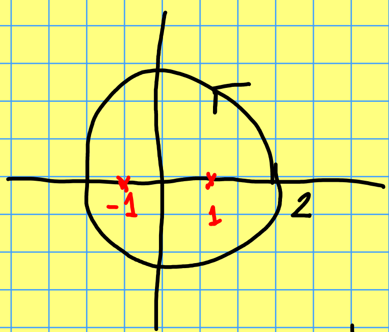
EXERCÍCIOS

22/11/22

1. $$\int_{|z|=2} \frac{2z}{z^2-1}\,dz = 2\pi i \left(\operatorname{Res}_1 f + \operatorname{Res}_{-1} f\right) = 2\pi i\,(1+1) = 4\pi i$$



Pontos singulares de f : $z^2 - 1 = 0 \iff z = \pm 1$

$$f(z) = \frac{2z}{z^2-1}$$

± 1 são pólos simples (ordem 1)

$$\frac{2z}{z^2-1} = \frac{1}{z-1}\ \underbrace{\frac{2z}{z+1}}_{=\varphi(z)} \qquad f(z) = \frac{\varphi(z)}{(z-1)^1}$$

φ é analítica em 1

$$\operatorname{Res}_1 f = \varphi(1) = 1$$

$$\frac{2z}{z^2-1} = \frac{1}{z+1} \cdot \underbrace{\frac{2z}{z-1}}_{=\psi(z)} \qquad f(z) = \frac{\psi(z)}{z+1}$$

ψ é analítica em -1

$$\operatorname{Res}_{-1} f = \psi(-1) = \frac{-2}{-2} = 1$$

Em geral, se $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$: z_0 é polo de ordem m de f

φ analítica em z_0

$m \geq 1$

$$\operatorname{Res}_{z_0} f = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

2.ª resolução:

$$\frac{2z}{z^2-1} = \frac{1}{z+1} + \frac{1}{z-1}$$

$$\int_{|z|=2} \frac{2z}{z^2-1}\,dz = \int_{|z|=2} \frac{1\,dz}{z+1} + \int_{|z|=2} \frac{1\,dz}{z-1}$$

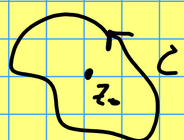
$$= 2\pi i\Big(\underbrace{f(-1)}_{=1} + \underbrace{f(1)}_{=1}\Big) = 4\pi i \;//$$

$f(z)=1$, $n=1$, $z_0=-1$

$f(z)=1$, $z_0=1$, $n=1$

Fórmula integral de Cauchy:

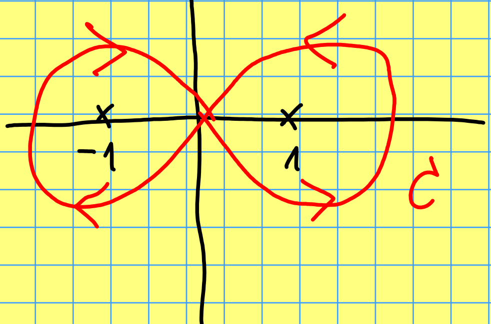
$$\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)$$

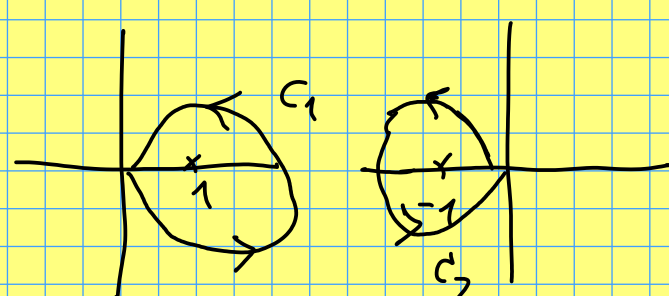


f analítica em $C \cup \operatorname{int}(C)$, $z_0 \in \operatorname{int}(C)$

2.
$$\int_C \underbrace{\frac{2z}{z^2-1}}_{f(z)}\,dz = \int_{C_1} \frac{2z}{z^2-1}\,dz - \int_{C_2} \frac{2z}{z^2-1}\,dz = (*)$$

$C = C_1 - C_2$ Ptos sings: ± 1





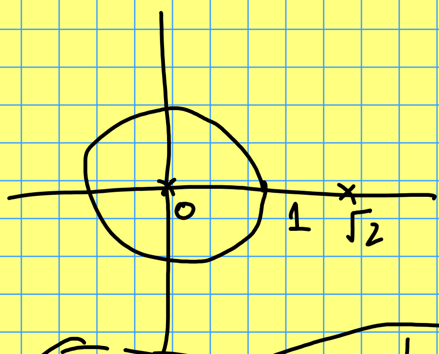
$1 \in \operatorname{int}(C_1)$
$-1 \in \operatorname{ext}(C_1)$

$$(*) = 2\pi i \operatorname{Res}_1 f - 2\pi i \operatorname{Res}_{-1} f$$

$$= 2\pi i \cdot 1 - 2\pi i \cdot 1 = 0 \;//$$

3. $I = \displaystyle\int_{|z|=1} \frac{\tanh(z^2)}{z^5(2-z^2)}\,dz = \ ?$

$$f(z) = \frac{\tanh(z^2)}{z^5(2-z^2)}$$ singularidades : $0, \pm\sqrt{2}$



$I = 2\pi i \operatorname{Res}_0 f$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$\sinh 0 = 0$

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$\cosh 0 = 1 \neq 0$

0 é um zero de $\tanh z$ de ordem $\boxed{1}$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = z^1\overbrace{\left(1 + \frac{z^2}{3!} + \dots\right)}^{\text{analítica em } 0,\ \neq 0 \text{ em } 0}$$

0 é um zero de ordem 1 de $\sinh z$

$$\boxed{\begin{array}{l} f(z) = (z-z_0)^k g(z),\ g \text{ analítica em } z_0,\ g(z_0) \neq 0 \\ \iff z_0 \text{ é zero de ordem } k \text{ de } f \end{array}}$$

$$\boxed{\iff f(z_0) = \dots = f^{(k-1)}(z_0)}$$

$$\tanh z = \sinh z \cdot \cosh z$$

$$= \underbrace{z\left(1 + \frac{z^2}{3!} + \dots\right)}\ \underbrace{\frac{1}{\cosh z}} \Rightarrow 0 \text{ é zero de ordem } 1 \text{ de } \tanh z$$

$z^{\perp}$ analítica em 0, $\neq 0$ em 0 — analítica em 0, $\neq 0$ em 0

$\tanh(z^2)$: 0 é zero de ordem $\boxed{2}$

$$\tanh(z^2) = z^2 \underbrace{\left(1 + \frac{z^4}{3!} + \dots\right)}_{\text{anal. em } 0,\ \neq 0 \text{ em } 0} \underbrace{\frac{1}{\cosh(z^2)}}_{\text{anal em } 0,\ \neq 0 \text{ em } 0}$$

$$f(z) = \frac{\tanh(z^2)}{z^5(2-z^2)} \begin{matrix} \leftarrow 2 \\ \leftarrow 5 \end{matrix}$$

$z^5(2-z^2)$: 0 é zero de ordem 5

0 é um pólo triplo de f

NUM 2
DEN 5

$$I = 2\pi i \operatorname{Res}_0 f$$

$$f(z) = \frac{1}{z^3} \underbrace{\frac{\tanh(z^2)}{z^2(2-z^2)}}$$ tem sing. removível em 0

$$\lim_{z\to 0} \frac{\tanh(z^2)}{z^2(2-z^2)} = \lim_{z\to 0} \frac{\cancel{z^2}\left(1+\frac{z^4}{3!}+\dots\right)\frac{1}{\cosh(z^2)}}{\cancel{z^2}(2-z^2)}$$

$$= \frac{1 \cdot 1}{2 - 0} = \frac{1}{2} \neq 0$$

$$\varphi(z) = \begin{cases} \dfrac{\tanh(z^2)}{z^2(2-z^2)}, & \text{se } z \neq 0 \\ \dfrac{1}{2}, & \text{se } z = 0 \end{cases}$$ é analítica em 0

$$\varphi(0) = \frac{1}{2} \neq 0$$

$f(z) = \frac{1}{z^3} \varphi(z)$ tem polo triplo em 0

$$\operatorname{Res}_0 f = \frac{\varphi''(0)}{2!} = \frac{1}{4} \quad \therefore \quad I = \frac{2\pi i}{4} = \frac{\pi i}{2} \,//$$

$$\varphi(z) = \left(1 + \frac{z^4}{3!} + \dots\right) \frac{1}{\cosh(z^2)(2-z^2)}$$

$$\varphi'(z) = \left(\frac{4z^3}{3!} + \dots\right) \cdot \frac{1}{\cosh(z^2)(2-z^2)} - \left(1 + \frac{z^4}{3!} + \dots\right) \frac{2z\sinh(z^2)(2-z^2) + \cosh(z^2)(-2z)}{\cosh^2(z^2)(2-z^2)^2}$$

$$\varphi''(0) = 0 - 1 \cdot \frac{4 \cdot (-2) - 0 \cdot \dots}{16}$$

$$\left.\frac{d}{dz}\right|_{z=0} \left\{ -2z \cosh(z^2) + 2z \sinh(z^2)(2-z^2) \right\} = -2 \cdot 1 + 2 \cdot 0 = -2$$

$$\varphi''(0) = \frac{1}{2}$$

$\tanh z$ Série de Taylor em 0?

$\sinh z = z + \frac{z^3}{3!} + \dots$

$$f(z) = \frac{\tanh(z^2)}{z^5(2-z^2)}$$

$$= \frac{1}{z^5} \cdot \frac{1}{2-z^2} \cdot \sinh(z^2) \cdot \frac{1}{\cosh(z^2)}$$

$$= \underbrace{\frac{1}{z^5}\left(z^2 + \frac{z^6}{3!} + \dots\right)}_{\text{afeta a singularidade em } 0} \underbrace{\frac{1}{2-z^2} \cdot \frac{1}{\cosh(z^2)}}_{\text{analítica em } 0,\ e \neq 0 \text{ em } 0}$$

é anal em 0

$$= \frac{1}{z^3}\left(1 + \frac{z^4}{3!} + \dots\right) \frac{1}{2-z^2} \frac{1}{\cosh(z^2)}$$

$$= \left(\frac{1}{z^3} + \frac{z}{3!} + \dots\right) \underbrace{\boxed{\frac{1}{2-z^2} \cdot \frac{1}{\cosh(z^2)}}}$$

$$= \sum_{n=0}^{\infty} a_n z^n, \quad a_0 \neq 0$$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

$$= \frac{a_0}{z^3} + \frac{a_1}{z^2} + \boxed{\frac{a_2}{z}} + a_3 + \left(a_4 + \frac{a_0}{3!}\right) z + \dots$$

$$\operatorname{Res}_0 f = a_2 = \frac{g''(0)}{2} = \frac{1}{4}$$

$$g(z) = \frac{1}{(2-z^2)\cosh(z^2)}$$

$$g'(z) = \frac{+\left(+2z\cosh(z^2) + (2-z^2)\sinh(z^2)\,2z\right)}{(2-z^2)^2\cosh^2(z^2)}$$

$$g'(0) = \frac{2\cdot 1\cdot 4 - 0\cdots}{16}$$

$$= \frac{1}{2}$$

$$\frac{1}{2-z^2} = \frac{1}{2}\cdot\frac{1}{1-\frac{z^2}{2}} = \frac{1}{2}\left(1+\frac{z^2}{2}+\frac{z^4}{4}+\frac{z^6}{8}+\cdots\right)$$

$$\left|\frac{z^2}{2}\right| < 1 \qquad |z| < \sqrt{2}$$

$$\cosh z = 1 + \frac{z^2}{2} + \frac{z^4}{4!} + \cdots \qquad |z| < \infty$$

$$\cosh(z^2) = 1 + \frac{z^4}{2} + \frac{z^8}{4!} + \cdots \qquad |z| < \infty$$

$$\begin{array}{l|l} 1 & 1 + \frac{z^4}{2} + \cdots \\ \hline -1 - \frac{z^4}{2} & 1 - \frac{z^4}{2} + \cdots \\ \hline -\frac{z^4}{2} & \\ +\frac{z^4}{2} & \\ \hline 0 & \end{array}$$

$$\frac{1}{\cosh(z^2)} = 1 - \frac{z^4}{2} + \cdots \qquad |z| < \infty$$

$$g(z) = \frac{1}{2-z^2} \cdot \frac{1}{\cosh(z^2)}$$

$$= \frac{1}{2}\left(1 + \frac{z^2}{2} + \frac{z^4}{4} + \dots\right)\left(1 - \frac{z^4}{2} + \dots\right)$$

$|z| < \sqrt{2}$

$$= \frac{1}{2} + \frac{1}{4} z^2 + \dots$$

$g(0) = \frac{1}{2}$ $\quad g'(0) = 0$ $\quad \frac{g''(0)}{2} = \frac{1}{4} \Rightarrow g''(0) = \frac{1}{2}$