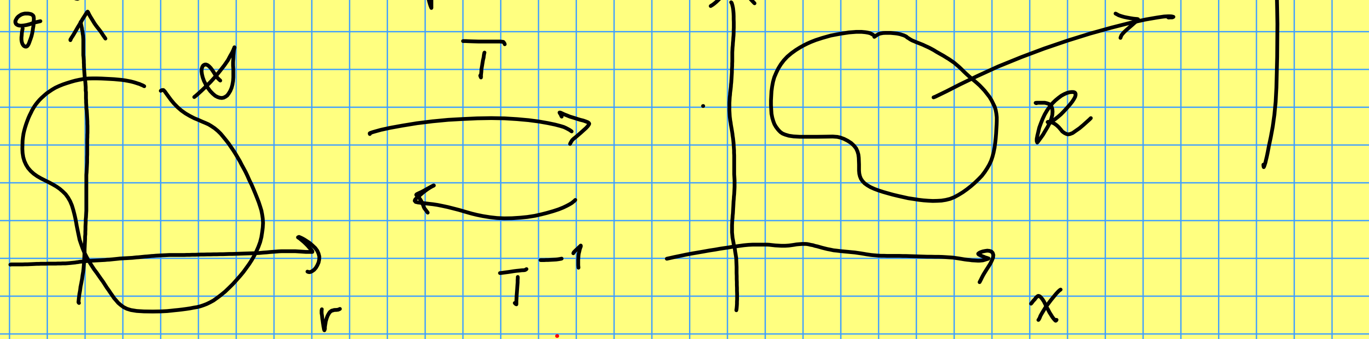
MUDANÇA DE VARIÁVEIS NA INTEGRAL DUPLA

30/08/21

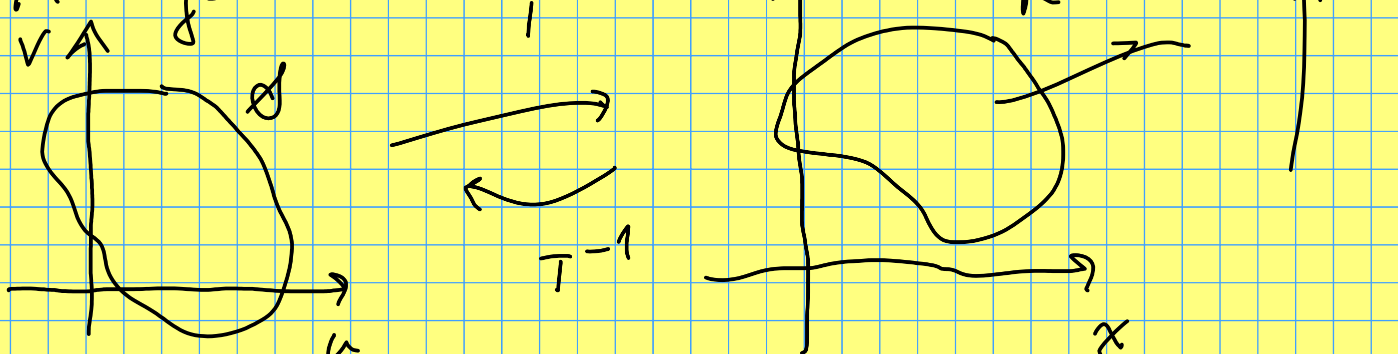
Lembrando as coords polares:



$$\iint_{R} f(x,y)\,\underbrace{dx\,dy}_{=dA} = \iint_{S} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}_{=dA}$$

$$T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Mais geralmente:

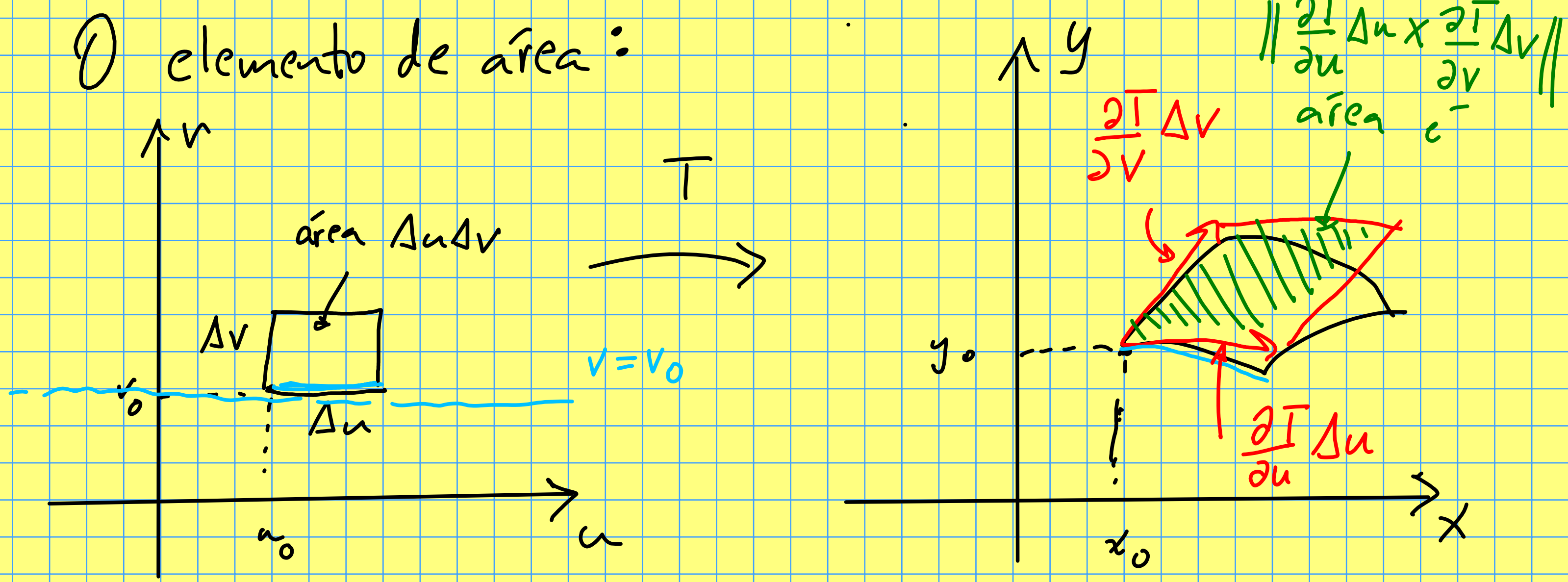


$$\iint_{R} f(x,y)\,\underbrace{dx\,dy}_{=dA} = \iint_{S} f(x(u,v), y(u,v))$$

dA em termos de u e v ?

$$T : \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

O elemento de área:



$$T(u_0,v_0) = (x_0,y_0)$$

$$\Big\| \underbrace{\frac{\partial T}{\partial u}}_{\text{vetor}} \underbrace{\Delta u}_{\text{escalar}} \times \underbrace{\frac{\partial T}{\partial v}}_{\text{vetor}} \underbrace{\Delta v}_{\text{escalar}} \Big\| = \Big\| \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \Big\| \Delta u \Delta v = (*)$$

$$T(u,v) = (x(u,v), y(u,v))$$

$$\frac{\partial T}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) \qquad \frac{\partial T}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right)$$

$$(*) = \left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{pmatrix} \right\| \Delta u \Delta v$$

$$= \left\| \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \vec{k} \right\| \Delta u \Delta v$$

$$= \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| \Delta u \Delta v$$

$$= \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \Delta u \Delta v$$

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} =: \frac{\partial (x, y)}{\partial (u, v)}$$

determinante Jacobiano

FÓRMULA DE MUDANÇA DE VARIÁVEIS NA INTEGRAL DUPLA:

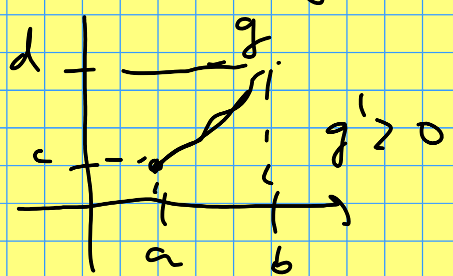
$$\boxed{\iint_R f(x,y)\, dx\, dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial (x,y)}{\partial (u,v)} \right| du\, dv}$$

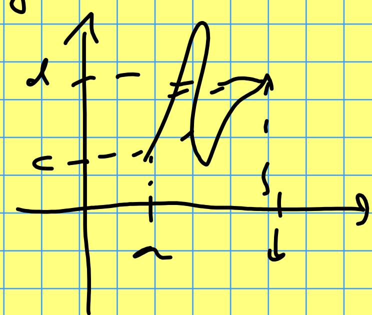
$$T: \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad T(S) = R$$

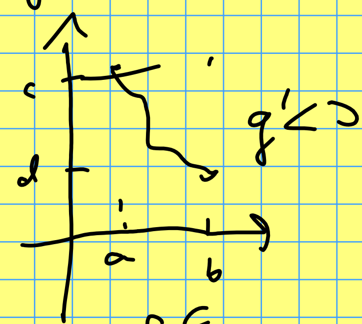
Lembrando, para efeito de comparação, que no caso de funções de uma variável:

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\,\underbrace{g'(t)\,dt}.$$

$$x = g(t) \qquad g(c) = a \qquad g(d) = b \qquad dx = g'(t)\,dt$$







$$\int_d^c = -\int_c^d$$

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Revisitando coords polares:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r \qquad \therefore \quad dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= r\,dr\,d\theta,$$

conforme a discussão na aula passada.

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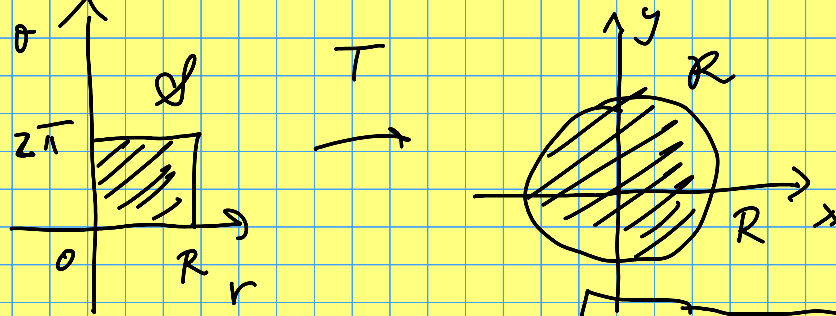
Exemplo



$$\boxed{0 \le r \le R}$$

Área do círculo de raio R $= \iint_{R} dA = \iint_{\mathcal{D}} r\,dr\,d\theta$

$$= \int_0^{2\pi}\left(\int_0^R r\,dr\right)d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=R} d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2}\,d\theta = \frac{R^2}{2}\left[\theta\right]_{\theta=0}^{\theta=2\pi} = 2\pi\,\frac{R^2}{2} = \pi R^2 \;//$$

—n—

Exemplo Calcular

$$\iint_{R} e^{\frac{x-y}{x+y}}\,dx\,dy$$

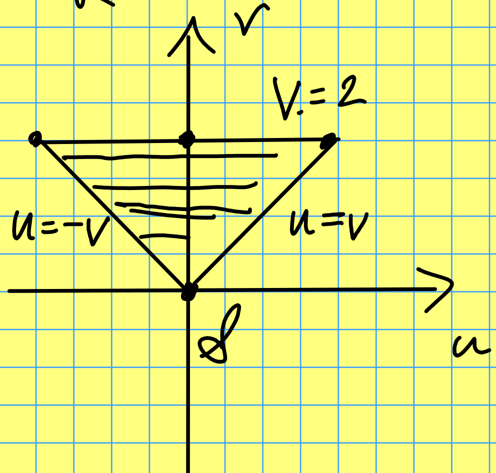
onde R é a região delimitada pelos eixos coordenados e pela reta $x+y=2$.

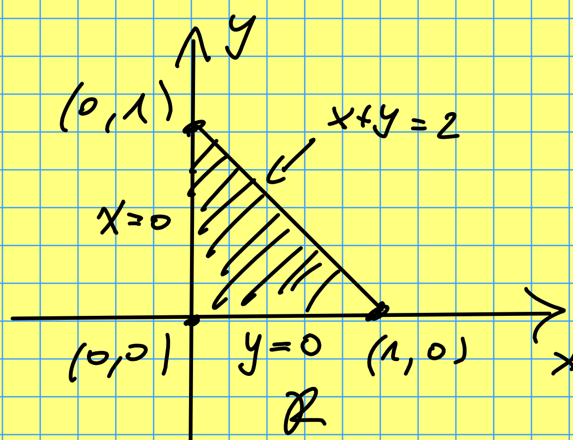
Resolução: $\frac{d}{dx} e^{x} = e^{x}$ $\qquad \frac{d}{dx} e^{\frac{x-y}{x+y}} = e^{\frac{x-y}{x+y}} \frac{d}{dx}\left(\frac{x-y}{x+y}\right)$

Mudança de variáveis: $\begin{cases} x-y = u \\ x+y = v \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(-u+v) \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & +1/2 \end{vmatrix} = +\frac{1}{4} + \frac{1}{4} = +\frac{1}{2}$$

$$\iint_{\mathcal{R}} e^{\frac{x-y}{x+y}} \, dx\,dy = \iint_{\mathcal{S}} e^{\frac{u}{v}} \left|\frac{1}{2}\right| du\,dv = (*)$$





$$\mathcal{S}: \begin{cases} -v \leq u \leq v \\ 0 \leq v \leq 2 \end{cases}$$

$$(*) = \frac{1}{2}\int_0^2 \int_{-v}^{v} e^{\frac{u}{v}} \, du\, dv = (**) \qquad \frac{\partial}{\partial u}\left(v e^{u/v}\right) = e^{u/v}$$

$$\frac{\partial}{\partial u}\left(e^{u/v}\right) = e^{u/v} \frac{\partial}{\partial u}\left(\frac{u}{v}\right) = e^{\frac{u}{v}} \frac{1}{v}$$

$$\int_{-v}^{v} e^{u/v}\,du = \left[v\, e^{u/v} \right]_{u=-v}^{u=v} = v\,(e - e^{-1})$$

$$(**) = \frac{1}{2}\int_0^2 v(e - e^{-1})\,dv = \frac{e-e^{-1}}{2}\ \frac{v^2}{2}\Big|_0^2 = e - e^{-1} \quad //.$$

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§20.9 versão em português:

Lista de problemas sobre mudança de variáveis na página do curso

www.ime.usp.br/~gorodski/teaching/mat2352/main.html

(não é para entregar).