Cálculo IV – Lista 8

1-) z_0 é um pólo de $f(z) \Leftrightarrow \lim\limits_{z\to z_0} |f(z)| = \infty$

A ordem de um pólo z_0 é m, quando para $f(z) = \dfrac{p(z)}{q(z)}$:

$p(z_0) \neq 0$ e $q(z_0) = q'(z_0) = \ldots = q^{(m-1)}(z_0) = 0$

mas $q^{(m)}(z_0) \neq 0$.

Para um pólo de ordem m, o resíduo A_{-1} é dado por:

$$A_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{(m-1)}}{dz^{(m-1)}}\left[(z-z_0)^m f(z)\right]$$

Para pólos de ordem 1:

$$A_{-1}(z_0) = \frac{p(z_0)}{q'(z_0)}$$

a) $f(z) = \dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z(z-2)} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z(z-2) = 0 \Rightarrow z_0 = 0$
$\quad z_1 = 2$

$\lim\limits_{z\to z_0} |f(z)| = \lim\limits_{z\to 0}\left|\dfrac{z+1}{z(z-2)}\right| = \infty \Rightarrow z_0 = 0$ é um pólo

$\lim\limits_{z\to z_1} |f(z)| = \lim\limits_{z\to 2}\left|\dfrac{z+1}{z(z-2)}\right| = \infty \Rightarrow z_1 = 2$ é um pólo

$p(z) = z+1$; $q(z) = z(z-2)$; $q'(z) = 2z-2$

$p(0) = 1 \neq 0$
$q(0) = 0$
$q'(0) = -2 \neq 0 \Rightarrow z_0 = 0$ é pólo de ordem 1

$p(2) = 3 \neq 0$
$q(2) = 0$
$q'(2) = 2 \neq 0 \Rightarrow z_1 = 2$ é pólo de ordem 1

$A_{-1}(0) = \lim\limits_{z\to 0} \cancel{z}\dfrac{(z+1)}{\cancel{z}(z-2)} = -\dfrac{1}{2}$

$A_{-1}(2) = \lim\limits_{z\to 2} \cancel{(z-2)}\dfrac{(z+1)}{z\cancel{(z-2)}} = \dfrac{3}{2}$

b) $f(z) = \tanh z = \dfrac{\operatorname{senh} z}{\cosh z} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$e^z + e^{-z} = 0 \Rightarrow z = i(2n+1)\dfrac{\pi}{2}$; $n \in \mathbb{Z}$

$\lim\limits_{z\to i(2n+1)\frac{\pi}{2}} \left|\dfrac{e^z - e^{-z}}{e^z + e^{-z}}\right| = \infty \Rightarrow z = i(2n+1)\dfrac{\pi}{2}$; $n \in \mathbb{Z}$ são pólos

$p(z) = e^z - e^{-z}$; $q(z) = e^z + e^{-z}$; $q'(z) = e^z - e^{-z}$

$p\left(i(2n+1)\frac{\pi}{2}\right) = \cos(2n+1)\frac{\pi}{2} + i\,\operatorname{sen}(2n+1)\frac{\pi}{2} - \cos(2n+1)\frac{\pi}{2} + i\,\operatorname{sen}(2n+1)\frac{\pi}{2} = 2i(-1)^n \neq 0$

$$q\left(i(2n+1)\frac{\pi}{2}\right) = \cos\left((2n+1)\frac{\pi}{2}\right) + i\,\text{sen}\left((2n+1)\frac{\pi}{2}\right) + \cos\left((2n+1)\frac{\pi}{2}\right) - i\,\text{sen}\left((2n+1)\frac{\pi}{2}\right) = 0$$

$q'\left(i(2n+1)\frac{\pi}{2}\right) = 2i(-1)^n \neq 0 \Rightarrow z = i(2n+1)\frac{\pi}{2}$ são pólos de ordem 1

$$A_{-1}\left(i(2n+1)\frac{\pi}{2}\right) = \frac{p\left(i(2n+1)\frac{\pi}{2}\right)}{q'\left(i(2n+1)\frac{\pi}{2}\right)} = \frac{2i(-1)^n}{2i(-1)^n} = 1 \quad ; n \in \mathbb{Z}$$

c) $f(z) = \dfrac{1-e^{2z}}{z^4}$

pontos singulares:

$z^4 = 0 \Rightarrow z_0 = 0$

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to 0}\left|\frac{1-e^{2z}}{z^4}\right| \overset{\text{L'Hospital}}{=} \lim_{z\to 0}\left|\frac{-2e^{2z}}{4z^3}\right| = \infty \Rightarrow z_0 = 0 \text{ é um pólo}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$e^{2z} = 1 + 2z + 2^2\frac{z^2}{2!} + 2^3\frac{z^3}{3!} + \dots$$

$$\frac{1-e^{2z}}{z^4} = -\frac{2}{z^3} - \frac{2^2}{2!}\frac{1}{z^2} - \frac{2^3}{3!}\frac{1}{z} - \frac{2^4}{4!} - \frac{2^5}{5!}z + \dots$$

$\Rightarrow z_0 = 0$ é um pólo de ordem 3 e o resíduo (coeficiente de $\frac{1}{z}$) é $A_{-1} = \dfrac{-2^3}{3!} = \dfrac{-4}{3}$

d) $f(z) = \dfrac{e^{2z}}{(z-1)^2} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z - 1 = 0 \Rightarrow z_0 = 1$

$$\lim_{z\to z_0}|f(z)| = \lim_{z\to 1}\left|\frac{e^{2z}}{(z-1)^2}\right| = \infty \Rightarrow z_0 = 1 \text{ é um pólo}$$

$p(z) = e^{2z}$; $q(z) = (z-1)^2$; $q'(z) = 2(z-1)$; $q''(z) = 2$

$p(1) = e^2 \neq 0$

$q(1) = 0$

$q'(1) = 0$

$q''(1) = 2 \neq 0 \Rightarrow z_0 = 1$ é pólo de ordem 2

$$A_{-1}(1) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{e^{2z}}{(z-1)^2}\right] = \lim_{z\to 1} 2e^{2z} = 2e^2$$

e) $f(z) = \dfrac{z}{\cos z} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$\cos z = 0 \Rightarrow z = (2n+1)\dfrac{\pi}{2}$; $n \in \mathbb{Z}$

$$\lim_{z \to (2n+1)\frac{\pi}{2}} \left| \frac{z}{\cos z} \right| = \infty \Rightarrow z = (2n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z} \text{ são pólos}$$

$p(z) = z \; ; \; q(z) = \cos z \; ; \; q'(z) = -\operatorname{sen} z$

$p\left((2n+1)\frac{\pi}{2}\right) = (2n+1)\frac{\pi}{2} \neq 0$

$q\left((2n+1)\frac{\pi}{2}\right) = 0$

$q'\left((2n+1)\frac{\pi}{2}\right) = -(-1)^n \neq 0 \Rightarrow z = (2n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z}$ são pólos de ordem 1

$$A_{-1}\left((2n+1)\frac{\pi}{2}\right) = \frac{p\left((2n+1)\frac{\pi}{2}\right)}{q'\left((2n+1)\frac{\pi}{2}\right)} = \frac{(2n+1)\frac{\pi}{2}}{(-1)^{n+1}} = (-1)^{n+1}(2n+1)\frac{\pi}{2}$$

f) $f(z) = \dfrac{e^z}{z^2+\pi^2} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z^2+\pi^2 = 0 \Rightarrow z^2 = -\pi^2 \Rightarrow z_0 = i\pi$

$z_1 = -i\pi$

$$\lim_{z \to i\pi} \left| \frac{e^z}{z^2+\pi^2} \right| = \infty \Rightarrow z_0 = i\pi \text{ é um pólo}$$

$$\lim_{z \to -i\pi} \left| \frac{e^z}{z^2+\pi^2} \right| = \infty \Rightarrow z_1 = -i\pi \text{ é um pólo}$$

$p(z) = e^z \; ; \; q(z) = z^2+\pi^2 \; ; \; q'(z) = 2z$

$p(i\pi) = e^{i\pi} = -1 \neq 0$

$q(i\pi) = 0$

$q'(i\pi) = 2i\pi \neq 0 \Rightarrow z_0 = i\pi$ é pólo de ordem 1

$p(-i\pi) = e^{-i\pi} = -1 \neq 0$

$q(-i\pi) = 0$

$q'(-i\pi) = -2i\pi \Rightarrow z_1 = -i\pi$ é pólo de ordem 1

$$A_{-1}(i\pi) = \lim_{z \to i\pi} \frac{(z-i\pi)\, e^z}{(z-i\pi)(z+i\pi)} = \frac{e^{i\pi}}{2i\pi} = \frac{-1}{2i\pi}$$

$$A_{-1}(-i\pi) = \lim_{z \to -i\pi} \frac{(z+i\pi)\, e^z}{(z-i\pi)(z+i\pi)} = \frac{e^{-i\pi}}{-2i\pi} = \frac{1}{2i\pi}$$

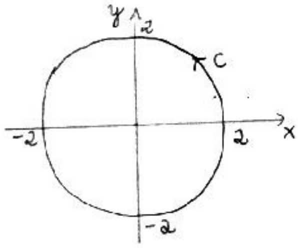
respostas dos itens g, h, i: (a resolução será feita em aula)

g) resíduo de $z_0 = 0$: $A_{-1}(0) = 0$

h) resíduo de $z_0 = 0$: $A_{-1}(0) = \frac{1}{6}$

i) A singularidade não é um pólo, mas sim um pólo essencial.
resíduo de $z_0 = 0$: $A_{-1}(0) = -\dfrac{1}{2}$

2-)



a) $f(z) = \tan z = \dfrac{\operatorname{sen} z}{\cos z} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$\cos z = 0 \Rightarrow z = (2n+1)\dfrac{\pi}{2} \;;\; n \in \mathbb{Z}$

somente $z_0 = \dfrac{\pi}{2}$ e $z_1 = -\dfrac{\pi}{2}$ são internos a C

$p(z) = \operatorname{sen} z \;;\; q(z) = \cos z \;;\; q'(z) = -\operatorname{sen} z$

$p\left(\frac{\pi}{2}\right) = 1 \neq 0$

$q\left(\frac{\pi}{2}\right) = 0$

$q'\left(\frac{\pi}{2}\right) = -1 \neq 0 \Rightarrow z_0 = \dfrac{\pi}{2}$ é pólo de ordem 1

$p\left(-\frac{\pi}{2}\right) = -1 \neq 0$

$q\left(-\frac{\pi}{2}\right) = 0$

$q'\left(-\frac{\pi}{2}\right) = 1 \neq 0 \Rightarrow z_1 = -\dfrac{\pi}{2}$ é pólo de ordem 1

$A_{-1}\left(\frac{\pi}{2}\right) = \dfrac{p\left(\frac{\pi}{2}\right)}{q'\left(\frac{\pi}{2}\right)} = \dfrac{1}{-1} = -1$

$A_{-1}\left(-\frac{\pi}{2}\right) = \dfrac{p\left(-\frac{\pi}{2}\right)}{q'\left(-\frac{\pi}{2}\right)} = \dfrac{-1}{1} = -1$

$\int_C \tan z \, dz = 2\pi i \left(A_{-1}\left(\frac{\pi}{2}\right) + A_{-1}\left(-\frac{\pi}{2}\right)\right) = 2\pi i(-1-1) = \underline{-4\pi i}$ //

b) $f(z) = \dfrac{1}{\operatorname{senh} 2z} = \dfrac{2}{e^{2z} - e^{-2z}} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z_0 = 0$

$2z = in\pi \Rightarrow z = in\dfrac{\pi}{2} \;;\; n \in \mathbb{Z}$

$\Rightarrow$ somente $z_0 = 0$, $z_1 = i\dfrac{\pi}{2}$, $z_2 = -i\dfrac{\pi}{2}$ são internos a C

$p(z) = 2 \;;\; q(z) = e^{2z} - e^{-2z} \;;\; q'(z) = 2\left(e^{2z} + e^{-2z}\right)$

$p(0) = 2 \neq 0$

$q(0) = 0$

$q'(0) = 2.2 = 4 \neq 0 \Rightarrow z_0 = 0$ é pólo de ordem 1

$A_{-1}(0) = \dfrac{p(0)}{q'(0)} = \dfrac{2}{4} = \dfrac{1}{2}$

$p\left(i\frac{\pi}{2}\right) = 2 \neq 0$

$q\left(i\frac{\pi}{2}\right) = 0$

$q'\left(i\frac{\pi}{2}\right) = -2.2 = -4 \Rightarrow z_1 = i\dfrac{\pi}{2}$ é pólo de ordem 1

$A_{-1}\left(i\frac{\pi}{2}\right) = \dfrac{p\left(i\frac{\pi}{2}\right)}{q'\left(i\frac{\pi}{2}\right)} = -\dfrac{2}{4} = -\dfrac{1}{2}$

$p\left(-i\frac{\pi}{2}\right) = 2 \neq 0$

$q\left(-i\frac{\pi}{2}\right) = 0$

$q'\left(-i\frac{\pi}{2}\right) = -4 \Rightarrow z_2 = -i\frac{\pi}{2}$ é pólo de ordem 1

$$A_{-1}\left(-i\frac{\pi}{2}\right) = \frac{p\left(-i\frac{\pi}{2}\right)}{q'\left(-i\frac{\pi}{2}\right)} = \frac{2}{-4} = -\frac{1}{2}$$

$$\int_C \frac{dz}{\operatorname{senh} 2z} = 2\pi i\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right) = -\pi i$$

c) $f(z) = \dfrac{\cosh \pi z}{z(z^2+1)} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z_0 = 0$

$z^2 + 1 = 0 \Rightarrow z = \pm i$ $\Big\}$ $z_0 = 0$, $z_1 = i$, $z_2 = -i$ são internos a C

$p(z) = \cosh \pi z$; $q(z) = z(z^2+1)$; $q'(z) = 3z^2 + 1$

$p(0) = 1 \neq 0$	$p(i) = -1 \neq 0$	$p(-i) = -1 \neq 0$
$q(0) = 0$	$q(i) = 0$	$q(-i) = 0$
$q'(0) = 1 \neq 0 \Rightarrow z_0 = 0$ é pólo de ordem 1	$q'(i) = -2 \neq 0 \Rightarrow z_1 = i$ é pólo de ordem 1	$q'(-i) = -2 \neq 0 \Rightarrow z_2 = -i$ é pólo de ordem 1

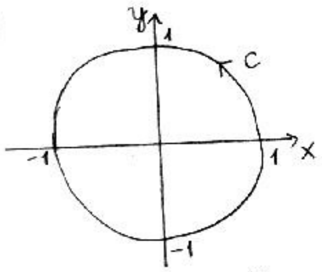
$$A_{-1}(0) = \frac{p(0)}{q'(0)} = 1$$

$$A_{-1}(i) = \frac{p(i)}{q'(i)} = \frac{-1}{-2} = \frac{1}{2}$$

$$A_{-1}(-i) = \frac{p(-i)}{q'(-i)} = \frac{-1}{-2} = \frac{1}{2}$$

$$\Rightarrow \int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz = 2\pi i\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 4\pi i$$

3-)



a) $f(z) = z^{-2} e^{-z} = \dfrac{1}{z^2 e^z} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$z_0 = 0$ é interno a C

$p(z) = 1$; $q(z) = z^2 e^z$; $q'(z) = 2z\, e^z + z^2 e^z$; $q''(z) = (2 + 2z)\, e^z + (2z + z^2)\, e^z$

$p(0) = 1 \neq 0$	$q'(0) = 0$
$q(0) = 0$	$q''(0) = 2 \neq 0 \Rightarrow z_0 = 0$ é pólo de ordem 2

$$A_{-1}(0) = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \frac{1}{z^2 e^z}\right] = \lim_{z\to 0} -e^{-z} = -1$$

$$\int_C z^{-2} e^{-z}\, dz = 2\pi i(-1) = \underline{-2\pi i}$$

b) $f(z) = z^{-1} \csc z = \dfrac{1}{z \operatorname{sen} z} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$\left.\begin{array}{l} z_0 = 0 \\ \operatorname{sen} z = 0 \Rightarrow z = n\pi \;;\; n \in \mathbb{Z} \end{array}\right\}$ somente $z_0 = 0$ é interno a C

$p(z) = 1$; $q(z) = z \operatorname{sen} z$; $q'(z) = \operatorname{sen} z + z\cos z$; $q''(z) = 2\cos z - z \operatorname{sen} z$

$\begin{array}{l|l} p(0) = 1 \neq 0 & q'(0) = 0 \\ q(0) = 0 & q''(0) = 2 \neq 0 \Rightarrow z_0 = 0 \text{ é pólo de ordem } 2 \end{array}$

$$A_{-1}(0) = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \frac{1}{z \operatorname{sen} z}\right] = \lim_{z\to 0} \frac{\operatorname{sen} z - z\cos z}{\operatorname{sen}^2 z} \underset{\text{L'Hospital}}{=} \lim_{z\to 0} \frac{\cos z - \cos z + z \operatorname{sen} z}{2 \operatorname{sen} z \cos z} = 0$$

$$\int_C z^{-1} \csc z\, dz = 2\pi i . 0 = \underline{0}$$

c) $f(z) = z^{-2} \csc z = \dfrac{1}{z^2 \operatorname{sen} z} = \dfrac{p(z)}{q(z)}$

pontos singulares:

$\left.\begin{array}{l} z_0 = 0 \\ \operatorname{sen} z = 0 \Rightarrow z = n\pi \;;\; n \in \mathbb{Z} \end{array}\right\}$ somente $z_0 = 0$ é interno a C

Para $z_0 = 0$:

$$\operatorname{sen} z = 0 + 1z + 0 - 1\frac{z^3}{3!} + 0 + 1\frac{z^5}{5!} + 0 - 1\frac{z^7}{7!} + \ldots$$

$$z^2 \operatorname{sen} z = z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \frac{z^9}{7!} + \ldots$$

$$f(z) = \frac{1}{z^2 \operatorname{sen} z} = \frac{1}{z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \frac{z^9}{7!} + \ldots}$$

$$\begin{array}{l|l} 1 & z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \frac{z^9}{7!} + \ldots \\ \hline -1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \ldots & z^{-3} + \frac{1}{3!} z^{-1} + \ldots \\ \hline \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \ldots & \end{array}$$

$$f(z) = \frac{1}{z^2 \operatorname{sen} z} = z^{-3} + \frac{1}{3!}\frac{1}{z} + \ldots \Rightarrow z_0 = 0 \text{ é pólo de ordem } 3$$

$$A_{-1}(0) = \frac{1}{3!}$$

$$\int_C z^{-2} \csc z\, dz = 2\pi i \frac{1}{3!} = \underline{\frac{\pi}{3} i}$$

3-) d) $f(z) = z\,e^{\frac{1}{z}}$

pontos singulares:

$z_0 = 0$ é interno a C

Para $z_0 = 0$:

$$e^{z} = 1 + 1z + 1\frac{z^2}{2!} + 1\frac{z^3}{3!} + \dots$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots$$

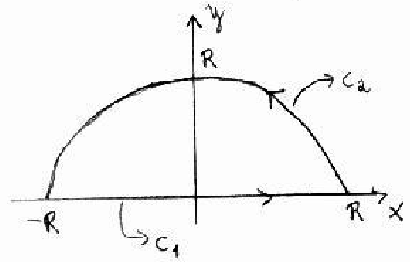
$$f(z) = z\,e^{\frac{1}{z}} = z + 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \dots$$ $\Rightarrow z_0 = 0$ é um pólo essencial

$$A_{-1}(0) = \frac{1}{2!} = \frac{1}{2}$$

$$\int_C z\,e^{\frac{1}{z}}\,dz = 2\pi i\,\frac{1}{2} = \pi i$$

4-) a) $$\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$

onde usou-se o fato de $f(x) = \dfrac{x^2}{(x^2+1)(x^2+4)}$ ser uma função par.



$C = C_1 + C_2$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty} \oint_C \frac{z^2\,dz}{(z^2+1)(z^2+4)}$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{p(z)}{q(z)}$$

pontos singulares:

$z^2+1 = 0 \Rightarrow z = \pm i$

$z^2+4 = 0 \Rightarrow z = \pm 2i$

somente $z_0 = i$, $z_1 = 2i$ são internos a C

$p(z) = z^2$; $q(z) = (z^2+1)(z^2+4)$; $q'(z) = 2z(z^2+4) + (z^2+1)2z$

$P(i) = -1 \neq 0$ | $q'(i) = 2i \cdot 3 = 6i \Rightarrow z_0 = i$ é pólo de ordem 1

$q(i) = 0$ | $A_{-1}(i) = \dfrac{p(i)}{q'(i)} = \dfrac{-1}{6i}$

$p(2i) = -4 \neq 0$

$q(2i) = 0$

$q'(2i) = -3.2.2i = -12i$

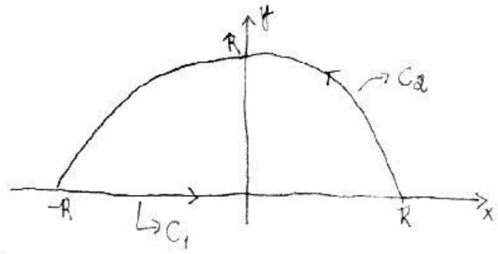
$z_1 = 2i$ é pólo de ordem 1

$$A_{-1}(2i) = \frac{p(2i)}{q'(2i)} = \frac{-4}{-12i} = \frac{1}{3i}$$

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\lim_{R\to\infty} \oint_C \frac{z^2\,dz}{(z^2+1)(z^2+4)} = \frac{1}{2}\cdot 2\pi i\left(\frac{1}{3i} - \frac{1}{6i}\right)$$

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$$

b) $$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \qquad \left(f(x) = \frac{1}{x^4+1} \text{ é par} \right)$$



$C = C_1 + C_2$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \lim_{R\to\infty} \oint_C \frac{dz}{z^4+1}$$

$$f(z) = \frac{1}{z^4+1} = \frac{p(z)}{q(z)}$$

pontos singulares:

$$z^4+1=0 \Rightarrow (z^2-i)(z^2+i)=0 \Rightarrow \left.\begin{array}{l} z = \pm\sqrt{i} = \pm e^{i\frac{\pi}{4}} \\ z = \pm\sqrt{-i} = \pm e^{i\frac{3\pi}{4}} \end{array}\right\} \text{somente } \begin{array}{l} z_0 = e^{i\frac{\pi}{4}} \\ z_1 = e^{i\frac{3\pi}{4}} \end{array} \text{ são internos a } C$$

$p(z) = 1$; $q(z) = z^4+1$; $q'(z) = 4z^3$

$p(e^{i\frac{\pi}{4}}) = 1 \neq 0$

$q(e^{i\frac{\pi}{4}}) = 0$

$q'(e^{i\frac{\pi}{4}}) = 4\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \neq 0 \Rightarrow z_0 = e^{i\frac{\pi}{4}}$ é pólo de ordem 1

$$A_{-1}(e^{i\frac{\pi}{4}}) = \frac{p(e^{i\frac{\pi}{4}})}{q'(e^{i\frac{\pi}{4}})} = \frac{\sqrt{2}}{4(-1+i)}$$

$p(e^{i\frac{3\pi}{4}}) = 1 \neq 0$

$q(e^{i\frac{3\pi}{4}}) = 0$

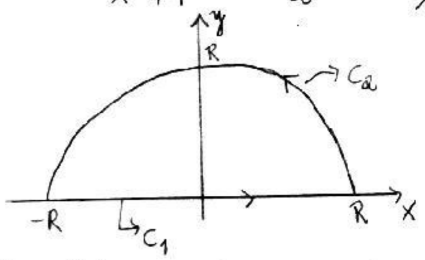
$q'(e^{i\frac{3\pi}{4}}) = 4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \neq 0 \Rightarrow z_0 = e^{i\frac{3\pi}{4}}$ é pólo de ordem 1

$$A_{-1}(e^{i\frac{3\pi}{4}}) = \frac{p(e^{i\frac{3\pi}{4}})}{q'(e^{i\frac{3\pi}{4}})} = \frac{\sqrt{2}}{4(1+i)}$$

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\lim_{R\to\infty}\oint_C \frac{dz}{z^4+1} = \frac{1}{2}\,2\pi i\,\frac{\sqrt{2}}{4}\left[\frac{1}{(i-1)} + \frac{1}{(i+1)}\right] = \frac{\sqrt{2}}{4}\pi i\,\frac{(i+1+i-1)}{-2}$$

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}$$

4-) c) $\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{x^6+1}$ $\left(f(x)=\frac{x^2}{x^6+1} \text{ é par}\right)$



$C = C_1 + C_2$

$$\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1} = \lim_{R\to\infty}\oint_C \frac{z^2\,dz}{z^6+1}$$

$$f(z)=\frac{z^2}{z^6+1}=\frac{p(z)}{q(z)}$$

pontos singulares:

$z^6+1=0 \Rightarrow (z^3+i)(z^3-i)=0 \Rightarrow z = i^{\frac{1}{3}} = \left[e^{i\left(\frac{\pi}{2}+2n\pi\right)}\right]^{\frac{1}{3}} = e^{i\left(\frac{\pi}{6}+n\frac{2\pi}{3}\right)}$

$z=-i^{\frac{1}{3}} = -e^{i\left(\frac{\pi}{6}+n\frac{2\pi}{3}\right)}$

somente $z_0=e^{i\frac{\pi}{6}}$
$z_1=e^{i\frac{5\pi}{6}}$ são internos a C
$z_3=-e^{i\frac{3\pi}{2}}$

$p(z)=z^2$; $q(z)=z^6+1$; $q'(z)=6z^5$

$p\left(e^{i\frac{\pi}{6}}\right)=e^{i\frac{\pi}{3}}\neq0$

$q\left(e^{i\frac{\pi}{6}}\right)=0$

$q'\left(e^{i\frac{\pi}{6}}\right)=6e^{i\frac{5\pi}{6}}\neq0 \Rightarrow z_0=e^{i\frac{\pi}{6}}$ é pólo de ordem 1

$A_{-1}\left(e^{i\frac{\pi}{6}}\right) = \frac{p\left(e^{i\frac{\pi}{6}}\right)}{q'\left(e^{i\frac{\pi}{6}}\right)} = \frac{e^{i\frac{\pi}{3}}}{6e^{i\frac{5\pi}{6}}} = \frac{e^{-i\frac{\pi}{2}}}{6} = \frac{-i}{6}$

$p\left(e^{i\frac{5\pi}{6}}\right)=e^{i\frac{5\pi}{3}}\neq0$

$q\left(e^{i\frac{5\pi}{6}}\right)=0$

$q'\left(e^{i\frac{5\pi}{6}}\right)=6e^{i\frac{25\pi}{6}}\neq0 \Rightarrow z_1=e^{i\frac{5\pi}{6}}$ é pólo de ordem 1

$A_{-1}\left(e^{i\frac{5\pi}{6}}\right) = \frac{p\left(e^{i\frac{5\pi}{6}}\right)}{q'\left(e^{i\frac{5\pi}{6}}\right)} = \frac{e^{i\frac{5\pi}{3}}}{6e^{i\frac{25\pi}{6}}} = \frac{e^{-i\frac{\pi}{2}}}{6} = \frac{-i}{6}$

$p\left(-e^{i\frac{3\pi}{2}}\right)=e^{i3\pi}\neq0$

$q\left(-e^{i\frac{3\pi}{2}}\right)=0$

$q'\left(-e^{i\frac{3\pi}{2}}\right)=-6e^{i\frac{15\pi}{2}}\neq0 \Rightarrow z_2=-e^{i\frac{3\pi}{2}}$ é pólo de ordem 1

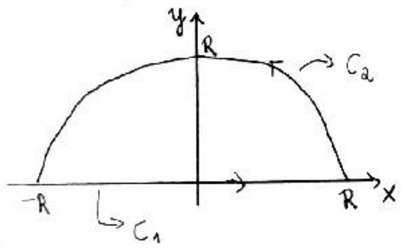
$A_{-1}\left(-e^{i\frac{3\pi}{2}}\right) = \frac{p\left(-e^{i\frac{3\pi}{2}}\right)}{q'\left(-e^{i\frac{3\pi}{2}}\right)} = \frac{e^{i3\pi}}{-6e^{i\frac{15\pi}{2}}} = \frac{-1}{-6(-i)}$

$A_{-1}\left(-e^{i\frac{3\pi}{2}}\right) = \frac{i}{6}$

$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\lim_{R\to\infty}\oint_C\frac{z^2\,dz}{z^6+1} = \frac{1}{\cancel{2}}\,\cancel{2}\pi i\left(\frac{-i}{6}-\frac{i}{6}+\frac{i}{6}\right)$$

$$\boxed{\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}}$$

4-) d) $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ $\quad \left(f(x) = \frac{1}{(x^2+1)^2} \text{ é par} \right)$



$C = C_1 + C_2$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \lim_{R\to\infty} \oint_C \frac{dz}{(z^2+1)^2}$$

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{p(z)}{q(z)}$$

pontos singulares:

$z^2+1=0 \Rightarrow z_0 = i$, $z_1 = -i$ $\Big\}$ somente $z_0 = i$ é interno a C

$p(z) = 1$; $q(z) = (z^2+1)^2$; $q'(z) = 2(z^2+1)2z$; $q''(z) = 4(z^2+1) + 4z.2z$

$p(i) = 1 \neq 0$
$q(i) = 0$
$q'(i) = 0$

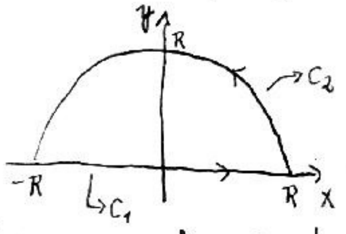
$q''(i) = -8 \neq 0 \Rightarrow z_0 = i$ é pólo de ordem 2

$$A_{-1}(i) = \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right] = \lim_{z\to i} \frac{d}{dz}\left[\frac{(z-i)^2}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i} \frac{-2}{(z+i)^3}$$

$$A_{-1}(i) = \frac{-2}{-2^3 i} = \frac{1}{4i}$$

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2} \lim_{R\to\infty} \oint_C \frac{dz}{(z^2+1)^2} = \frac{1}{2} 2\pi i \frac{1}{4i} = \frac{\pi}{4}$$

5-) a)



$C = C_1 + C_2$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{R\to\infty} \oint_C \frac{dz}{z^2+2z+2}$$

$$f(z) = \frac{1}{z^2+2z+2} = \frac{p(z)}{q(z)}$$

pontos singulares:

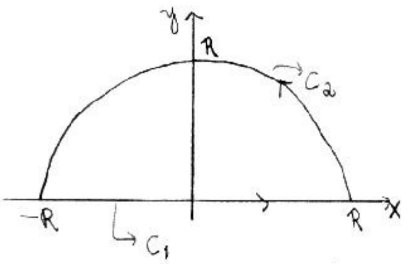
$$z^2+2z+2=0 \Rightarrow z = \frac{-2 \pm \sqrt{4-8}}{2} \Rightarrow \left. \begin{array}{l} z_0 = -1+i \\ z_1 = -1-i \end{array} \right\} \text{ somente } z_0 = -1+i \text{ é interno a } C$$

$$p(z) = 1 \;;\; q(z) = z^2+2z+2 \;;\; q'(z) = 2z+2$$

$$\begin{array}{l|l} p(-1+i) = 1 \neq 0 & q'(-1+i) = 2(-1+i)+2 = 2i \neq 0 \Rightarrow z_0 = -1+i \text{ é pólo de ordem } 1 \\ q(-1+i) = 0 & A_{-1}(-1+i) = \dfrac{p(-1+i)}{q'(-1+i)} = \dfrac{1}{2i} \end{array}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{R\to\infty} \oint_C \frac{dz}{z^2+2z+2} = 2\pi i \frac{1}{2i} = \pi$$

b)



$$C = C_1 + C_2$$

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = \lim_{R\to\infty} \oint_C \frac{z\,dz}{(z^2+1)(z^2+2z+2)}$$

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)} = \frac{p(z)}{q(z)}$$

pontos singulares:

$$\left. \begin{array}{l} z^2+1=0 \Rightarrow z = \pm i \\ z^2+2z+2 = 0 \Rightarrow z = -1 \pm i \end{array} \right\} \text{ somente } \begin{array}{l} z_0 = i \\ z_1 = -1+i \end{array} \text{ são internos a } C$$

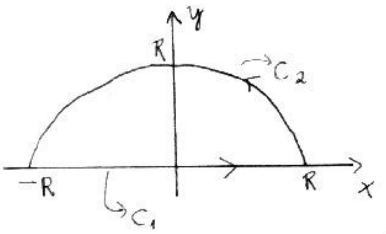
$$p(z) = z \;;\; q(z) = (z^2+1)(z^2+2z+2) \;;\; q'(z) = 2z(z^2+2z+2) + (z^2+1)(2z+2)$$

$$\begin{array}{l|l} p(i) = i \neq 0 & p(-1+i) = -1+i \neq 0 \\ q(i) = 0 & q(-1+i) = 0 \\ q'(i) = 2i(2i+1) \neq 0 \Rightarrow z_0 = i \text{ é pólo de ordem } 1 & q'(-1+i) = (-2i+1).2i \neq 0 \Rightarrow z_1 = -1+i \text{ é pólo de ordem } 1 \\ A_{-1}(i) = \dfrac{p(i)}{q'(i)} = \dfrac{i}{2i(2i+1)} = \dfrac{1}{2(2i+1)} & A_{-1}(-1+i) = \dfrac{p(-1+i)}{q'(-1+i)} = \dfrac{-1+i}{2i(1-2i)} \end{array}$$

$$\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)} = \lim_{R\to\infty}\oint_C \frac{z\,dz}{(z^2+1)(z^2+2z+2)} = 2\pi i\left[\frac{1}{2(2i+1)}+\frac{(-1+i)}{2i(1-2i)}\right] =$$

$$= \pi \cancel{i}\left[\frac{i(1-2i)+(2i+1)(-1+i)}{\cancel{i}(1+4)}\right] = \pi\,\frac{\cancel{i}+\cancel{2}-\cancel{2i}-\cancel{2}-1+\cancel{i}}{5} = \frac{-\pi}{5}$$

5-) c) $\displaystyle\int_0^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2}$ $\quad\left(f(x)=\dfrac{x^2}{(x^2+1)^2} \text{ é par}\right)$



$C = C_1 + C_2$

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \lim_{R\to\infty}\oint_C \frac{z^2\,dz}{(z^2+1)^2}$$

$$f(z) = \frac{z^2}{(z^2+1)^2} = \frac{p(z)}{q(z)}$$

pontos singulares:

$z^2+1=0 \Rightarrow \left.\begin{matrix} z_0 = i \\ z_1 = -i\end{matrix}\right\}$ somente $z_0 = i$ é interno a C

$p(z) = z^2$; $q(z) = (z^2+1)^2$; $q'(z) = 4z(z^2+1)$; $q''(z) = 4(z^2+1)+4z.2z$

$p(i) = -1 \neq 0$ $\quad|\quad q'(i) = 0$

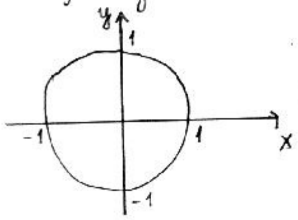
$q(i) = 0$ $\quad|\quad q''(i) = -8 \neq 0 \Rightarrow z_0 = i$ é pólo de ordem 2

$$A_{-1}(i) = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{z^2}{(z^2+1)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2 z^2}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i}\frac{2z(z+i)^2 - z^2.2(z+i)}{(z+i)^4} =$$

$$= \frac{2i(-4)+2.2i}{(2i)^4} = \frac{-4i}{16} = -\frac{i}{4}$$

$$\int_0^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \frac{1}{2}\lim_{R\to\infty}\oint_C\frac{z^2\,dz}{(z^2+1)^2} = \frac{1}{2}\,2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{4}$$

6-) a) Seja C a circunferência de raio 1, centrada na origem:



Sobre C: $z = e^{ix}$; $-\pi \leqslant x \leqslant \pi$

$$dz = i e^{ix} dx$$

$$dx = \frac{dz}{iz}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\int_{-\pi}^{\pi} \frac{\cos x \, dx}{5+4\cos x} = \oint_C \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)\frac{dz}{iz}}{5+4.\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_C \frac{(z^2+1) \quad dz}{\frac{2iz^2}{z}\left(5z+2z^2+2\right)} = \oint_C \frac{(z^2+1) \quad dz}{2iz\left(2z^2+5z+2\right)}$$

pontos singulares:

$z_0 = 0$

$2z^2+5z+2=0 \Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{4} = \frac{-5 \pm 3}{4} \Rightarrow z_1 = -\frac{1}{2}$ e $z_2 = -2$

somente $z_0 = 0$, $z_1 = -\frac{1}{2}$ são internos a C

$p(z) = z^2+1$; $q(z) = 2iz(2z^2+5z+2)$; $q'(z) = 2i(2z^2+5z+2)+2iz(4z+5)$

$p(0) = 1 \neq 0$

$q(0) = 0$

$q'(0) = 4i \neq 0 \Rightarrow z_0 = 0$ é pólo de ordem 1

$A_{-1}(0) = \frac{p(0)}{q'(0)} = \frac{1}{4i}$

$p\left(-\frac{1}{2}\right) = \frac{5}{4} \neq 0$

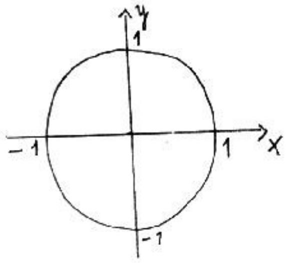
$q\left(-\frac{1}{2}\right) = 0$

$q'\left(-\frac{1}{2}\right) = -3i \neq 0 \Rightarrow z_1 = -\frac{1}{2}$ é pólo de ordem 1

$A_{-1}\left(-\frac{1}{2}\right) = \frac{p\left(-\frac{1}{2}\right)}{q'\left(-\frac{1}{2}\right)} = \frac{5}{4} \cdot \frac{1}{(-3i)} = -\frac{5}{12i}$

$$\int_{-\pi}^{\pi} \frac{\cos x \, dx}{5+4\cos x} = 2\pi \cancel{i}\left(\frac{1}{4\cancel{i}} - \frac{5}{12\cancel{i}}\right) = 2\pi\left(\frac{3-5}{12}\right) = \boxed{-\frac{\pi}{3}}$$

b)



Sobre C: $z = e^{ix}$; $-\pi \leqslant x \leqslant \pi$

$$dz = i e^{ix} dx$$

$$dx = \frac{dz}{iz}$$

$$\operatorname{sen} x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_{-\pi}^{\pi} \frac{dx}{1+\operatorname{sen}^2 x} = \oint_C \frac{\frac{dz}{iz}}{1+\left(\frac{z-\frac{1}{z}}{2i}\right)^2} = \oint_C \frac{dz}{iz\left[1-\frac{1}{4}\left(z^2-2+\frac{1}{z^2}\right)\right]} = \oint_C \frac{4z^2 dz}{i\cancel{z}\left(-z^4+6z^2-1\right)} =$$

$$= \oint_C \frac{4iz \, dz}{z^4-6z^2+1}$$

pontos singulares:

$z^4-6z^2+1=0 \Rightarrow z^2 = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} \Rightarrow z_0 = \sqrt{3+2\sqrt{2}}$, $z_1 = \sqrt{3-2\sqrt{2}}$, $z_3 = -\sqrt{3+2\sqrt{2}}$, $z_4 = -\sqrt{3-2\sqrt{2}}$

Somente $z_1 = \sqrt{3-2\sqrt{2}}$ são internos a C

$z_4 = -\sqrt{3-2\sqrt{2}}$

$p(z) = 4iz \; ; \; q(z) = z^4 - 6z^2 + 1 \; ; \; q'(z) = 4z^3 - 12z$

$p\left(\sqrt{3-2\sqrt{2}}\right) = 4i\sqrt{3-2\sqrt{2}}$

$q\left(\sqrt{3-2\sqrt{2}}\right) = 0$

$q'\left(\sqrt{3-2\sqrt{2}}\right) = 4\left(3-2\sqrt{2}\right)\sqrt{3-2\sqrt{2}} - 12\sqrt{3-2\sqrt{2}} = -8\sqrt{6-4\sqrt{2}} \neq 0 \Rightarrow z_1 = \sqrt{3-2\sqrt{2}}$ é pólo de ordem 1

$$A_{-1}\left(\sqrt{3-2\sqrt{2}}\right) = \frac{p\left(\sqrt{3-2\sqrt{2}}\right)}{q'\left(\sqrt{3-2\sqrt{2}}\right)} = \frac{4i\sqrt{3-2\sqrt{2}}}{-8\sqrt{6-4\sqrt{2}}} = \frac{-i}{2\sqrt{2}}$$

$p\left(-\sqrt{3-2\sqrt{2}}\right) = -4i\sqrt{3-2\sqrt{2}}$

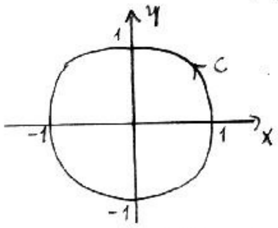
$q\left(-\sqrt{3-2\sqrt{2}}\right) = 0$

$q'\left(-\sqrt{3-2\sqrt{2}}\right) = -4\left(3-2\sqrt{2}\right)\sqrt{3-2\sqrt{2}} + 12\sqrt{3-2\sqrt{2}} = 8\sqrt{6-4\sqrt{2}} \neq 0 \Rightarrow z_4 = -\sqrt{3-2\sqrt{2}}$ é pólo de ordem 1

$$A_{-1}\left(-\sqrt{3-2\sqrt{2}}\right) = \frac{p\left(-\sqrt{3-2\sqrt{2}}\right)}{q'\left(-\sqrt{3-2\sqrt{2}}\right)} = \frac{-4i\sqrt{3-2\sqrt{2}}}{8\sqrt{6-4\sqrt{2}}} = \frac{-i}{2\sqrt{2}}$$

$$\int_{-\pi}^{\pi} \frac{dx}{1+\operatorname{sen}^2 x} = 2\pi i\left(\frac{-i}{2\sqrt{2}} \; \frac{-i}{2\sqrt{2}}\right) = \pi i\left(\frac{-2i}{\sqrt{2}}\right) = \underline{\pi\sqrt{2}}$$

6-) c)



Sobre C: $z = e^{ix}$; $0 \leqslant x \leqslant 2\pi$

$dz = i e^{ix} dx$

$dx = \frac{dz}{iz}$ $\qquad \cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\int_0^{2\pi} \frac{dx}{1+k\cos x} = \int_0^{2\pi} \frac{\frac{dz}{iz}}{1+\frac{k}{2}\left(z+\frac{1}{z}\right)} = \int_0^{2\pi} \frac{dz}{iz\left[\frac{2z+k(z^2+1)}{2z}\right]} = \int_0^{2\pi} \frac{2\,dz}{i\left(kz^2+2z+k\right)}$$

pontos singulares:

$kz^2 + 2z + k = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4-4k^2}}{2k} \Rightarrow z_1 = \frac{-1+\sqrt{1-k^2}}{k}$, $z_2 = \frac{-1-\sqrt{1-k^2}}{k}$ somente $z_1 = \frac{-1+\sqrt{1-k^2}}{k}$ é interno a C

$p(z) = 2 \; ; \; q(z) = i\left(kz^2+2z+k\right) \; ; \; q'(z) = i\left(2kz+2\right)$

$p\left(\frac{-1+\sqrt{1-k^2}}{k}\right) = 2 \neq 0$

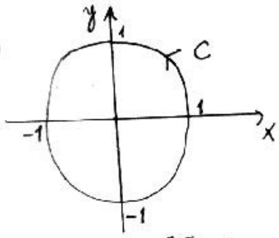
$q\left(\frac{-1+\sqrt{1-k^2}}{k}\right) = 0$

$q'\left(\frac{-1+\sqrt{1-k^2}}{k}\right) = 2i\left(-1+\sqrt{1-k^2}+1\right) = 2i\sqrt{1-k^2} \neq 0 \Rightarrow z_1 = \frac{-1+\sqrt{1-k^2}}{k}$ é pólo de ordem 1

$$A_{-1}\left(\frac{-1+\sqrt{1-k^2}}{k}\right) = \frac{p\left(\frac{-1+\sqrt{1-k^2}}{k}\right)}{q'\left(\frac{-1+\sqrt{1-k^2}}{k}\right)} = \frac{2}{2i\sqrt{1-k^2}} = \frac{1}{i\sqrt{1-k^2}}$$

$$\int_0^{2\pi}\frac{dx}{1+k\cos x} = 2\pi\cancel{i}\,\frac{1}{\cancel{i}\sqrt{1-k^2}} = \boxed{\frac{2\pi}{\sqrt{1-k^2}}}\ ;\ (k^2<1)$$

6-) d)



Sobre C: $z = e^{ix}$; $0 \leq x \leq 2\pi$

$dz = i\,e^{ix}dx$

$dx = \frac{dz}{iz}$ $\qquad \operatorname{sen} x = \frac{e^{ix}-e^{-ix}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$$\int_0^{2\pi}\frac{dx}{1+k\operatorname{sen}x} = \int_0^{2\pi}\frac{\frac{dz}{iz}}{1+\frac{k}{2i}\left(z-\frac{1}{z}\right)} = \int_0^{2\pi}\frac{dz}{iz\left[\frac{2iz+kz^2-k}{2iz}\right]} = \int_0^{2\pi}\frac{2\,dz}{kz^2+2iz-k}$$

pontos singulares:

$kz^2+2iz-k=0 \Rightarrow z = \frac{-2i\pm\sqrt{-4+4k^2}}{2k} \Rightarrow z_1 = \frac{-i+i\sqrt{1-k^2}}{k}$, $z_2 = \frac{-i-i\sqrt{1-k^2}}{k}$ $\Bigg\}$ somente $z_1 = \frac{-i+i\sqrt{1-k^2}}{k}$ é interno a C

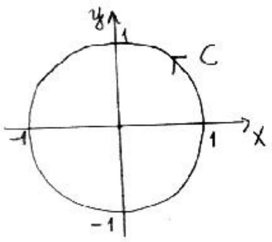
$p(z)=2$; $q(z) = kz^2+2iz-k$; $q'(z) = 2kz+2i$

$p(z_1) = 2 \neq 0$ $\quad\Big|\quad q'(z_1) = 2\left(-i+i\sqrt{1-k^2}\right)+2i = 2i\sqrt{1-k^2} \neq 0 \Rightarrow z_1$ é polo de ordem 1

$q(z_1) = 0$ $\quad\Big|\quad A_{-1}(z_1) = \frac{p(z_1)}{q'(z_1)} = \frac{2}{2i\sqrt{1-k^2}} = \frac{1}{i\sqrt{1-k^2}}$

$$\int_0^{2\pi}\frac{dx}{1+k\operatorname{sen}x} = 2\pi i\cdot\frac{1}{i\sqrt{1-k^2}} = \boxed{\frac{2\pi}{\sqrt{1-k^2}}}\ ;\ (k^2<1)$$

e)



Sobre C: $z = e^{ix}$; $0 \leq x \leq 2\pi$

$dz = i\,e^{ix}dx$

$dx = \frac{dz}{iz}$

$$\cos 3x = \frac{e^{3ix}+e^{-3ix}}{2} = \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$$

$$\cos 2x = \frac{e^{2ix}+e^{-2ix}}{2} = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$$

$$\int_0^{2\pi}\frac{\cos^2 3x\,dx}{5-4\cos 2x} = \int_0^{2\pi}\frac{\frac{1}{4}\left(z^6+2+\frac{1}{z^6}\right)\frac{dz}{iz}}{5-\frac{4}{2}\left(z^2+\frac{1}{z^2}\right)} = \int_0^{2\pi}\frac{\frac{z^{12}+2z^6+1}{z^6}}{4iz\left(\frac{5z^2-2z^4-2}{z^2}\right)}dz = \int_0^{2\pi}\frac{\left(z^{12}+2z^6+1\right)dz}{4iz^5\left(5z^2-2z^4-2\right)}$$

pontos singulares:

$z_0 = 0$

$-2z^4+5z^2-2=0 \Rightarrow z^2 = \frac{-5\pm\sqrt{25-16}}{-4} \Rightarrow z^2 = \frac{1}{2}$, $z^2 = 2$ $\Rightarrow z_1 = \frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}}$, $z_3 = \sqrt{2}$, $z_4 = -\sqrt{2}$

Somente $z_0 = 0$, $z_1 = \frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}}$ são internos a C

Para $z_0 = 0$:

$$
\begin{array}{l|l}
z^{12} + 2z^6 + 1 & -2z^9 + 5z^7 - 2z^5 \\
\cline{2-2}
-z^{12} + \frac{5}{2}z^{10} - z^8 & -\frac{1}{2}z^3 - \frac{5}{4}z - \frac{21}{8}z^{-1} - \dots \\
\cline{1-1}
\frac{5}{2}z^{10} - z^8 + 2z^6 + 1 & \\
-\frac{5}{2}z^{10} + \frac{25}{4}z^8 - \frac{5}{2}z^6 & \\
\cline{1-1}
\frac{21}{4}z^8 - \frac{1}{2}z^6 + 1 &
\end{array}
$$

$$f(z) = \frac{z^{12} + 2z^6 + 1}{4i z^5(-2z^4 + 5z^2 - 2)} = \frac{1}{4i}\left(-\frac{1}{2}z^3 - \frac{5}{4}z - \frac{21}{8}\frac{1}{z} - \dots\right)$$

$$A_{-1}(0) = \frac{1}{4i}\left(-\frac{21}{8}\right) = -\frac{21}{32i}$$

Para $z_1 = \frac{1}{\sqrt{2}}$ e $z_2 = \frac{-1}{\sqrt{2}}$:

$$p(z) = z^{12} + 2z^6 + 1 \; ; \; q(z) = 4i z^5(-2z^4 + 5z^2 - 2)$$

$$q'(z) = 20i z^4(-2z^4 + 5z^2 - 2) + 4i z^5(-8z^3 + 10z)$$

$$p\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2^6} + \frac{1}{2^2} + 1 = \frac{1 + 16 + 64}{64} = \frac{81}{64} \neq 0$$

$$q\left(\frac{1}{\sqrt{2}}\right) = 0$$

$$q'\left(\frac{1}{\sqrt{2}}\right) = \frac{\cancel{4}i}{\cancel{4}\sqrt{2}}\left(\frac{-\cancel{8}^{4}}{\cancel{2}\sqrt{2}} + \frac{10}{\sqrt{2}}\right) = 3i \neq 0$$

$\Rightarrow z_1 = \frac{1}{\sqrt{2}}$ é pólo de ordem 1

$$A_{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{p\left(\frac{1}{\sqrt{2}}\right)}{q'\left(\frac{1}{\sqrt{2}}\right)} = \frac{81}{64} \cdot \frac{1}{3i} = \frac{27}{64i}$$

$$p\left(\frac{-1}{\sqrt{2}}\right) = \frac{81}{64} \neq 0$$

$$q\left(\frac{-1}{\sqrt{2}}\right) = 0$$

$$q'\left(\frac{-1}{\sqrt{2}}\right) = \frac{-\cancel{4}i}{\cancel{4}\sqrt{2}}\left(\frac{+\cancel{8}^{4}}{\cancel{2}\sqrt{2}} - \frac{10}{\sqrt{2}}\right) = +3i \neq 0$$

$\Rightarrow z_2 = \frac{-1}{\sqrt{2}}$ é pólo de ordem 1

$$A_{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{p\left(\frac{-1}{\sqrt{2}}\right)}{q'\left(\frac{-1}{\sqrt{2}}\right)} = \frac{81}{64} \cdot \frac{1}{3i} = \frac{27}{64i}$$

$$\int_0^{2\pi} \frac{\cos^2 3x}{5 - 4\cos 2x}\,dx = 2\pi\cancel{i}\left(\frac{-21}{32\cancel{i}} + \frac{27}{64\cancel{i}} + \frac{27}{64\cancel{i}}\right) = 2\pi\left(\frac{6}{32}\right) = \frac{6}{16}\pi = \frac{3}{8}\pi$$