MAT220 - Cálculo IV Lista 2

1-) $z = x + iy$ $\qquad f(z) = u + iv$

a) $f(z) = z^2 - 4z + 2 = (x+iy)^2 - 4(x+iy) + 2 = x^2 + 2xyi - y^2 - 4x - 4yi + 2$

$f(z) = (x^2 - y^2 - 4x + 2) + i(2xy - 4y)$

$u(x,y) = x^2 - y^2 - 4x + 2$

$v(x,y) = 2xy - 4y$

b) $f(z) = \dfrac{2}{z-7} = \dfrac{2}{(x-7)+iy} = \dfrac{2(x-7-iy)}{(x-7)^2+y^2} = \dfrac{2x-14-2iy}{(x-7)^2+y^2}$

$u(x,y) = \dfrac{2x-14}{(x-7)^2+y^2}$ $\qquad v(x,y) = \dfrac{-2y}{(x-7)^2+y^2}$

c) $f(z) = e^z(z-i) = e^x e^{iy}(x+iy-i) = e^x(\cos y + i \operatorname{sen} y)[x + i(y-1)]$

$f(z) = e^x[x\cos y - (y-1)\operatorname{sen} y + i x \operatorname{sen} y + i(y-1)\cos y]$

$u(x,y) = e^x[x\cos y - (y-1)\operatorname{sen} y]$ $\qquad v(x,y) = e^x[x\operatorname{sen} y + (y-1)\cos y]$

2-) a) $\lim\limits_{z \to -i} (z^2 - 3z) = -1 + 3i$

b) $\lim\limits_{z \to 2i} \dfrac{9}{z^2+4} = \lim\limits_{z \to 2i} \dfrac{9}{(z+2i)(z-2i)} = \infty$

c) $\lim\limits_{z \to \infty} \dfrac{z+2}{z^2-3} = \lim\limits_{z \to \infty} \dfrac{z\left(1+\overset{\to 0}{\frac{2}{z}}\right)}{z^2\left(1-\underset{\to 0}{\frac{3}{z^2}}\right)} = 0$

d) $\lim\limits_{z \to \infty} \dfrac{4z+5}{2z-1} = \lim\limits_{z \to \infty} \dfrac{z\left(4+\overset{0}{\frac{5}{z}}\right)}{z\left(2-\underset{\to 0}{\frac{1}{z}}\right)} = \dfrac{4}{2} = 2$

3-) Fórmulas de Diferenciação:

(1) $\dfrac{d}{dz}(c) = 0$ $\quad$; c = constante complexa

(2) $\dfrac{d}{dz}(cw) = c\dfrac{dw}{dz}$ $\quad$; $w(z)$ = função complexa

(3) $\dfrac{d}{dz}(z^n) = n z^{n-1}$

$(4)\ \frac{d}{dz}(w_1+w_2) = \frac{dw_1}{dz} + \frac{dw_2}{dz}$

w_1 e w_2 são duas funções cujas derivadas existem

$(5)\ \frac{d}{dz}(w_1 w_2) = w_1\frac{dw_2}{dz} + w_2\frac{dw_1}{dz}$

$(6)\ \frac{d}{dz}\left(\frac{w_1}{w_2}\right) = \frac{w_2\left(dw_1/dz\right) - w_1\left(dw_2/dz\right)}{w_2^2}$; $w_2 \neq 0$

$(7)\ \frac{d}{dz}\left\{w_1\left[w_2(z)\right]\right\} = \frac{dw_1}{dw_2}\frac{dw_2}{dz}$

a) $f(z) = 3z^2 - 4z + 1$

$f'(z) = 6z - 4$

b) $f(z) = (2+z^2)^7$

$f'(z) = 7(2+z^2)^6 \cdot 2z = 14z(2+z^2)^6$

c) $f(z) = \frac{z-1}{2z+1}$

$f'(z) = \frac{1.(2z+1)-(z-1)2}{(2z+1)^2} = \frac{2z+1-2z+2}{(2z+1)^2} = \frac{3}{(2z+1)^2}$; $z \neq -\frac{1}{2}$

4-) $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\bar{z}+\overline{\Delta z}-\bar{z}}{\Delta z} = \lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$

$$f'(z) = \lim_{\Delta z\to 0}\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

tomando o caminho $x = y$:

$$f'(z) = \lim_{\Delta x\to 0}\frac{\Delta x - i\Delta x}{\Delta x + i\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(1-i)}{\Delta x(1+i)} = \frac{1+i}{1-i}$$

tomando o caminho $y = 2x$:

$$f'(z) = \lim_{\Delta x\to 0}\frac{\Delta x - i2\Delta x}{\Delta x + i2\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(1-2i)}{\Delta x(1+2i)} = \frac{1-2i}{1+2i}$$

$\therefore$ $f'(z)$ não existe, pois depende da maneira em que $\Delta z \to 0$

5-) a) $f(z) = \bar{z} = x - iy$

$u(x,y) = x \qquad v(x,y) = -y$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = 1 \\ \frac{\partial v}{\partial y} = -1 \end{array}\right\} \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \; ; \forall z \in \mathbb{C}$$

$\Rightarrow$ As Condições de Cauchy-Riemann não são satisfeitas em nenhum $z \in \mathbb{C}$

$\Downarrow$

$f(z)$ não é derivável em nenhum ponto

b) $f(z) = \mathrm{Im}(z) = \mathrm{Im}(x+iy) = y$

$u(x,y) = y \qquad v(x,y) = 0$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial v}{\partial y} = 0 \end{array}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \forall z \in \mathbb{C}$$

$$\left.\begin{array}{l} \frac{\partial v}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = 1 \end{array}\right\} \Rightarrow \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y} \; ; \forall z \in \mathbb{C}$$

$\Rightarrow$ As Condições de Cauchy-Riemann não são satisfeitas em nenhum $z \in \mathbb{C}$

$\Downarrow$

$f(z)$ não é derivável em nenhum ponto

c) $f(z) = 2x + ixy^2$

$u(x,y) = 2x \qquad v(x,y) = xy^2$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = 2 \\ \frac{\partial v}{\partial y} = 2xy \end{array}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \text{p/ } \forall z \text{ tal que } xy = 1$$

$$\left.\begin{array}{l} \frac{\partial v}{\partial x} = y^2 \\ \frac{\partial u}{\partial y} = 0 \end{array}\right\} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \; ; \text{p/ } \forall z \text{ tal que } y = 0$$

$\Rightarrow$ As Condições de Cauchy-Riemann não são satisfeitas em nenhum $z \in \mathbb{C}$

$\Downarrow$

$f(z)$ não é derivável em nenhum ponto

d) $f(z) = e^{\bar{z}} = e^x \cdot e^{-iy} = e^x(\cos y - i \,\mathrm{sen}\, y)$

$u(x,y) = e^x \cos y \qquad v(x,y) = -e^x \,\mathrm{sen}\, y$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = e^x \cos y \\ \frac{\partial v}{\partial y} = -e^x \cos y \end{array}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \text{p/ } \forall z \text{ tal que } y = (2n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z}$$

$$\left.\begin{array}{l} \frac{\partial v}{\partial x} = -e^x \,\mathrm{sen}\, y \\ \frac{\partial u}{\partial y} = -e^x \,\mathrm{sen}\, y \end{array}\right\} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \; ; \text{p/ } \forall z \text{ tal que } y = n\pi \; ; n \in \mathbb{Z}$$

$\Rightarrow$ As Condições de Cauchy-Riemann não são satisfeitas em nenhum $z \in \mathbb{C}$

$\Downarrow$

$f(z)$ não é derivável em nenhum ponto

6-) $f(z) = z^3 = x^3 - 3xy^2 + 3ix^2y - iy^3$

a)

$u(x,y) = x^3 - 3xy^2 \qquad v(x,y) = 3x^2y - y^3$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \\ \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 \end{array}\right\} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial u}{\partial x}$ e $\dfrac{\partial v}{\partial y}$ são contínuas em todo o plano

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial x} = 6xy \\ \dfrac{\partial u}{\partial y} = -6xy \end{array}\right\} \Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial v}{\partial x}$ e $\dfrac{\partial u}{\partial y}$ são contínuas em todo o plano

$\Rightarrow f(z)$ é derivável em todos os pontos

$f'(z) = 3z^2$

b) $f(z) = e^{-x}(\cos y - i \operatorname{sen} y)$

$u(x,y) = e^{-x}\cos y \qquad v(x,y) = -e^{-x}\operatorname{sen} y$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = -e^{-x}\cos y \\ \dfrac{\partial v}{\partial y} = -e^{-x}\cos y \end{array}\right\} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial u}{\partial x}$ e $\dfrac{\partial v}{\partial y}$ são contínuas em todo o plano

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial x} = e^{-x}\operatorname{sen} y \\ \dfrac{\partial u}{\partial y} = -e^{-x}\operatorname{sen} y \end{array}\right\} \Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial v}{\partial x}$ e $\dfrac{\partial u}{\partial y}$ são contínuas em todo o plano

$\Rightarrow f(z)$ é derivável em todos os pontos

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = e^{-x}(-\cos y + i \operatorname{sen} y)$$

c) $f(z) = \cos x \cosh y - i \operatorname{sen} x \operatorname{senh} y$

$u(x,y) = \cos x \cosh y \qquad v(x,y) = -\operatorname{sen} x \operatorname{senh} y$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = -\operatorname{sen} x \cosh y \\ \dfrac{\partial v}{\partial y} = -\operatorname{sen} x \cosh y \end{array}\right\} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial u}{\partial x}$ e $\dfrac{\partial v}{\partial y}$ são contínuas em todo o plano

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial x} = -\cos x \operatorname{senh} y \\ \dfrac{\partial u}{\partial y} = \cos x \operatorname{senh} y \end{array}\right\} \Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \ ; \ p/ \ \forall z \in \mathbb{C}$$

$\dfrac{\partial v}{\partial x}$ e $\dfrac{\partial u}{\partial y}$ são contínuas em todo o plano

$\Rightarrow f(z)$ é derivável em todos os pontos

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = -(\operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y$$

7) a) $f(x+iy) = x^2 + iy^2$

$u(x,y) = x^2 \qquad v(x,y) = y^2$

$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = 2x \\ \dfrac{\partial v}{\partial y} = 2y \end{array}\right\} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$; p/ $\forall z$ tal que $x = y$

$\dfrac{\partial u}{\partial x}$ e $\dfrac{\partial v}{\partial y}$ são contínuas em todo plano

$\left.\begin{array}{l} \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial u}{\partial y} = 0 \end{array}\right\} \Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$; p/ $\forall z \in \mathbb{C}$

$\dfrac{\partial v}{\partial x}$ e $\dfrac{\partial u}{\partial y}$ são contínuas em todo plano

$\Rightarrow$ $f(x+iy)$ é derivável $\forall z$ tal que $x = y$

$\Rightarrow f'(x+ix) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 2x$

$f'(x+ix) = 2x$

b) $f(x+iy) = x^2 + y^2$

$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$

$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = 2x \\ \dfrac{\partial v}{\partial y} = 0 \end{array}\right\} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$; p/ $\forall z$ tal que $x = 0$

$\dfrac{\partial u}{\partial x}$ e $\dfrac{\partial v}{\partial y}$ são contínuas em todo plano

$\left.\begin{array}{l} \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial u}{\partial y} = 2y \end{array}\right\} \Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$; p/ $\forall z$ tal que $y = 0$

$\dfrac{\partial v}{\partial x}$ e $\dfrac{\partial u}{\partial y}$ são contínuas em todo plano

$\Rightarrow$ $f(x+iy)$ é derivável somente em $z = 0$

$f'(0) = 0$

c) $f(x+iy) = \sqrt{|xy|}$

$u(x,y) = \begin{cases} \sqrt{xy} & ; p/\ xy \geq 0 \\ \sqrt{-xy} & ; p/\ xy < 0 \end{cases} \qquad v(x,y) = 0$

$\dfrac{\partial v}{\partial y} = 0 \qquad \dfrac{\partial v}{\partial x} = 0$

para $xy \neq 0$:

$\dfrac{\partial u}{\partial x} = \begin{cases} \dfrac{y}{2\sqrt{xy}} & ; p/\ xy > 0 \\ \dfrac{-y}{2\sqrt{-xy}} & ; p/\ xy < 0 \end{cases} \Rightarrow \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$; $\forall z$ tal que $xy \neq 0$

As Condições de Cauchy-Riemann não são satisfeitas

para $(x,y) = (0,0)$:

$f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$

tomando o caminho: $y = 0$, $x \neq 0$

$f'(0) = \lim\limits_{\substack{x \to 0 \\ y = 0}} \dfrac{\sqrt{|xy|}}{x+iy} = 0$

tomando o caminho: $x = y$.

$f'(0) = \lim\limits_{x \to 0} \dfrac{\sqrt{|x^2|}}{x+ix} = \lim\limits_{x \to 0} \dfrac{\pm x}{x(1+i)} = \dfrac{\pm 1}{1+i}$

$\Rightarrow f'(0)$ não existe em $(x,y) = (0,0)$

para $x=0$ e $y\neq 0$:

$$\frac{\partial u}{\partial x}(0,y)=\lim_{x\to 0}\frac{u(x,y)-u(0,y)}{x}=\lim_{x\to 0}\frac{\sqrt{|xy|}}{x}=\lim_{x\to 0}\frac{\sqrt{|x|}\sqrt{|y|}}{x}$$

$$\lim_{x\to 0+}\frac{\sqrt{|x|}\sqrt{|y|}}{x}=+\infty \qquad \lim_{x\to 0-}\frac{\sqrt{|x|}\sqrt{|y|}}{x}=-\infty$$

$\Rightarrow \frac{\partial u}{\partial x}(0,y)$ não existe

para $x\neq 0$ e $y=0$:

$$\frac{\partial u}{\partial y}(x,0)=\lim_{y\to 0}\frac{u(x,y)-u(x,0)}{y}=\lim_{y\to 0}\frac{\sqrt{|xy|}}{y}=\lim_{y\to 0}\frac{\sqrt{|x|}\sqrt{|y|}}{y}$$

$$\lim_{y\to 0+}\frac{\sqrt{|x|}\sqrt{|y|}}{y}=+\infty \qquad \lim_{y\to 0-}\frac{\sqrt{|x|}\sqrt{|y|}}{y}=-\infty$$

$\Rightarrow \frac{\partial u}{\partial y}(x,0)$ não existe

$\therefore f(x+iy)$ não é derivável em nenhum ponto.

8) Condições de Cauchy-Riemann em coordenadas polares:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$z=r(\cos\theta+i\,\mathrm{sen}\,\theta)$$

$$f(z)=z^{1/2}=\sqrt{r}\left(\cos\frac{\theta}{2}+i\,\mathrm{sen}\frac{\theta}{2}\right)$$

$$u(r,\theta)=\sqrt{r}\cos\frac{\theta}{2} \qquad v(r,\theta)=\sqrt{r}\,\mathrm{sen}\frac{\theta}{2}$$

$$\frac{\partial u}{\partial r}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta}=\frac{\sqrt{r}}{2r}\cos\frac{\theta}{2}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}$$

$\Rightarrow \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$; p/ $\forall z$ tal que $r>0$

$\frac{\partial u}{\partial r}$ e $\frac{\partial v}{\partial \theta}$ são contínuas

$$\frac{\partial v}{\partial r}=\frac{1}{2\sqrt{r}}\,\mathrm{sen}\frac{\theta}{2}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta}=+\frac{1}{r}\frac{\sqrt{r}}{2}\,\mathrm{sen}\frac{\theta}{2}=\frac{1}{2\sqrt{r}}\,\mathrm{sen}\frac{\theta}{2}$$

$\Rightarrow \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$; p/ $\forall z$ tal que $r>0$

$\frac{\partial v}{\partial r}$ e $\frac{\partial u}{\partial \theta}$ são contínuas

$\Downarrow$

$f'(z)$ existe $\forall z$ tal que $r>0$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right)+i\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

$$f'(z)=\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\mathrm{sen}\,\theta}{r}\right)+i\left(\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\frac{\mathrm{sen}\,\theta}{r}\right)=\frac{1}{2\sqrt{r}}\left[\left(\cos\frac{\theta}{2}\cos\theta+\mathrm{sen}\frac{\theta}{2}\,\mathrm{sen}\,\theta\right)+i\left(\mathrm{sen}\frac{\theta}{2}\cos\theta-\cos\frac{\theta}{2}\,\mathrm{sen}\,\theta\right)\right]=$$

$$=\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}-i\,\mathrm{sen}\frac{\theta}{2}\right) \Rightarrow f'(z)=\frac{1}{2f(z)}$$

9-) a) $\operatorname{sen} z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\qquad$ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$\operatorname{sen}(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i\,\frac{e^{z} - e^{-z}}{2} = i \operatorname{senh} z \Rightarrow \operatorname{sen} iz = i \operatorname{senh} z$$

b) $$\operatorname{sen} z_1 \cos z_2 + \operatorname{sen} z_2 \cos z_1 = \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_2} - e^{-iz_2}}{2i} \cdot \frac{e^{iz_1} + e^{-iz_1}}{2} =$$

$$= \frac{1}{4i}\left[e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)} + e^{i(z_1+z_2)} + e^{-i(z_1-z_2)} - e^{i(z_1-z_2)} - e^{-i(z_1+z_2)}\right] =$$

$$= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)}\right] = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \operatorname{sen}(z_1+z_2)$$

$$\Rightarrow \operatorname{sen}(z_1+z_2) = \operatorname{sen} z_1 \cos z_2 + \operatorname{sen} z_2 \cos z_1$$

c) $$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z \Rightarrow \cos iz = \cosh z$$

$$\operatorname{sen}(x+iy) = \operatorname{sen} x \underbrace{\cos iy}_{\cosh y} + \underbrace{\operatorname{sen} iy}_{i \operatorname{senh} y} \cos x = \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y$$

d) $$\operatorname{senh} z_1 \cosh z_2 + \operatorname{senh} z_2 \cosh z_1 = \frac{e^{z_1} - e^{-z_1}}{2} \cdot \frac{e^{z_2} + e^{-z_2}}{2} + \frac{e^{z_2} - e^{-z_2}}{2} \cdot \frac{e^{z_1} + e^{-z_1}}{2} =$$

$$= \frac{1}{4}\left[e^{(z_1+z_2)} + e^{(z_1-z_2)} - e^{(z_2-z_1)} - e^{-(z_1+z_2)} + e^{(z_1+z_2)} + e^{(z_2-z_1)} - e^{(z_1-z_2)} - e^{-(z_1+z_2)}\right] =$$

$$= \frac{1}{4}\left[2e^{(z_1+z_2)} - 2e^{-(z_1+z_2)}\right] = \frac{e^{(z_1+z_2)} - e^{(z_1+z_2)}}{2} = \operatorname{senh}(z_1+z_2)$$

$$\Rightarrow \operatorname{senh}(z_1+z_2) = \operatorname{senh} z_1 \cosh z_2 + \operatorname{senh} z_2 \cosh z_1$$

e) $$\operatorname{senh}(x+iy) = \operatorname{senh} x \underbrace{\cosh iy}_{\cos y} + \underbrace{\operatorname{senh} iy}_{i \operatorname{sen} y} \cosh x = \operatorname{senh} x \cos y + i \cosh x \operatorname{sen} y$$

$$\cosh z = \cos iz \Rightarrow \cosh iy = \cos i(iy) = \cos(-y) = \cos y$$

$$\operatorname{senh} z = \frac{\operatorname{sen} iz}{i} \Rightarrow \operatorname{senh} iy = \frac{\operatorname{sen} i(iy)}{i} = \frac{\operatorname{sen}(-y)}{i} = -\frac{\operatorname{sen} y}{i} = i \operatorname{sen} y$$

f) $$\exp(z+i\pi) = e^{z} \cdot e^{i\pi} = e^{z}(\underbrace{\cos \pi}_{-1} + i \overbrace{\operatorname{sen} \pi}^{0}) = -e^{z}$$

10-) $\boxed{\operatorname{sen} z = 0}$

$$\operatorname{sen}(x+iy) = \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y = 0$$

$$\Downarrow$$

$$\begin{cases} \operatorname{sen} x \cosh y = 0 \Longrightarrow x = n\pi \quad ; n \in \mathbb{Z} \\ \quad e \\ \cos x \operatorname{senh} y = 0 \Longrightarrow x = (2n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z} \quad \text{ou } y = 0 \end{cases}$$

$$\therefore \begin{cases} x = n\pi \quad ; n \in \mathbb{Z} \\ y = 0 \end{cases}$$

$$z = n\pi + i.0$$

$$\underline{z = n\pi} \quad ; n \in \mathbb{Z}$$

$\boxed{\cos z = 0}$

$$\cos(x+iy) = \cos x \cos iy - \operatorname{sen} x \operatorname{sen} iy = \cos x \cosh y - i \operatorname{sen} x \operatorname{senh} y = 0$$

$$\Downarrow$$

$$\begin{cases} \cos x \cosh y = 0 \Longrightarrow x = (2n+1)\frac{\pi}{2} \quad ; n \in \mathbb{Z} \\ \quad e \\ \operatorname{sen} x \operatorname{senh} y = 0 \Longrightarrow x = n\pi \quad ; n \in \mathbb{Z} \quad \text{ou} \quad y = 0 \end{cases}$$

$$\therefore \begin{cases} x = (2n+1)\frac{\pi}{2} \quad ; n \in \mathbb{Z} \\ \quad e \\ y = 0 \end{cases}$$

$$z = (2n+1)\frac{\pi}{2} + i.0$$

$$\underline{z = (2n+1)\frac{\pi}{2}} \quad ; n \in \mathbb{Z}$$

11-) a) $e^z = -2$

$$e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i \operatorname{sen} y) = -2$$

$$\Downarrow$$

$$\begin{cases} \cos y = -1 \\ \operatorname{sen} y = 0 \\ \quad e \\ e^x = 2 \end{cases} \Rightarrow \begin{matrix} y = (2n+1)\pi \; ; n \in \mathbb{Z} \\ x = \ln 2 \end{matrix} \quad \underline{\text{ou}} \quad \begin{cases} \cos y = 1 \\ \operatorname{sen} y = 0 \\ \quad e \\ e^x = -2 \end{cases} \Rightarrow \text{não há solução para } x$$

$$\underline{z = \ln 2 + i(2n+1)\pi} \quad ; n \in \mathbb{Z}$$

b) $\exp(2z-1) = 1$

$$e^{2x+2iy-1} = e^{2x-1} . e^{i2y} = e^{2x-1}(\cos 2y + i \operatorname{sen} 2y) = 1$$

$$\Downarrow$$

$$\begin{cases} \cos 2y = 1 \\ \operatorname{sen} 2y = 0 \\ \quad e \\ e^{2x-1} = 1 \end{cases} \Rightarrow \begin{matrix} y = n\pi \; ; n \in \mathbb{Z} \\ x = \frac{1}{2} \end{matrix} \quad \underline{\text{ou}} \quad \begin{cases} \cos 2y = -1 \\ \operatorname{sen} 2y = 0 \\ \quad e \\ e^{2x-1} = -1 \end{cases} \Rightarrow \text{não há solução para } x$$

$$\underline{z = \frac{1}{2} + i \; n\pi} \quad ; n \in \mathbb{Z}$$