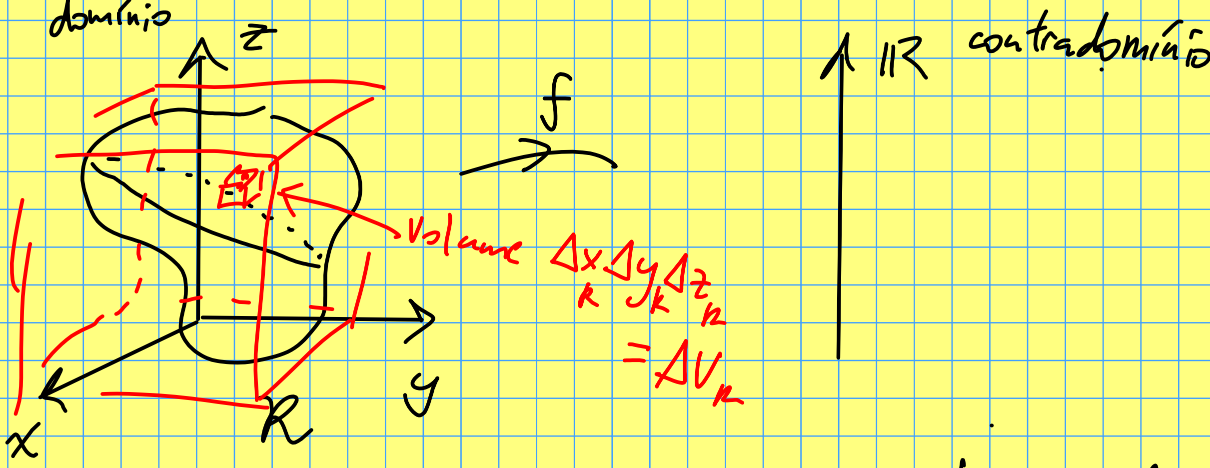
INTEGRAIS TRIPLAS

02/09/21



$$f : R \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$

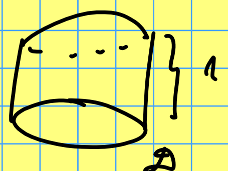
Gráfico de f é subconjunto de $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$!

$$\iiint_R f(x,y,z) \underbrace{dV}_{\text{elemento de volume}} := \lim \sum_k f(x_k, y_k, z_k)\, \Delta V_k$$

O limite é sobre todas partições do bloco, quando o tamanho das subdivisões $\to 0$

<u>Roteiro:</u> • Calcular integrais triplas por meio de integrais iteradas
• Aplicações à Geometria e à Física.

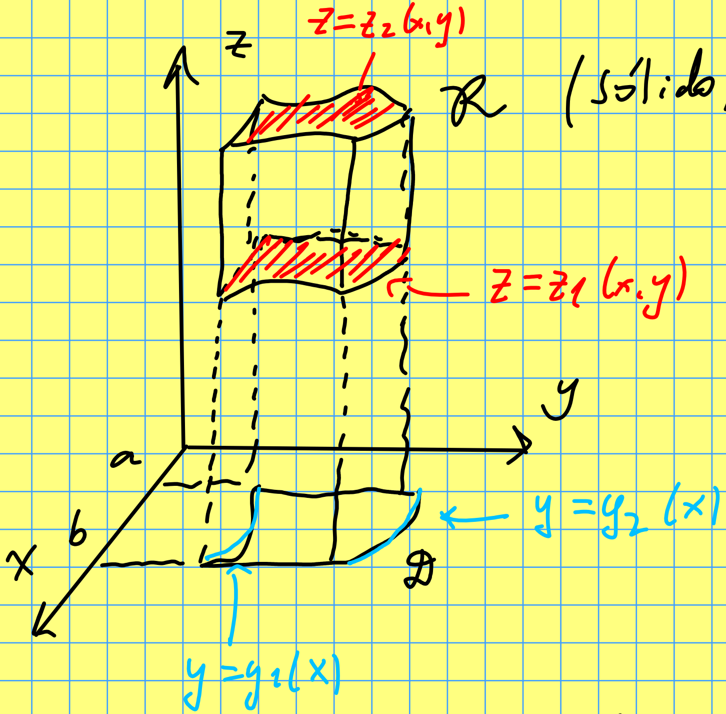
Integrais iteradas



$$\text{Vol}\begin{pmatrix}\text{Cilindro} \\ \text{altura } 1 \\ \text{sobre } \mathcal{D}\end{pmatrix} = \iint_{\mathcal{D}} dA = A(\mathcal{D})$$

$$\iint_{\mathcal{D}} f\, dA = \text{Vol}\begin{pmatrix}\text{Sólido} \\ \text{debaixo} \\ \text{do gráfico}\end{pmatrix}$$

$$\text{Área}(\mathcal{D}) = \text{Vol}(\text{Cilindro})$$



$$\mathcal{D}:\begin{cases} a \le x \le b \\ y_1(x) \le y \le y_2(x) \end{cases}$$

$$\mathcal{R}:\begin{cases} (x,y) \in \mathcal{D} \\ z_1(x,y) \le z \le z_2(x,y) \end{cases}$$

$$\iiint_{\mathcal{R}} f(x,y,z)\,dV = \iint_{\mathcal{D}} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$$

$$= \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dy \right] dx$$

ordem de integração

$3! = 6$ ordens de integração possíveis.

—//—

Interpretacão como grandezas geométricas e físicas

$$\boxed{\iiint_{\mathcal{R}} dV = \text{Vol}(\mathcal{R})} \quad (f \equiv 1)$$

(= Volume quadri-dimensional do cilindro do $\mathbb{R}^4$ de altura 1 e base $\mathcal{R} \subset \mathbb{R}^3$)

$\mathcal{R} \subset \mathbb{R}^3$

$$\left[\iiint_R f(x,y,t)\, dV = \text{Volume quadri-dimensional da região do } \mathbb{R}^4 \text{ debaixo do gráfico de } f\right]$$

$\delta(x,y,z)$: densidade volumétrica

$$\boxed{\iiint_R \delta(x,y,z)\, dV = \text{Massa}(R)}$$

Momento de R em relação ao plano xy

$$\boxed{M_{xy} = \iiint_R z\,\delta(x,y,z)\, dV}$$

$$M_{yz} = \iiint_R x\,\delta\, dV \qquad M_{xz} = \iiint_R y\,\delta\, dV$$

Momento de inércia de R em relação ao eixo z

$$\boxed{I_z = \iiint_R (x^2+y^2)\,\delta\, dV}$$

$$I_x = \iiint_R (y^2+z^2)\,\delta\, dV \qquad I_y = \iiint_R (x^2+z^2)\,\delta\, dV$$

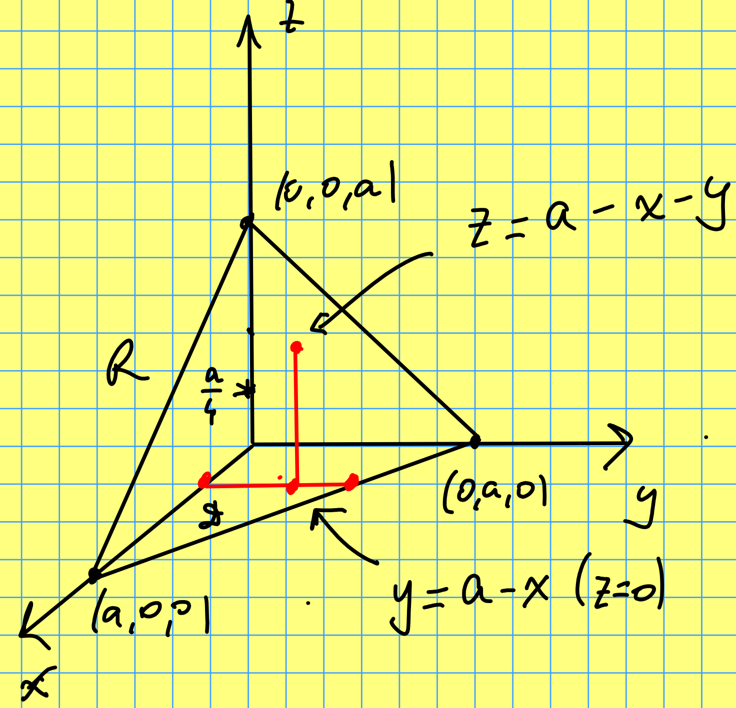
Centro de massa de R : $(\bar{x}, \bar{y}, \bar{z})$ onde

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Exemplo. Calcular o centro de massa do sólido homogêneo delimitado pelos planos coordenados e pelo plano $x+y+z=a$ $(a>0)$.

$\delta \equiv$ constante

Resolução.



$$R: \begin{cases} 0 \leq x \leq a \\ 0 \leq y \leq a - x \\ 0 \leq z \leq a - x - y \end{cases}$$

$\delta(x,y,z) = k > 0$

$$\text{Vol}(R) = \frac{1}{3}(\text{Área } \triangle)(\text{altura}) = \frac{1}{3}\frac{1}{2}a^2 \cdot a = \frac{a^3}{6}$$

Centro de massa $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\iiint_R z \cdot k \, dV}{\iiint_R k \, dV} = \frac{\cancel{k}}{\cancel{k}} \frac{\iiint_R z \, dV}{\text{Vol}(R)}$$

$$\bar{z} = \frac{6}{a^3} \iiint_R z \, dV$$

$$= \frac{6}{a^3} \int_0^a \int_0^{a-x} \int_0^{a-x-y} z \, dz \, dy \, dx$$

$$= \frac{6}{a^3} \int_0^a \int_0^{a-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=a-x-y} dy \, dx$$

$$= \frac{3}{a^3} \int_0^a \int_0^{a-x} (a-x-y)^2 \, dy \, dx$$

$$= \frac{\cancel{3}}{a^3} \int_0^a \left[\frac{-(a-x-y)^3}{\cancel{3}} \right]_{y=0}^{y=a-x} dx$$

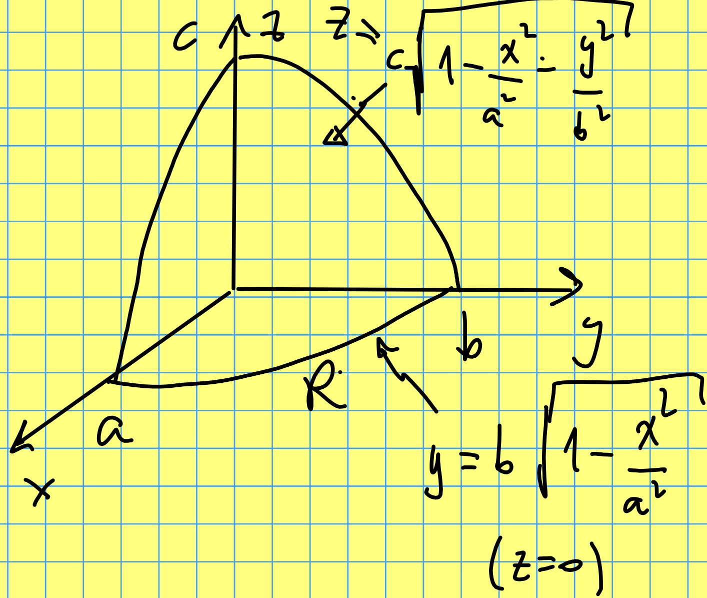
$$= \frac{1}{a^3} \int_0^a +(a-x)^3 dx = \frac{1}{a^3} \left[\frac{-(a-x)^4}{4} \right]_{x=0}^{x=a} = \frac{1}{a^3} \frac{a^4}{4} = \frac{a}{4}$$

Por simetria e homogeneidade, $\bar{x} = \bar{y} = \frac{a}{4}$

$\therefore$ CM é $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4} \right)$ //

Exemplo. Calcular o volume do sólido delimitado pelo elipsóide $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a, b, c > 0)$.

Resolução.



Volume desejado $= 8 \, Vol(R)$

$$= 8 \iiint_R dV = (*)$$

$$R: \begin{cases} 0 \leq x \leq a \\ 0 \leq y \leq b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2} \\ 0 \leq z \leq c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \end{cases}$$

$$(*) = 8 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz \, dy \, dx$$

$$= \frac{8c}{b} \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2} = A} \sqrt{\underbrace{\frac{b^2}{a^2}\left(a^2-x^2\right)}_{= A^2} - y^2} \, dy \, dx = (**)$$

$$\int_0^A \sqrt{A^2-y^2} \, dy = \int_0^{\pi/2} \sqrt{A^2 - A^2\sin^2\theta} \, A\cos\theta \, d\theta$$

$y = A\sin\theta \quad dy = A\cos\theta \, d\theta$

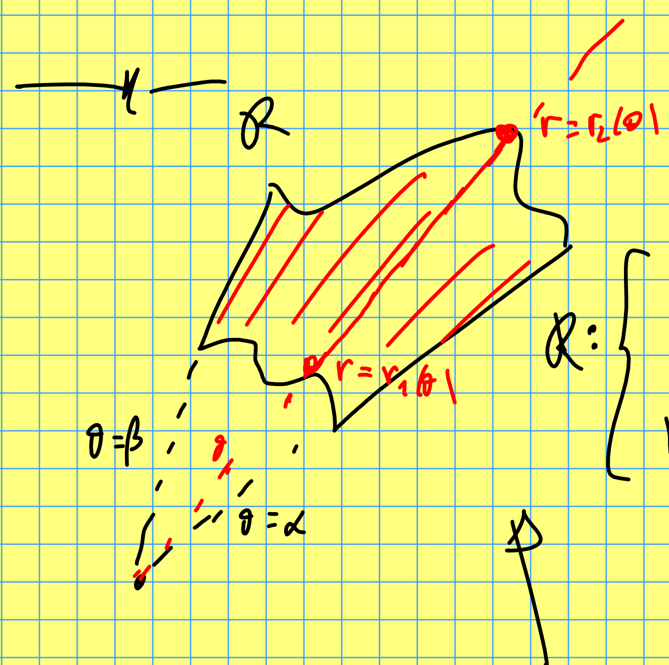
$$= A^2 \int_0^{\pi/2} \underbrace{\cos^2\theta}_{=\frac{1+\cos 2\theta}{2}}\, d\theta = A^2 \left[\frac{\theta}{2} + \cancel{\frac{\sin 2\theta}{4}} \right]_0^{\pi/2} = \frac{\pi A^2}{4}$$

$$(**) = \frac{8c}{b} \int_0^a \frac{\pi A^2}{4}\, dx = \frac{2\pi c}{b} \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{2\pi c b}{a^2} \int_0^a a^2 - x^2\, dx = \frac{2\pi b c}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_{x=0}^{x=a}$$

$$= \frac{2\pi b c}{a^2} \underbrace{\left(a^3 - \frac{a^3}{3} \right)}_{= \frac{2a^3}{3}} = \frac{4\pi}{3} abc \quad //$$

Caso especial: $a = b = c = R \Rightarrow Vol = \frac{4\pi R^3}{3}$ //

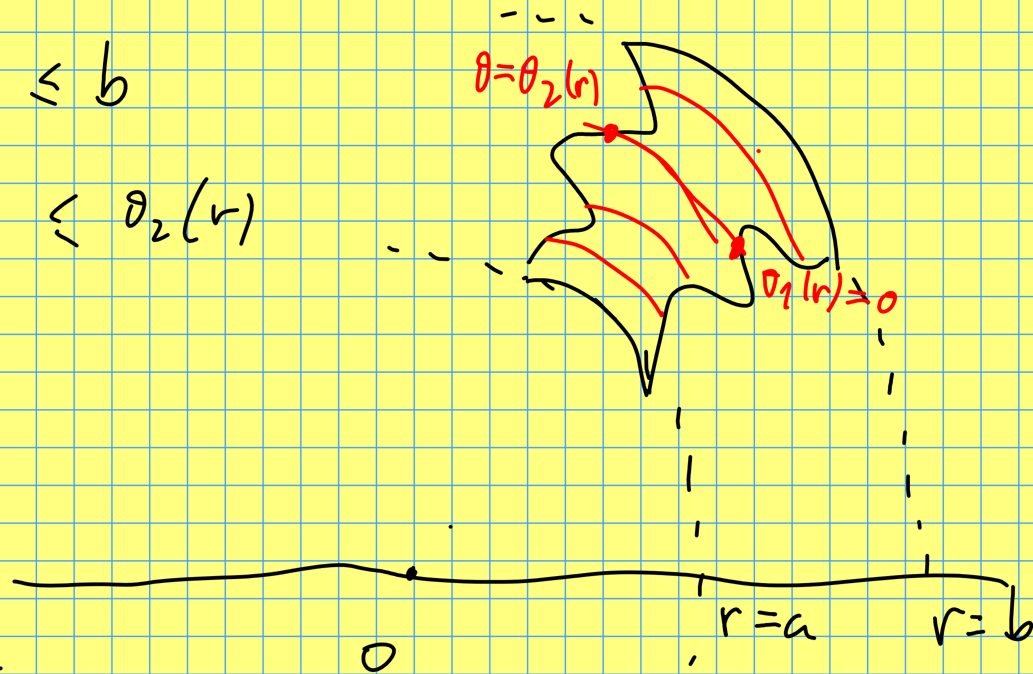


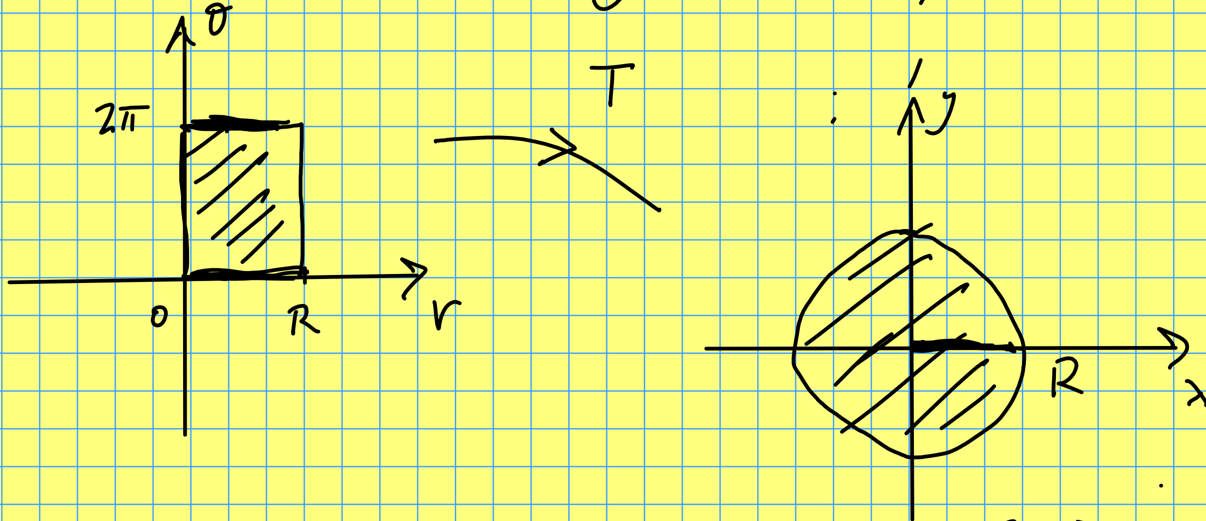
(§20.5)

$$R: \begin{cases} \alpha \le \theta \le \beta \\ r_1(\theta) \le r \le r_2(\theta) \end{cases}$$

$a \le r \le b$

$\theta_1(r) \le \theta \le \theta_2(r)$





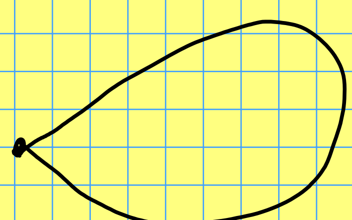
$T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$r^2 = x^2 + y^2 \le R^2$

ou

$\boxed{0 \le r \le R}$

θ livre



$\begin{cases} 0 \le r \le R \\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \end{cases}$