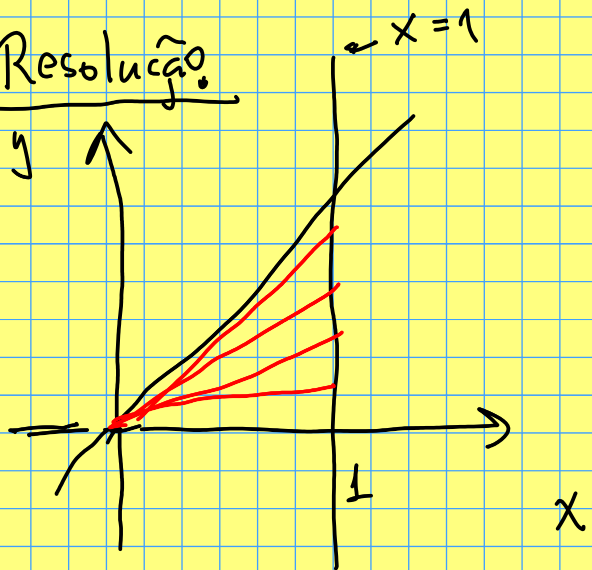
09/09/21

LISTA 3

$$I = \iint_R \frac{dx\,dy}{(1+x^2+y^2)^{3/2}} = ?\,,$$

R : região delimitada por $y=0$, $y=x$, $x=1$

Resolução:



Coords polares:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad 0 \le \theta \le \frac{\pi}{4}$$

$$0 \le r \le \sec\theta$$

$$1 = r\cos\theta$$

$$r = \frac{1}{\cos\theta} = \sec\theta$$

$$x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

$$I = \iint_{R'} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}} = \int_0^{\pi/4}\left[\int_0^{\sec\theta} \frac{r\,dr}{(1+r^2)^{3/2}}\right] d\theta$$

$$\int \frac{r\,dr}{(1+r^2)^{3/2}} = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}\int u^{-3/2}\,du$$

$u = 1+r^2$

$du = 2r\,dr$

$$= \frac{1}{2}\,\frac{u^{-1/2}}{-1/2} = -u^{-1/2} + C$$

$$= -(1+r^2)^{-1/2} + C$$

$$= \frac{-1}{\sqrt{1+r^2}} + C$$

$$\int_0^{\sec\theta} \frac{r\,dr}{(1+r^2)^{3/2}} = \left[-\frac{1}{\sqrt{1+r^2}}\right]_0^{\sec\theta} = -\frac{1}{\sqrt{1+\sec^2\theta}} + 1$$

$$I = \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{\sqrt{1+\sec^2\theta}}\right) d\theta = \frac{\pi}{4} - \int_0^{\pi/4} \frac{\cos\theta}{\sqrt{\cos^2\theta + 1}}\, d\theta$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\cos\theta\, d\theta}{\sqrt{2-\operatorname{sen}^2\theta}} = \frac{\pi}{4} - \int_0^{1/2} \frac{dt}{\sqrt{1-t^2}}$$

$$\int \frac{\cos\theta\, d\theta}{\sqrt{2-\sin^2\theta}} = \int \frac{\sqrt{2}\,dt}{\sqrt{2-2t^2}} = \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \arcsin t + C$$

$$\sin\theta = \sqrt{2}\,t$$

$$\cos\theta\, d\theta = \sqrt{2}\,dt$$

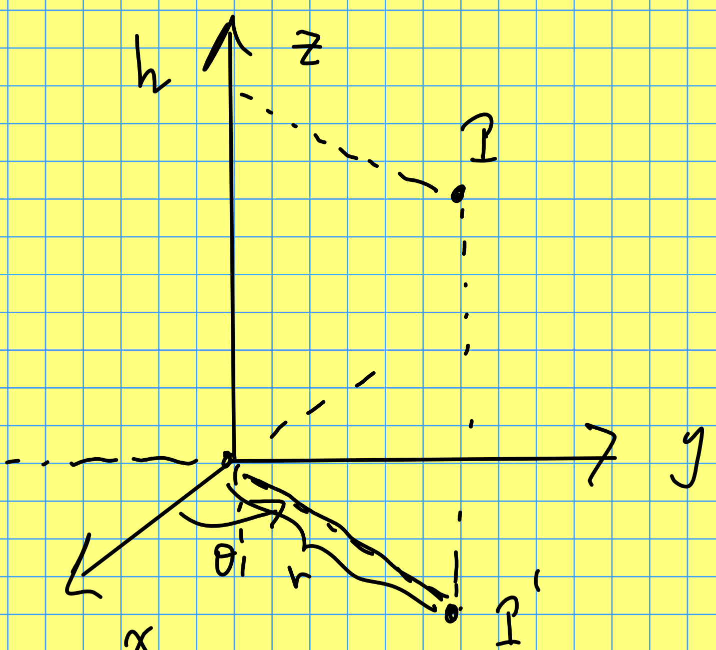
$$y = \arcsin x \iff \sin y = x \text{ e } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{+\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

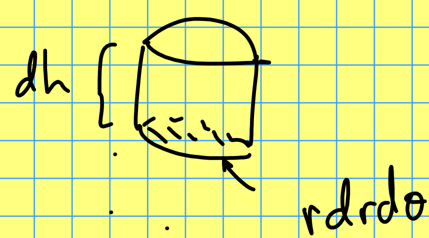
$$I = \frac{\pi}{4} - \left[\arcsin t\right]_0^{1/2} = \frac{\pi}{4} - \arcsin\frac{1}{2}$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \quad //.$$

(seção 20.6) COORDENADAS CILÍNDRICAS

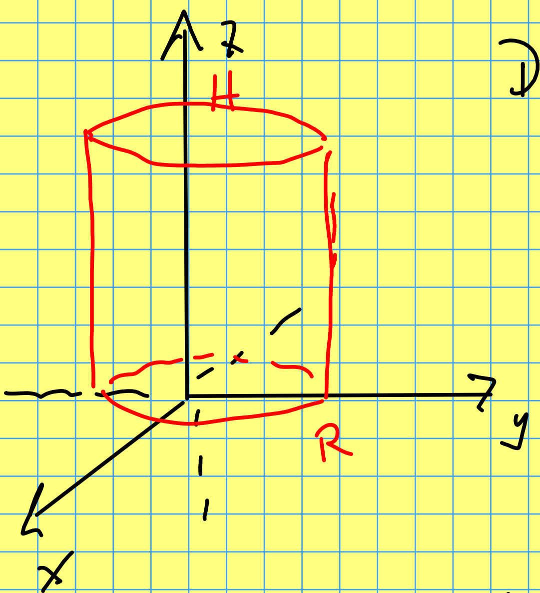


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = h \end{cases}$$



$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dh$$

Exemplo. Calcular o momento de inércia de um cilindro sólido homogêneo de altura H, raio da base R e massa M em torno de seu eixo.



Densidade δ constante

$$I_z = \iiint_{\mathcal{R}} r^2 \delta \, dV$$

$$= \iiint_{\mathcal{R}'} r^2 \, r \, dr \, d\theta \, dh$$

eixo z = eixo do cilindro

$\mathcal{R}'$: $0 \leq h \leq H$
$0 \leq r \leq R$
$0 \leq \theta \leq 2\pi$

$$= \delta \int_0^H \int_0^{2\pi} \int_0^R r^3 \, dr \, d\theta \, dh$$

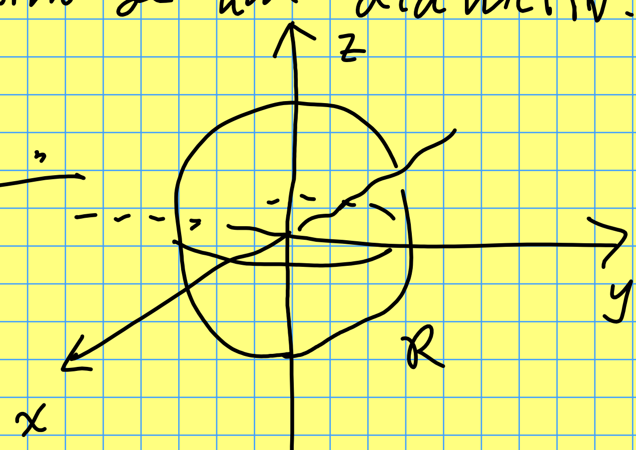
$$= \delta H 2\pi \frac{R^4}{4} = \frac{\delta \pi R^4 H}{2}$$

$M = \delta . V = \delta . H . \pi R^2$

$$= \frac{1}{2} M R^2 \; //$$

Exemplo. Calcular o momento de inércia de uma esfera sólida homogênea de raio R e massa M em torno de um diâmetro.

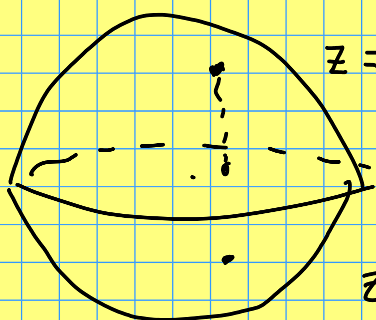
Resolução.



$$I_z = \iiint_{\mathcal{R}} r^2 \delta \, dV$$

Coords cilíndricas: $0 \le r \le R \quad -\sqrt{R^2-r^2} \le h \le \sqrt{R^2-r^2}$

$$0 \le \theta \le 2\pi$$



$z = \sqrt{R^2-x^2-y^2} = \sqrt{R^2-r^2}$

$dV = r\,dr\,d\theta\,dh$

$z = -\sqrt{R^2-r^2}$

$$I_z = 2\delta \int_0^{2\pi}\int_0^{R}\int_0^{\sqrt{R^2-r^2}} r^2 r \, dh\, dr\, d\theta$$

$$= 2\delta \int_0^{2\pi}\int_0^{R} r^3\sqrt{R^2-r^2}\, dr\, d\theta$$

$$= 2\delta \cdot 2\pi \int_0^{R} r^3 \sqrt{R^2-r^2}\, dr$$

$$= 4\pi\delta \int_0^{\pi/2} R^3 \sin^3 t \cdot R\cos t\, R\cos t\, dt \qquad \begin{aligned} r &= R\sin t \\ dr &= R\cos t\, dt \end{aligned}$$

$$= 4\pi\delta R^5 \int_0^{\pi/2} \underbrace{\sin^3 t}_{=\sin t(1-\cos^2 t)} \cos^2 t\, dt$$

$$= 4\pi\delta R^5 \int_0^{\pi/2} (\cos^2 t - \cos^4 t)\,\sin t\,dt$$

$$= 4\pi\delta R^5 \left[-\frac{\cos^3 t}{3} + \frac{\cos^5 t}{5} \right]_0^{\pi/2}$$

$\cos\frac{\pi}{2} = 0$

$\cos 0 = 1$

$$= 4\pi\delta R^5 \left(\frac{1}{3} - \frac{1}{5}\right) = 4\pi\delta R^5 \frac{2}{15} = \frac{8\pi\delta R^5}{15}$$

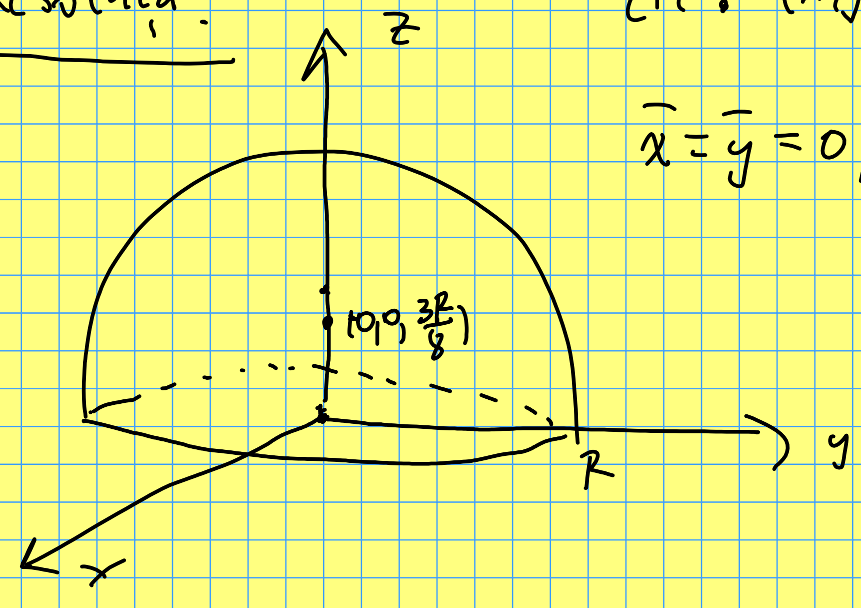
$$M = \delta V = \delta \frac{4}{3}\pi R^3 \Rightarrow R^3 = \frac{3M}{4\pi\delta}$$

$$I_z = \frac{\cancel{8}^{2}\,\cancel{\pi}\,\cancel{\delta}\,R^2}{\cancel{15}_{5}} \frac{\cancel{3}M}{\cancel{4\pi\delta}} = \frac{2}{5}MR^2 \;//$$

Ex.9 [Versão em inglês]

Calcular o CM de um hemisfério sólido homogêneo de raio R.

<u>Resolução</u>



CM : $(\bar{x}, \bar{y}, \bar{z})$

$\bar{x} = \bar{y} = 0$, por simetria rotacional em torno do eixo z, e por homogeneidade.

$\delta \equiv k$ (const)

$$\bar{z} = \frac{I_{xy}}{M} = \frac{\iiint_R z\,\delta\, dV}{\iiint_R \delta\, dV} = \frac{\iiint_R z\, dV}{\frac{2}{3}\pi R^3}$$

$$\iiint_R z\, dV = \int_0^{2\pi}\int_0^{R}\int_0^{\sqrt{R^2-r^2}} h\, r\, dh\, dr\, d\theta$$

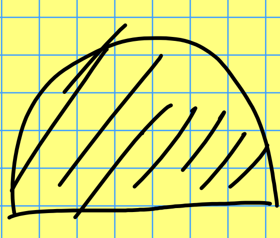
Coords cilíndricas

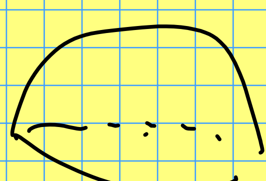
$$= 2\pi \int_0^R r\left(\int_0^{\sqrt{R^2-r^2}} h\, dh\right) dr$$

$$= \cancel{2}\pi \int_0^R r \left[\frac{h^2}{\cancel{2}}\right]_{h=0}^{h=\sqrt{R^2-r^2}} dr = \pi \int_0^R r(R^2-r^2)\, dr$$

$$= \pi\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R} = \pi\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\pi R^4}{4}$$

$$\bar{z} = \frac{\pi R^4/4}{\frac{2}{3}\pi R^3} = \frac{\pi R^4}{4}\cdot\frac{3}{2\pi R^3} = \frac{3}{8}R \quad //$$





$$\frac{4}{3\pi} > \frac{3}{8} \qquad 32 > 9\pi$$

$$\frac{32}{3} > 10 > 3\pi$$