MAT220 - Cálculo IV <u>Lista 1</u>

1-) a) $\dfrac{5i}{2+i} = \dfrac{5\, e^{i\frac{\pi}{2}}}{\sqrt{5}\, e^{i(\operatorname{arctg}\frac{1}{2})}} = \sqrt{5}\, e^{i\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{2}\right)} =$

$\left.\begin{array}{l} \cos\theta = \frac{2}{\sqrt{5}} \\ \operatorname{sen}\theta = \frac{1}{\sqrt{5}} \end{array}\right\} \Rightarrow \theta = \operatorname{arctg}\frac{1}{2}$

$= \sqrt{5}\left[\cos\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{2}\right) + i \operatorname{sen}\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{2}\right)\right] =$

$= \sqrt{5}\left[\cos\frac{\pi}{2}\cancel{\cos\left(\operatorname{arctg}\frac{1}{2}\right)}^{0} + \operatorname{sen}\frac{\pi}{2}\operatorname{sen}\left(\operatorname{arctg}\frac{1}{2}\right) + \right.$

$\left. + i \operatorname{sen}\frac{\pi}{2}\cos\left(\operatorname{arctg}\frac{1}{2}\right) - \cancel{i \operatorname{sen}\left(\operatorname{arctg}\frac{1}{2}\right)\cos\frac{\pi}{2}}^{0}\right] =$

$= \sqrt{5}\left[\frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}\right] = \boxed{1+2i}$

b) $(-1+i)^7 = \left(\sqrt{2}\, e^{i\frac{3\pi}{4}}\right)^7 = 8\sqrt{2}\, e^{i\frac{21\pi}{4}} = 8\sqrt{2}\left(\cos\frac{21\pi}{4} + i \operatorname{sen}\frac{21\pi}{4}\right) =$

$\left.\begin{array}{l} \cos\theta = \frac{-1}{\sqrt{2}} \\ \operatorname{sen}\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \theta = \frac{3\pi}{4}$

$= 8\sqrt{2}\left(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \boxed{-8(1+i)}$

c) $(1+i\sqrt{3})^{-10} = \left(2\, e^{i\frac{\pi}{6}}\right)^{-10} = 2^{-10}\, e^{-i\frac{5\pi}{3}} = 2^{-10}\left(\cos\frac{5\pi}{3} - i \operatorname{sen}\frac{5\pi}{3}\right) =$

$\left.\begin{array}{l} \cos\theta = \frac{1}{2} \\ \operatorname{sen}\theta = \frac{\sqrt{3}}{2} \end{array}\right\} \theta = \frac{\pi}{6}$

$= 2^{-10}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \boxed{2^{-11}(1+i\sqrt{3})}$

2-) $z_1 = |z_1|\, e^{i(\arg z_1)}$ $\qquad z_1 \bar{z}_2 = |z_1||z_2|\, e^{i(\arg z_1 - \arg z_2)}$

$z_2 = |z_2|\, e^{i(\arg z_2)}$

$R_e(z_1\bar{z}_2) = |z_1||z_2| \cos(\arg z_1 - \arg z_2) = |z_1||z_2|$

$\Downarrow$

$\cos(\arg z_1 - \arg z_2) = 1 \Rightarrow \arg z_2 = \arg z_1 + 2k\pi;\ k \in \mathbb{Z}$

$\arg z_2 = \arg z_1 + 2k\pi \Rightarrow R_e(z_1\bar{z}_2) = |z_1||z_2|\cos(\arg z_1 - \arg z_2) = |z_1||z_2|$

$\boxed{\therefore\ R_e(z_1\bar{z}_2) = |z_1||z_2| \iff \arg z_2 = \arg z_1 + 2k\pi \ ;\ k \in \mathbb{Z}}$

3-) a) $(-i)^{\frac{1}{3}} = \left(e^{i\left(\frac{3\pi}{2}+2k\pi\right)}\right)^{\frac{1}{3}} = e^{i\left(\frac{\pi}{2}+k\frac{2\pi}{3}\right)}$; $k \in \mathbb{Z}$ ②

$$\boxed{\left.\begin{array}{l}\cos\theta = 0\\ \operatorname{sen}\theta = -1\end{array}\right\} \theta = \frac{3\pi}{2}+2k\pi}$$

$$(-i)^{\frac{1}{3}} = \begin{cases} k=0 \Rightarrow e^{i\frac{\pi}{2}} = i \\ k=1 \Rightarrow e^{i\frac{7\pi}{6}} = -\frac{\sqrt{3}}{2} - i\frac{1}{2} \\ k=2 \Rightarrow e^{i\frac{11\pi}{6}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{cases}$$

b) $8^{\frac{1}{6}} = \left(8\, e^{i2k\pi}\right)^{\frac{1}{6}} = 8^{\frac{1}{6}}\, e^{ik\frac{\pi}{3}}$; $k \in \mathbb{Z}$

$$\boxed{\left.\begin{array}{l}\cos\theta = 1\\ \operatorname{sen}\theta = 0\end{array}\right\} \theta = 2k\pi}$$

$$8^{\frac{1}{6}} = \begin{cases} k=0 \Rightarrow 8^{\frac{1}{6}} \\ k=1 \Rightarrow 8^{\frac{1}{6}} e^{i\frac{\pi}{3}} = 8^{\frac{1}{6}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ k=2 \Rightarrow 8^{\frac{1}{6}} e^{i\frac{2\pi}{3}} = 8^{\frac{1}{6}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ k=3 \Rightarrow 8^{\frac{1}{6}} e^{i\pi} = -8^{\frac{1}{6}} \\ k=4 \Rightarrow 8^{\frac{1}{6}} e^{i\frac{4\pi}{3}} = 8^{\frac{1}{6}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\ k=5 \Rightarrow 8^{\frac{1}{6}} e^{i\frac{5\pi}{3}} = 8^{\frac{1}{6}}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \end{cases}$$

4-) Fórmula de Moivre:

$(\cos\theta + i\operatorname{sen}\theta)^n = \cos n\theta + i\operatorname{sen} n\theta$

$(\cos\theta + i\operatorname{sen}\theta)^3 = \cos 3\theta + i\operatorname{sen} 3\theta \qquad (I)$

utilizando a fórmula do binômio:

$(\cos\theta + i\operatorname{sen}\theta)^3 = \cos^3\theta + 3i\operatorname{sen}\theta\cos^2\theta - 3\operatorname{sen}^2\theta\cos\theta - i\operatorname{sen}^3\theta \qquad (II)$

Sabemos também que:

$\operatorname{sen} 3\theta = 3\operatorname{sen}\theta\cos^2\theta - \operatorname{sen}^3\theta \qquad (III)$

De (I) e (II):

$$\cos 3\theta + i\,\text{sen}\,3\theta = \cos^3\theta + 3i\,\text{sen}\,\theta\cos^2\theta - 3\,\text{sen}^2\theta\cos\theta - i\,\text{sen}^3\theta$$

Substituindo (III):

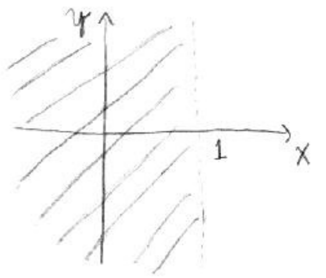
$$\cos 3\theta + i\,3\,\cancel{\text{sen}\,\theta}\cos^2\theta - \cancel{i\,\text{sen}^3\theta} = \cos^3\theta + 3i\,\cancel{\text{sen}\,\theta}\cos^2\theta - 3\,\text{sen}^2\theta\cos\theta - \cancel{i\,\text{sen}^3\theta}$$

$$\cos 3\theta = \cos^3\theta - 3\,\text{sen}^2\theta\cos\theta - 3\cos^3\theta + 3\cos^3\theta$$

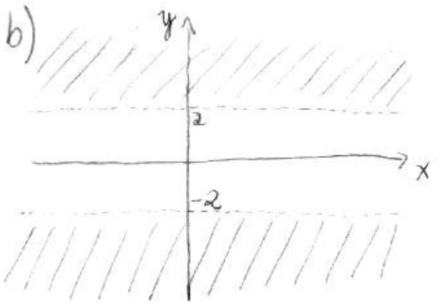
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

5-) $e^{(3+7\pi i)} = e^3 \cdot e^{7\pi i} = e^3(\underbrace{\cos 7\pi}_{-1} + i\,\cancel{\text{sen}\,7\pi}^{\,0}) = -e^3$

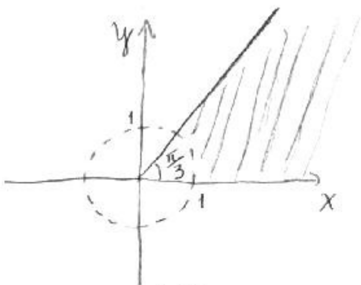
6-) a)



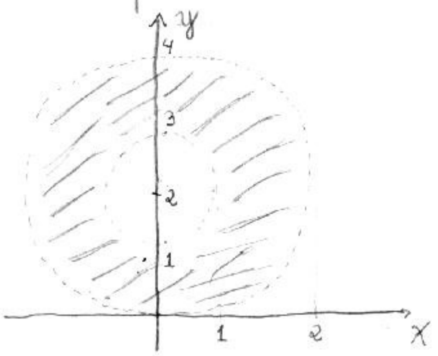
b)



c)



d)



④

e) $|z-1| = |z-i|$

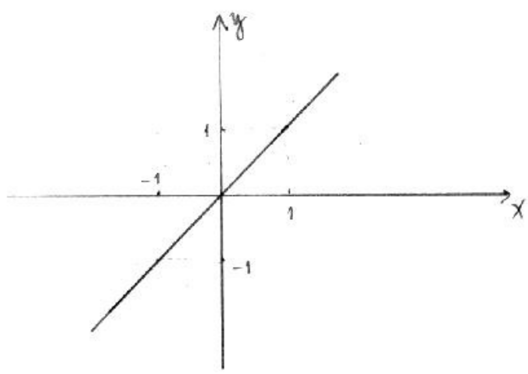
$|(x-1)+iy| = |x+i(y-1)|$

$\sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y-1)^2}$

$\cancel{x^2}-2x+\cancel{1}+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}-2y+\cancel{1}$

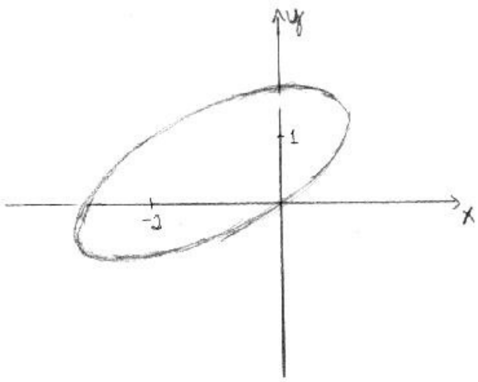
$-2x = -2y$

$x = y$



f) $|z-i|+|z+2|=3$

↳ elipse com focos em i e -2



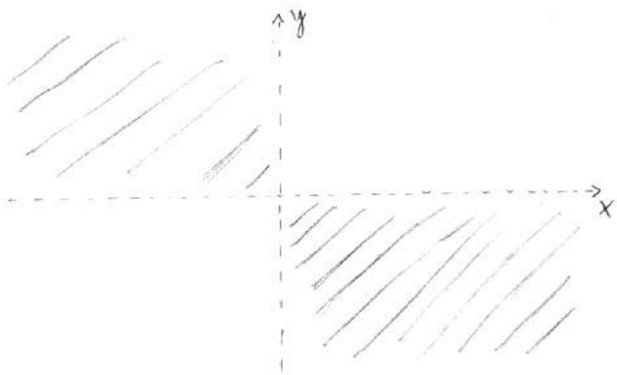
g) $\mathrm{Im}(z^2) < 0$

$\mathrm{Im}(x^2-y^2+2xyi) < 0$

$2xy < 0$

$\Downarrow$

$\begin{cases} x<0 \text{ e } y>0 & \text{ou} \\ x>0 \text{ e } y<0 \end{cases}$



h) $\mathrm{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

$\mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right) < \frac{1}{2}$

$\frac{x}{x^2+y^2} < \frac{1}{2}$

$\frac{x^2}{2}+\frac{y^2}{2}-x > 0$

$x^2-2x+y^2 > 0$

$(x-1)^2+y^2 > 1$