EXERCÍCIOS, 2.ª PARTE

22/07/21

Ex. 3.21, p 199 [port]

Determinar uma base o.n. do plano $x-y+z=0$ em $\mathbb{R}^3$ e calcular a matriz (em relação à base canônica) da projeção ortogonal P sobre o plano. Qual é o $N(P)=$?

Resolução.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -x+y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ e $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ são 2 vetores L.I. e geram o plano

$\therefore$ formam uma base do plano.

Gram-Schmidt

$$v_2^t v_1 = v_1^t v_2 = (1\ 0\ -1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -1 \qquad \|v_1\|^2 = 2$$

$$v_2' := v_2 - \frac{v_2^t v_1}{\|v_1\|^2} v_1 = v_2 - \frac{(-1)}{2} v_1 = v_2 + \frac{1}{2} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} \qquad \|v_2'\|^2 = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$$

$v_1^t v_2' = (1\ \ 0\ \ -1)\begin{pmatrix}1/2\\1\\1/2\end{pmatrix} = 0$ ✓.

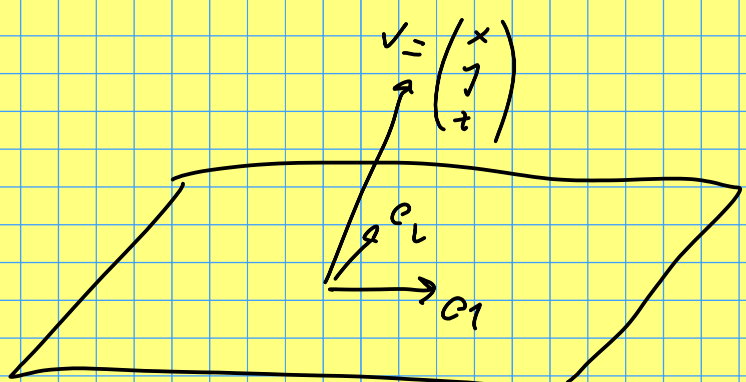
$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \leftarrow \qquad e_2 = \frac{v_2'}{\|v_2'\|} = \frac{1}{\sqrt{\frac{3}{2}}}\begin{pmatrix}1/2\\1\\1/2\end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\,\frac{1}{\cancel{2}}\,\frac{\cancel{\sqrt{2}}}{\sqrt{3}}\begin{pmatrix}1\\2\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix} \leftarrow$$

$\therefore$ e_1, e_2 é uma base ortonormal do plano dado.

$\|e_1\| = 1 = \|e_2\| \qquad e_1 \perp e_2$

As coordenadas de e_1, e_2 satisfazem $x - y + z = 0$



$P v = \text{proj}_{e_1} v + \text{proj}_{e_2} v \qquad$ pois e_1, e_2 é base o.n. do plano

$= (v^t e_1)\, e_1 + (v^t e_2)\, e_2$

$$= \left[(x\ y\ z)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \left[(x\ y\ z)\,\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}\right]\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \frac{1}{2}(x-z)\begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{1}{6}(x+2y+z)\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}x - \frac{1}{2}z + \frac{1}{6}x + \frac{1}{3}y + \frac{1}{6}z \\ \frac{1}{3}x + \frac{2}{3}y + \frac{1}{3}z \\ -\frac{1}{2}x + \frac{1}{2}z + \frac{1}{6}x + \frac{1}{3}y + \frac{1}{6}z \end{pmatrix}$$

$$P\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} \frac{2}{3}x + \frac{1}{3}y - \frac{1}{3}z \\ \frac{1}{3}x + \frac{2}{3}y + \frac{1}{3}z \\ -\frac{1}{3}x + \frac{1}{3}y + \frac{2}{3}z \end{pmatrix}$$

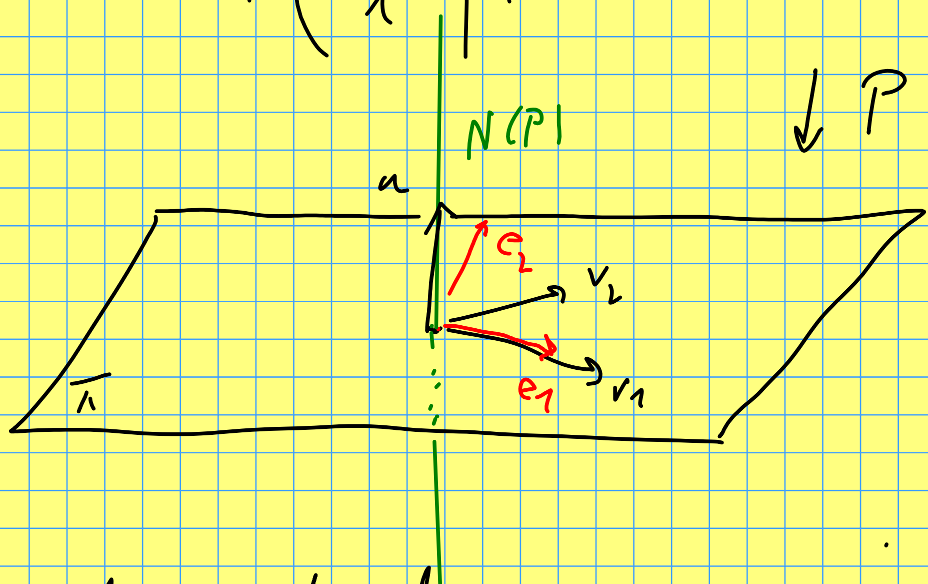
$$\begin{array}{cc} & \begin{array}{ccc} P\begin{pmatrix}1\\0\\0\end{pmatrix} & P\begin{pmatrix}0\\1\\0\end{pmatrix} & P\begin{pmatrix}0\\0\\1\end{pmatrix} \end{array} \\ (P) = & \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{array} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$N(P) = ? \quad \begin{cases} 2x + y - z = 0 \\ x + 2y + z = 0 \\ -x + y + 2z = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{x-2} \begin{pmatrix} 0 & -3 & -3 \\ 1 & 2 & 1 \\ 0 & 3 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + z = 0 \\ \quad\;\; y + z = 0 \end{cases} \qquad z = 1 \quad y = -1 \quad x = 1$$

$$N(P) = \left\langle u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle \qquad \underline{\text{reta}}$$



$\underline{\text{Obs}}$: As coordenadas $1, -1, 1$ de u são exatamente os coeficientes da eq do plano

$$x - y + z = 0.$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \pi \iff x - y + z = 0 \quad \text{(eq. definidora do plano em } \mathbb{R}^3\text{)}$$

$$\iff 1.x + (-1)y + 1.z = 0$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\iff u^t v = 0$$

$$\iff u \perp v,$$

ou seja, n é um vetor normal ao plano

Em $\mathbb{R}^n$:

$U: a_1 x_1 + \dots + a_n x_n = 0 \leftarrow$

subespaço de $\mathbb{R}^n$ de dimensão $n-1$

$\Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ é um vetor normal a U

$$U: \begin{aligned} a_{11} x_1 + \dots + a_{1n} x_n &= 0 \leftarrow \\ &\vdots \\ a_{r1} x_1 + \dots + a_{rn} x_n &= 0 \leftarrow \end{aligned}$$

subespaço de $\mathbb{R}^n$ de dim $\geq n-r$

e $\begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix}, \dots, \begin{pmatrix} a_{r1} \\ \vdots \\ a_{rn} \end{pmatrix}$ são vetores normais a U.

Ex 8, cj 5.6, p 302 [port]

Qual é a matriz M que realiza a mudança da base $u_1 = (1,1)$, $u_2 = (1,4)$ para $v_1 = (2,5)$, $v_2 = (1,4)$?

Resolução

$B_u: u_1, u_2$

$B_v: v_1, v_2$

$u_1 = 1\, v_1 + (-1)\, v_2$

$u_2 = 0\, v_1 + 1\, v_2$

$$M = \left(\boxed{\begin{matrix} 1 \\ -1 \end{matrix}} \; \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \right)$$

$$1(1,1) + \underbrace{2(1,4)}_{=(2,8)} = (3,9)$$

Exemplo:

$$w = (3,9) \qquad (w)_{B_v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= v_1 + v_2$$

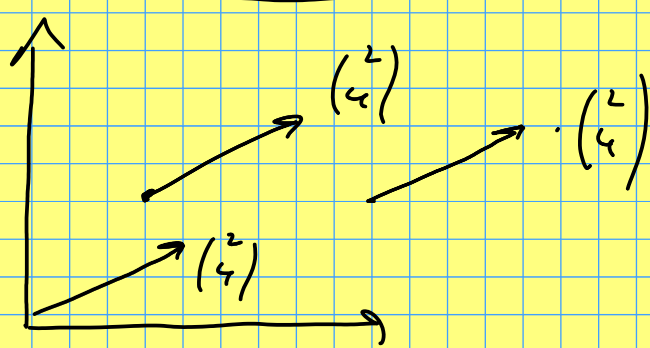
$$= 1 v_1 + 1 v_2$$

$$= u_1 + 2u_2 \qquad (w)_{B_u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 1 u_1 + 2 u_2$$

$$M\,(w)_{B_u} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (w)_{B_v}$$

$$\boxed{M_{B_v}^{B_u} \, (w)_{B_u} = (w)_{B_v}}$$

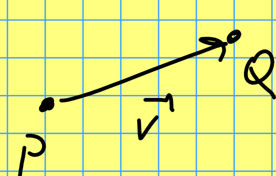


$\mathbb{R}^2$

$$Q - P = \vec{v}$$

$$B - A = \vec{v}$$

$$P + \vec{v} = Q$$



Matriz simétrica (real) : $A = A^t$

T. Espectral $\exists$ Q matriz ortogonal t.q.

$$Q^{-1} A Q = Q^t A Q = \Lambda$$

é diagonal.

As colunas de Q são os autovetores de A ortonormalizados.

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Se $\lambda_i \neq \lambda_j$ então autovetores associados a λ_i e λ_j são automaticamente ortogonais.

$\lambda_i = \lambda_j$ (caso em que λ_i tem uma multiplicidade > 1)

$$A = \left(\begin{array}{ccc|cc} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ \hline & & & 1 & \\ & & & & 3 \end{array}\right) \quad \begin{array}{lll} 2 & \text{mult } 3 & v_1 \; v_2 \; v_3 \\ 1 & \text{mult } 1 & v_4 \\ 3 & \text{mult } 1 & v_5 \end{array}$$

$\to N(A - 2I) \leftarrow$ dim 3 Sist. lin homog Esp. sol. dim 3

$\to N(A - I)$

$\to N(A - 3I)$

$$\boxed{A \text{ é diagonalizável} \iff \text{Para todo autovalor } \lambda \text{ de } A, \text{ vale que } \dim N(A - \lambda I) = \text{multiplicidade de } \lambda \text{ como autovalor de } A}$$

!!!

$N(A - \lambda I)$ = espaço de autovetores de A associados ao autovalor λ de A

Equivalências

(i) A é diagonalizável

(ii) $\dim N(A - \lambda I)$ = mult de λ como autovalor, para todo autovalor λ de A

(iii) $\exists$ matriz M inversível t.q. $M^{-1}AM$ é diagonal

(iv) $\exists$ base do espaço formada por autovetores de A.

$A \in M(n \times n, \mathbb{R})$ · base de $\mathbb{R}^n$.

$\mathbb{R}^n$

$A \in M(n \times n, \mathbb{C})$ base de $\mathbb{C}^n$.

$\mathbb{C}^n$

$-\eta-$

$$\det \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\det \begin{pmatrix} - v_1 - \\ - v_2 - \\ \vdots \end{pmatrix} = - \det \begin{pmatrix} - v_2 - \\ - v_1 - \\ \vdots \end{pmatrix} \leftarrow$$

$$\det \begin{pmatrix} - \lambda v_1 - \\ \hline \vdots \end{pmatrix} = \lambda \det \begin{pmatrix} - v_1 - \\ \hline \vdots \end{pmatrix} \leftarrow$$

$$\underbrace{\det \begin{pmatrix} v_1 \\ v_2 + \lambda v_1 \\ \vdots \end{pmatrix}}_{} = \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} \leftarrow$$

$$= \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} + \cancelto{0}{\lambda \det \begin{pmatrix} v_1 \\ v_1 \\ \vdots \end{pmatrix}} \Big] =$$