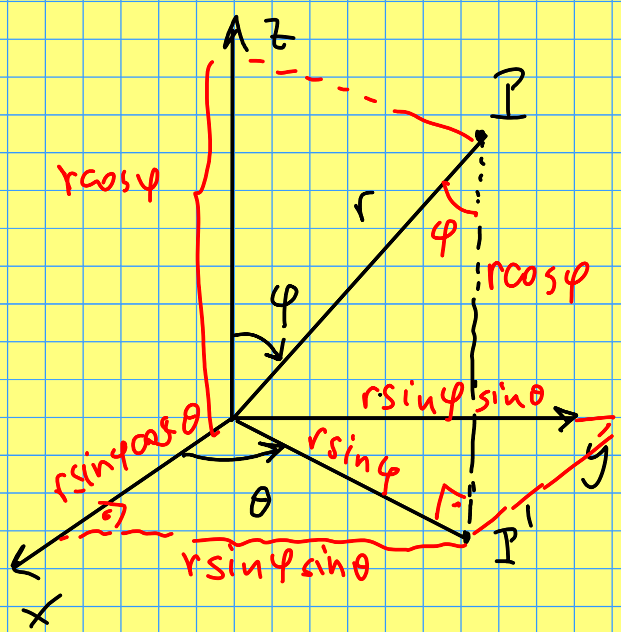
COORDENADAS ESFÉRICAS

13/09/21



(r, θ, φ) : coordenadas esféricas do ponto P

Em geral:

$$r \geq 0$$

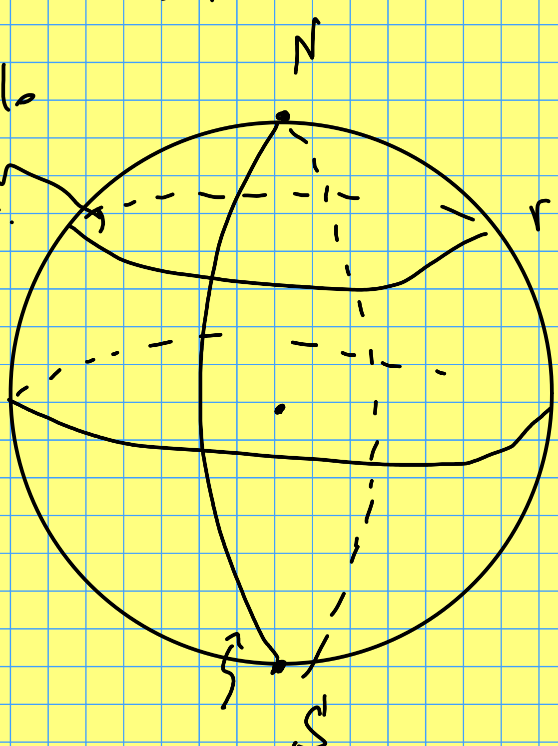
$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

$$\begin{cases} x = r \sin\varphi \cos\theta \\ y = r \sin\varphi \sin\theta \\ z = r \cos\varphi \end{cases}$$

paralelo

$\varphi = \text{const.}$



$r = R > 0$ (constante)

meridiano $\theta = \text{const}$

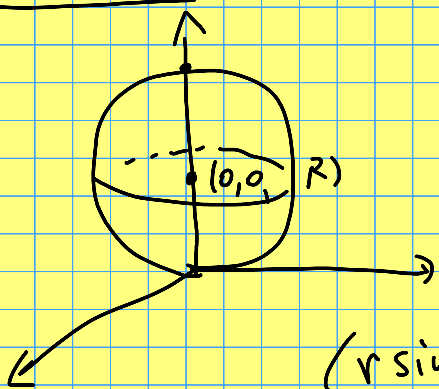
superfície esférica de raio R

θ : longitude

φ : co-latitude

Exemplos

1.



$$x^2+y^2+(z-R)^2=R^2$$

$$x^2+y^2+z^2-2Rz=0$$

$$\underbrace{(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2}$$

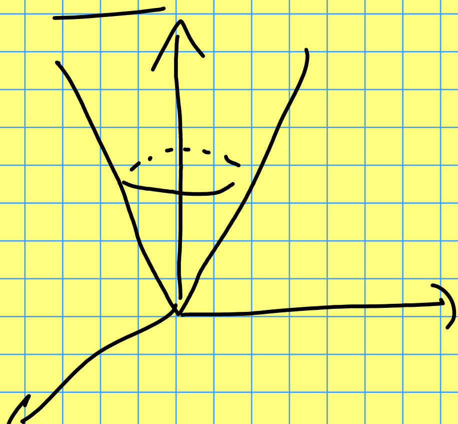
$$+(r\cos\varphi)^2-2Rr\cos\varphi=0$$

$$r^2\sin^2\varphi+r^2\cos^2\varphi-2Rr\cos\varphi=0$$

$$r^2-2Rr\cos\varphi=0$$

$$\boxed{r=2R\cos\varphi}\qquad \left(0\leq\varphi\leq\frac{\pi}{2}\right)$$

2. Cone $\quad z=a\sqrt{x^2+y^2}\quad (a>0)$



$$r\cos\varphi=a\sqrt{r^2\sin^2\varphi}$$

$$=a r.|\sin\varphi|$$

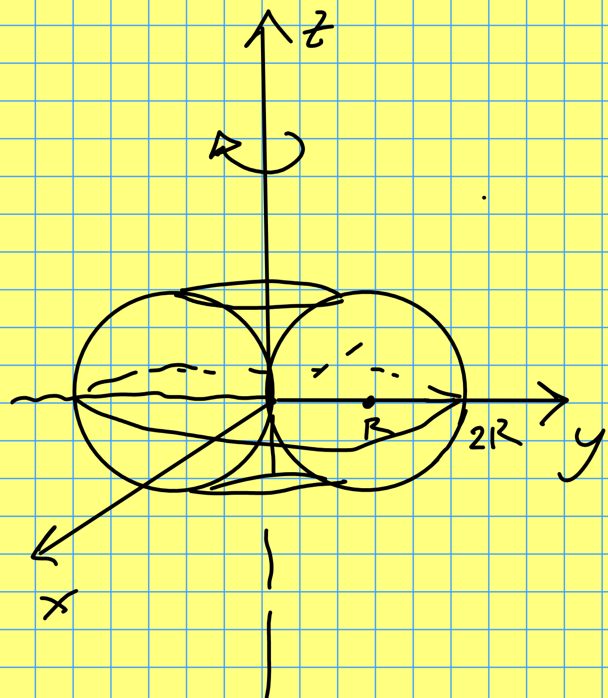
$$=ar\sin\varphi$$

$$\cos\varphi=a\sin\varphi\quad (0<\varphi<\pi)$$

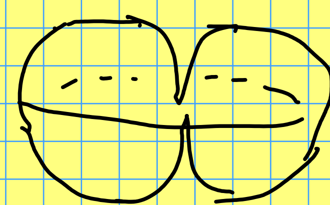
$$\cot\varphi=a$$

$$\varphi=\operatorname{arccot} a=\text{const}$$

3. $r = 2R\sin\varphi$



φ	r
0	0
$\frac{\pi}{4}$	$R\sqrt{2}$
$\frac{\pi}{2}$	$2R$
$\frac{3\pi}{4}$	$R\sqrt{2}$
π	0



$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$\theta = \pi/2$

$x = 0$

$y = r\sin\varphi$

$z = r\cos\varphi$

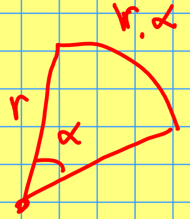
$$(y-R)^2 + z^2 = R^2$$

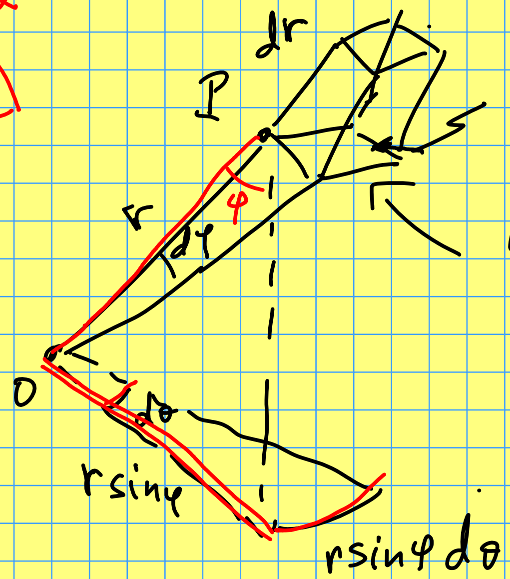
$$\underbrace{y^2} - 2Ry + \underbrace{z^2} = 0$$

$$r^2 - 2Rr\sin\varphi = 0$$

$$r = 2R\sin\varphi$$

<u>Elemento de volume em coordenadas esféricas</u>





Elemento de volume

$$dV = (dr)(r\sin\varphi\, d\theta)(r\, d\varphi)$$
$$= r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$V = \iiint_R dV = \iiint_R r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

Exemplos.

1. Calcular o volume da região delimitada pela esfera $x^2+y^2+z^2 = R^2$ (bola sólida)

Resolução.

$$V = \iiint_R dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$= \left(\int_0^R r^2 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\, d\varphi\right)$$

$$= \frac{R^3}{3}\, 2\pi \;\, 2 = \frac{4\pi R^3}{3} \;//$$

2. Calcular o volume do toro sólido delimitado por $r = 2R\sin\varphi$.

Resolução

$$V = \iiint_{\mathcal{R}} dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{2R\sin\varphi} r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$= 2\pi \int_0^{\pi} \frac{(2R\sin\varphi)^3}{3} \sin\varphi \;\, d\varphi$$

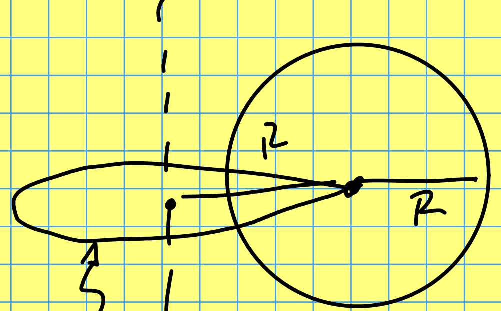
$$= \frac{2\pi}{3}\int_0^{\pi} 8R^3 \sin^4\varphi \; d\varphi \qquad \sin^4\varphi = (\sin^2\varphi)^2 = \left(\frac{1-\cos 2\varphi}{2}\right)^2$$

$$= \frac{16\pi}{3} R^3 \int_0^{\pi} \frac{1 - 2\cos 2\varphi + \cos^2 2\varphi}{4}\; d\varphi$$

$$= \frac{4\pi}{3} R^3 \int_0^{\pi} 1 - 2\cos 2\varphi + \frac{1+\cos 4\varphi}{2} \;\, d\varphi$$

$$= \frac{4\pi}{\cancel{3}} R^3 \cdot \frac{\cancel{3}}{2}\pi = 2\pi^2 R^3$$

$$= (2\pi R)(\pi R^2)$$



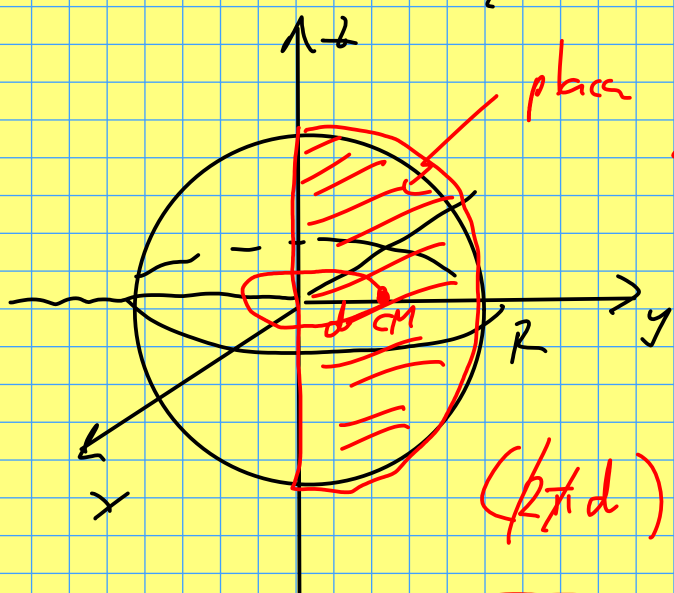
Área $= \pi R^2$
da
seção transversal

comprimento da trajetória do CM $= 2\pi R$

$$\text{Volume do sólido de revolução} = \begin{pmatrix}\text{compr.}\\ \text{da}\\ \text{trajetória}\\ \text{do CM}\end{pmatrix}\begin{pmatrix}\text{área da}\\ \text{seção}\\ \text{transversal}\end{pmatrix}$$

(Teorema de Pappus)

Outra ilustração do T. Pappus:

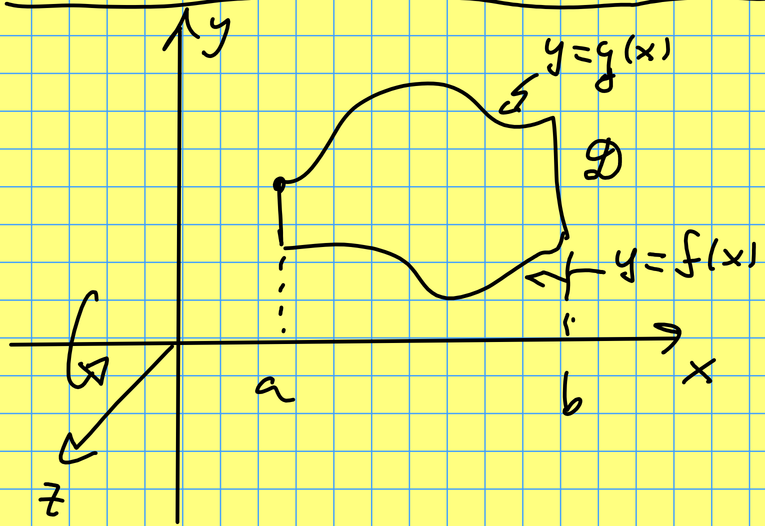


placa semi-circular
área $= \frac{1}{2}\pi R^2$

$$(\cancel{2}\cancel{\pi} d)\left(\frac{1}{\cancel{2}}\pi R^2\right) = \frac{4}{3}\cancel{\pi} R^3$$

$$\boxed{d = \frac{4R}{3\pi}}$$

Demonstração do T. Pappus



$$\mathcal{D}: \begin{cases} a \leq x \leq b \\ f(x) \leq y \leq g(x) \end{cases}$$

R: sólido produzido pela rotação de $\mathcal{D}$ ao redor do eixo x.

$A = \text{área}(\mathcal{D})$ $\qquad$ $V = \text{volume}(R)$

$V = V_1 - V_2$ $\qquad$ V_1 = volume da região R_1 (R_2) (V_2) delimitada pela rotação do gráfico $y = g(x)$ ($y = f(x)$) em torno do eixo x

$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases} \qquad dV = r\,dr\,d\theta\,dx$$

$$V_1 = \iiint_{R_1} dV = \int_a^b \int_0^{2\pi} \int_0^{g(x)} r\,dr\,d\theta\,dx$$

$$= \int_a^b \int_0^{2\pi} \frac{g(x)^2}{2} \, d\theta \, dx = \pi \int_a^b g(x)^2 \, dx$$

$$V_2 = \pi \int_a^b f(x)^2 \, dx$$

$$V = \pi \int_a^b \left[g(x)^2 - f(x)^2 \right] dx$$

$$= \pi \int_a^b \left[\int_{f(x)}^{g(x)} 2y \, dy \right] dx$$

$$= 2\pi \int_a^b \int_{f(x)}^{g(x)} y \, dy \, dx$$

$$= 2\pi \iint_{\mathcal{D}} y \, dx \, dy$$

$$\bar{y} = \frac{M_x}{M}$$

$$= 2\pi \, M_x$$

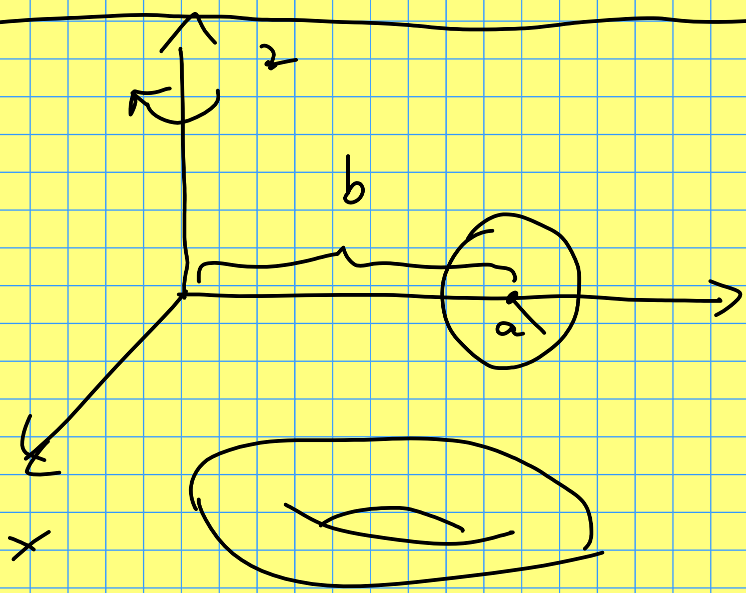
$$= (2\pi \bar{y}) A \quad //$$

$$M = A \ (\delta = 1)$$

Mais uma aplicação:

Volume do toro sólido de revolução

$0 < a < b$



$$V = (2\pi b)(\pi a^2) = 2\pi^2 a^2 b$$ //.