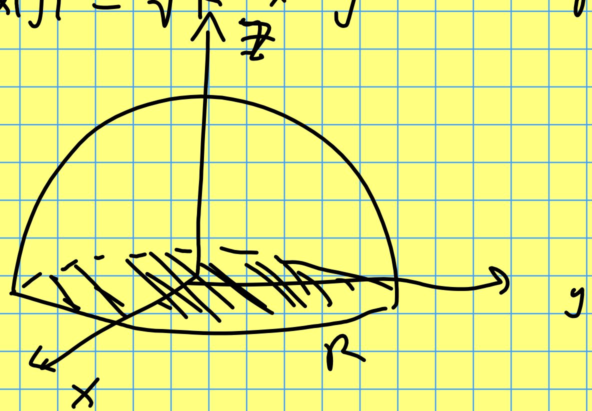
Exemplo Calcular a área da esfera

$x^2 + y^2 + z^2 = R^2 \quad (R > 0)$.

Resolução.

$$\text{Área}(S) = 2 \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$$

$f(x,y) = \sqrt{R^2 - x^2 - y^2}$ $\quad R: \; x^2 + y^2 \leq R^2$



$$\text{Área}(S) = 2 \iint_R \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dxdy$$

$$= 2 \iint_R \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}\, dxdy = 2R \iint_R \frac{dxdy}{\sqrt{R^2 - x^2 - y^2}}$$

Coords polares: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad dxdy = r\, drd\theta$

$$\text{Área}(S) = 2R \int_0^{2\pi}\int_0^R \frac{r\,dr\,d\theta}{\sqrt{R^2-r^2}}$$

$$= \frac{2R\cdot 2\pi}{-2} \int_0^R -2r\,(R^2-r^2)^{-1/2}\,dr$$

$$= -2\pi R \left[2\,(R^2-r^2)^{1/2}\right]_{r=0}^{r=R}$$

$$= -2\pi R\left(0 - 2\,(R^2)^{1/2}\right)$$

$$= +4\pi R^2 \quad //$$

Plano tangente : § 19.3

Área de gráfico : § 20.8