EXERCÍCIOS, 3.A PARTE

LISTA 10

27/07/21

Diagonalizar $A = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}$.

Resolução. A é uma matriz simétrica real, portanto diagonalizável sobre $\mathbb{R}$.

Polinômio característico:

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} 4-\lambda & -2 & -4 \\ -2 & 1-\lambda & 2 \\ -4 & 2 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)^2(1-\lambda) + 16 + 16 - 16(1-\lambda)$$
$$- 4(4-\lambda) - 4(4-\lambda)$$
$$= (4-\lambda)^2(1-\lambda) + \cancel{16} + \cancel{16} - 16 + 16\lambda$$
$$-\cancel{16} + 4\lambda - \cancel{16} + 4\lambda$$
$$= (4-\lambda)^2(1-\lambda) + 24\lambda - 16$$
$$= (16 + \lambda^2 - 8\lambda)(1-\lambda) + 24\lambda - 16$$
$$= \cancel{16} + \lambda^2 - 8\cancel{\lambda} - 16\cancel{\lambda} - \lambda^3 + 8\lambda^2 + 2\cancel{4}\lambda - \cancel{16}$$

$= -\lambda^3 + 9\lambda^2 = \lambda^2(9-\lambda)$

Autovalores: 0, 0, 9

Cálculo dos autoespaços:

$\boxed{\lambda = 9}$ $\quad N(A-9I) = ?$

$$A - 9I = \begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} \times -2$$

$$\Rightarrow \begin{pmatrix} -5 & -2 & -4 \\ 1 & 4 & -1 \\ 0 & 18 & -9 \end{pmatrix} \times 5$$

$$\Rightarrow \begin{pmatrix} 0 & 18 & -9 \\ 1 & 4 & -1 \\ 0 & 2 & -1 \end{pmatrix} \times -9$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

$\dim N(A-9I) = 1$

$=$ multiplicidade de 9

$$\begin{cases} x + 4y - z = 0 \\ 2y - z = 0 \end{cases}$$

$z = 2$

$y = 1$

$x = -2$

$u_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ é uma base de $N(A-9I)$

$\boxed{\lambda = 0}$ $\quad N(A) = ?$

$$A = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix} \begin{matrix} \times 2 \\ \times -2 \end{matrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim N(A) = 2 =$ multiplicidade de 0

$$\boxed{-2x + y + 2z = 0}$$

$x = 1,\ y = 2,\ z = 0 \qquad x = 1,\ y = 0,\ z = 1$

$$u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

u_2, u_3 é uma base de $N(A)$

Agora u_1, u_2, u_3 é uma base de $\mathbb{R}^3$ formada por autovetores de A.

$$u_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$u_1^t u_2 = -2 + 2 = 0$ ✓

(claro, pois são autovetores associados a autovalores distintos)

$u_1^t u_3 = -2 + 2 = 0$ ✓

(idem)

$$u_3^t u_2 = 1$$

$$u_3' := u_3 - \frac{u_3^t u_2}{\|u_2\|^2} u_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}.$$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \qquad \|u_3'\|^2 = \frac{16+4+25}{25} = \frac{45}{25} = \frac{9}{5}$$

$$v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \frac{\sqrt{5}}{3} \cdot \frac{1}{5} = \frac{1}{3\sqrt{5}}$$

$$v_3 = \frac{u_3'}{\|u_3'\|} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \qquad \frac{25+4+16}{45} = 1$$

v_1, v_2, v_3 é uma base ortonormal de $\mathbb{R}^3$ formada por autovetores de A

$$Q = \begin{pmatrix} -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ 2/3 & 0 & \sqrt{5}/3 \end{pmatrix}$$

é ortogonal e

$$Q^{-1} A Q = \Lambda = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad //$$

Continuação da discussão

$$p_A(\lambda) = \begin{vmatrix} 4-\lambda & -2 & -4 \\ -2 & 1-\lambda & 2 \\ -4 & 2 & 4-\lambda \end{vmatrix} \quad \times -2$$

$$= \begin{vmatrix} 4-\lambda & -2 & -4 \\ -2 & 1-\lambda & 2 \\ 0 & 2\lambda & -\lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} 4-\lambda & -2 & -4 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -1 \end{vmatrix} \quad \times 2$$

$$= \lambda \begin{vmatrix} -\lambda & -2\lambda & 0 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -1 \end{vmatrix} \quad \cancel{4}-2\lambda-\cancel{4}$$

$$= \lambda^2 \begin{vmatrix} -1 & -2 & 0 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -1 \end{vmatrix} \quad \times -2$$

$$= \lambda^2 \begin{vmatrix} -1 & -2 & 0 \\ 0 & 5-\lambda & 2 \\ 0 & 2 & -1 \end{vmatrix} \quad \times 2$$

$$= -\lambda^2 \begin{vmatrix} +1 & +2 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -\lambda^2 \cdot (\lambda - 9) = \lambda^2 (9 - \lambda)$$

$-\!\!/\!\!/\!-$

$N(A)$:

$$\boxed{-2x + y + 2z = 0}$$

$$w_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad -2.2 + 2 + 2.1 = -4 + 2 + 2 = 0 ✓$$

$$w_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad -2.1 + 2 + 2.0 = 0$$

$$w_2' = w_2 - \frac{w_2^t w_1}{\|w_1\|^2} w_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{6}{9} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$$\tilde{w}_1 = \frac{w_1}{\|w_1\|} = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \tilde{w}_2 = w_2' = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$v_1, \tilde{w}_1, \tilde{w}_2$ é uma base ortonormal de $\mathbb{R}^3$ formada por autovetores de A

$$\tilde{Q} = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

$$\tilde{Q}\,\tilde{Q}^t = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I$$

$\tilde{Q}$ é ortogonal!

$$\tilde{Q}^{-1} A \tilde{Q} = \tilde{Q}^t A \tilde{Q}$$

$$= \frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}\tilde{Q}$$

$$= \frac{1}{3}\begin{pmatrix} -18 & 9 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\tilde{Q}$$

$$= \cancel{3}\begin{pmatrix} -2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\cancel{3}}\begin{pmatrix} -2 & 2 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Lambda \quad ✓.$$

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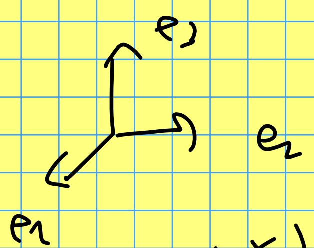
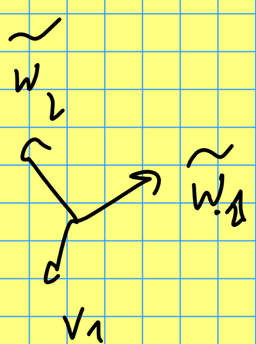
$$A = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}.$$



base canônica do $\mathbb{R}^3$

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x - 2y - 4z \\ -2x + y + 2z \\ -4x + 2y + 4z \end{pmatrix}$$

transformação linear do $\mathbb{R}^3$



base de autovetores

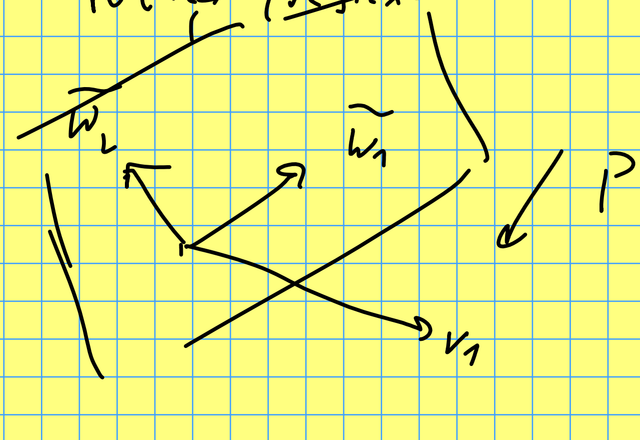
$$e_1, e_2, e_3 \xrightarrow{\tilde{Q}} v_1, \tilde{w}_1, \tilde{w}_2$$

rotação / ~~reflexão~~

$P \cdot v_1 = v_1$

$P \cdot \tilde{w}_1 = 0$

$P \cdot \tilde{w}_2 = 0$



$\langle \tilde{w}_1, \tilde{w}_2 \rangle = \langle v_1 \rangle^{\perp}$

P : projeção ortogonal de $\mathbb{R}^3$

sobre a reta $\langle v_1 \rangle$

$P v_1 = v_1 \quad P\tilde{w}_1 = P\tilde{w}_2 = 0$

$T = 9P$

$$A = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix} = \begin{pmatrix} (-2)(-2) & (-2).1 & (-2)\,2 \\ 1.(-2) & 1.1 & 1.2 \\ 2.(-2) & 2.1 & 2.2 \end{pmatrix}$$

$$v_1 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} -2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$= (3v_1)(3v_1)^t$$

$$\boxed{A = 9\, v_1 v_1^t}$$

$$Pu = \frac{v_1^t u}{\underbrace{\|v_1\|^2}_{=1}}\, v_1 = \underbrace{(v_1^t u)}_{\text{escalar}}\, v_1$$

$$= v_1 (v_1^t u) = (v_1^t v_1) u$$

$$Pu = (v_1^t v_1) u \qquad (P) = v_1^t v_1$$

$$A = 9(P)$$

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149
16

$$w = (10, 4, 7)$$

$$\frac{ww^t}{\|w\|^2} = \begin{pmatrix} 100 & 40 & 70 \\ 40 & 16 & 28 \\ 70 & 28 & 49 \end{pmatrix} \frac{1}{165}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$