FORMAS QUADRÁTICAS

15/07/21

$ax^2 + bxy + cy^2$ $\qquad x, y$

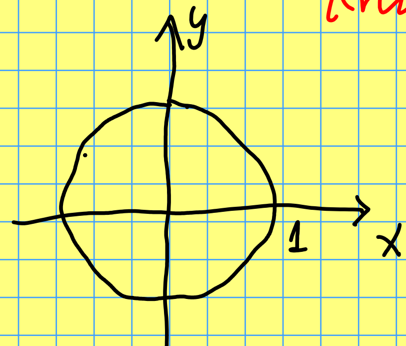
$a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{nn}x_n^2$ $\qquad x_1, \dots, x_n$

$Ax^2 + By^2 + C^2z^2 + Dxy + Exz + Fyz$ $\qquad x, y, z$

termos mistos

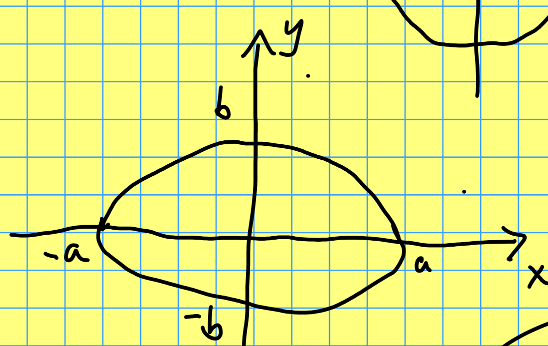
$\vdots$

Círculo: $x^2 + y^2 = 1$



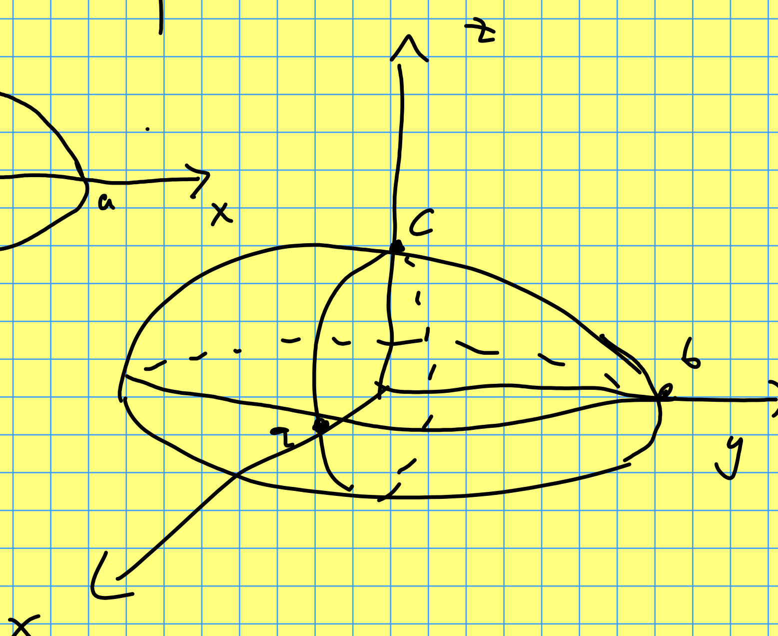
Elipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Elipsóide:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$\boxed{5x^2 + 8xy + 5y^2 = 1 \ ?}$$

Como desenhar esta curva em $\mathbb{R}^2$?

$\rightarrow \quad 5x^2 + 4xy + 4yx + 5y^2 = 1$

$$\rightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

Forma quadrática: $\quad v^t A v \quad$ onde $A = A^t$

$$\boxed{v^t A v = 1}$$

Vamos diagonalizar $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix}$$

$$= 25 + \lambda^2 - 10\lambda - 16 = \lambda^2 - 10\lambda + 9$$

Autovalores: $\lambda_1 = 1$ e $\lambda_2 = 9$

Cálculos dos autovetores:

$\boxed{\lambda_1 = 1} \quad N(A - I) = ? \quad \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}$ é múltiplo de $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Autovetor unitário: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_1$

$\boxed{\lambda_2 = 9} \quad N(A - 9I) = ? \quad \begin{pmatrix} -4 & 4 \\ +4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}$ é múltiplo de $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Autovetor unitário: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_2$

$\{v_1, v_2\}$ base ortonormal de $\mathbb{R}^2$ formada por autovetores de A

$$Q = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$ é uma matriz ortogonal $(Q^t = Q^{-1})$

$$Q^{-1} A Q = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^t$$

A equação

$$v^t A v = 1$$

$5x^2 + 8xy + 5y^2 = 1$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$

fica

$$v^t (Q \Lambda Q^t) v = 1$$

$$(Q^t v)^t \Lambda (Q^t v) = 1$$

$$u = Q^t v$$

$u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ xi eta

$$\rightarrow \quad w^t \Lambda w = 1$$

$$u = Q^t v \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

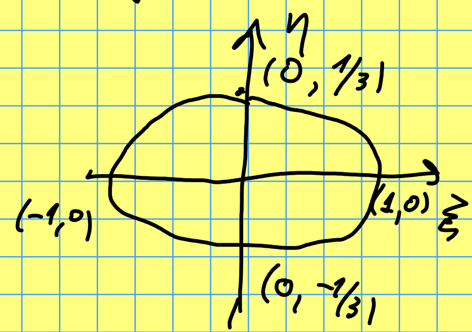
$$\begin{cases} \xi = \frac{1}{\sqrt{2}} (x - y) \\ \eta = \frac{1}{\sqrt{2}} (x + y) \end{cases}$$

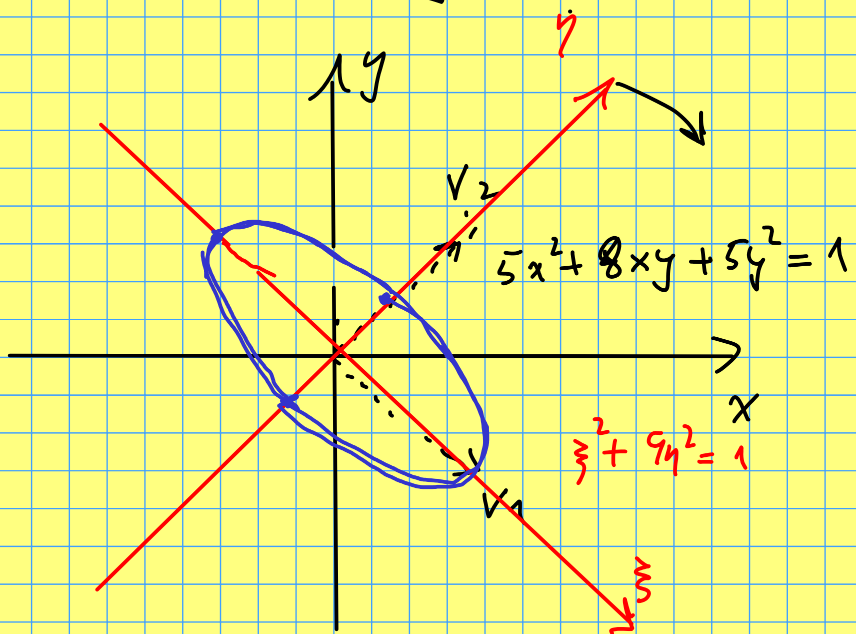
mudança de variáveis (coordenadas)

$$w^t \Lambda w = 1 \qquad (\xi \ \ \eta)\begin{pmatrix}1 & 0\\ 0 & 9\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 1$$

$$\boxed{\xi^2 + 9\eta^2 = 1}$$

elipse





$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Q é uma rotação de $45°$ no sentido horário

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \qquad Q\begin{pmatrix}1\\ 0\end{pmatrix} = v_1 \qquad Q\begin{pmatrix}0\\ 1\end{pmatrix} = v_2$$

As direções dos autovetores v_1, v_2 são os eixos principais da elipse.

$e_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$ base canônica de $\mathbb{R}^2$ $\xrightarrow{Q}$ v_1, v_2 base de autovetores de Λ

x, y coordenadas na base canônica $\xrightarrow{Q^t = Q^{-1}}$ ξ, η coordenadas na base v_1, v_2

1 -

$$5x^2 + 8xy + y^2 = 1\xi^2 + 9\eta^2 \geq 0$$

Autovalores

$1, 9 > 0$

soma de quadrados
com coeficientes positivos

$$\xi^2 + 9\eta^2 = 0 \iff \xi = \eta = 0$$

Dizemos que uma forma quadrática $v^t A v$ (a matriz A é definida positiva)
definida positiva se

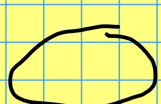
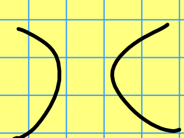
$$v^t A v \geq 0$$

e

$$v^t A v = 0$$

somente para $v = 0$.

Obs Se A é uma matriz $n \times n$ simétrica definida positiva (i.e. com autovalores (reais) positivos) então a equação $v^t A v = 1$ define um elipsóide em n dimensões

$n=2$   

Cônicas

$\lambda_1, \lambda_2 > 0$ elipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\lambda_1 > 0, \lambda_2 < 0$ hipérbole

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

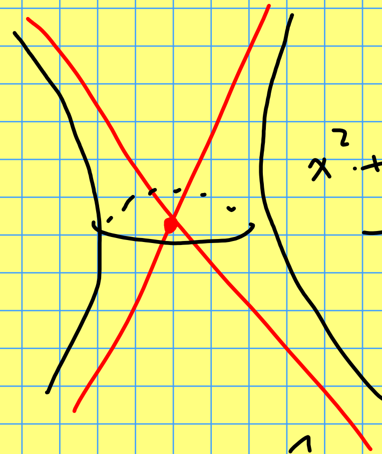
$$y = x^2$$

$n=3$ Quádricas

$x^2+y^2+z^2=1$

$+++$

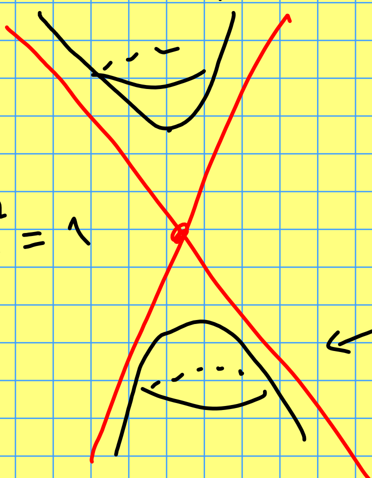
elipsoide



$x^2+y^2-z^2=1$

$-++$

hiperbolóides $\uparrow$ 1 folha



$-x^2-y^2+z^2=1$

$--+$

$\leftarrow$ 2 folhas

parabolóides

(elíptico, hiperbólico, ...)

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Ex. $$7x^2+\overbrace{4xy}^{=2xy+2yx}+6y^2+4yz+5z^2=1$$

Que superfície do $\mathbb{R}^3$ é essa?

$$(x\ y\ z)\underbrace{\begin{pmatrix}7&2&0\\2&6&2\\0&2&5\end{pmatrix}}_{=A}\begin{pmatrix}x\\y\\z\end{pmatrix}=1$$

$$p_A(\lambda)=\det(A-\lambda I)=\begin{vmatrix}7-\lambda&2&0\\2&6-\lambda&2\\0&2&5-\lambda\end{vmatrix}$$

$$= (7-\lambda)(6-\lambda)(5-\lambda) \underbrace{- 4(7-\lambda) - 4(5-\lambda)}$$

$$= -4(12-2\lambda)$$

$$= -8(6-\lambda)$$

$$= (6-\lambda)\left[(7-\lambda)(5-\lambda) - 8\right]$$

$$= (6-\lambda)(35+\lambda^2-12\lambda-8)$$

$$= (6-\lambda)(\lambda^2-12\lambda+27)$$

$$= (6-\lambda)(\lambda-9)(\lambda-3)$$

Autovalores: 3, 6, 9

Autovetores:

$\boxed{\lambda=3}$ $\quad N(A-3I)$ $\quad \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{matrix} \div 2 \\ \\ \div 2 \end{matrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \cdot(-1) \\ \\ \\ \end{matrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} 2x + y = 0 \\ y + z = 0 \end{cases} \quad \begin{matrix} x = 1 \\ y = -2 \\ z = 2 \end{matrix} \quad \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\left\| \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\|^2 = 1+4+4 = 9 \qquad v_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$\boxed{\lambda = 6}$ $N(A-6I)$ $A - 6I = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix} \div 2$

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow{x-1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ \cancel{0} & \cancel{2} & \cancel{-1} \end{pmatrix}$$

$x + 2y = 0$

$-2y + z = 0$

$y = 1 \quad x = -2$

$z = 2$

$$v_2 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$\boxed{\lambda = 9}$ $N(A-9I)$ $A - 9I = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{pmatrix}$

$$\begin{pmatrix} +1 & -1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow{x-2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ \cancel{0} & \cancel{1} & \cancel{-2} \end{pmatrix}$$

$x - y = 0$

$-y + 2z = 0$

$z = 1$

$y = 2 = x$

$$v_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$Q = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$Q^t = Q^{-1}$ Q é ortogonal

movimento rígido no $\mathbb{R}^3$

(rotação / reflexão)

$$Q^{-1} A Q = \Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = Q^t \begin{pmatrix}x\\ y\\ z\end{pmatrix} \qquad 3\xi^2 + 6\eta^2 + 9\zeta^2 = 1$$

xi eta zeta.

Elipsóide $3, 6, 9 > 0$

no $\mathbb{R}^3$ com direções dos eixos principais dadas por v_1, v_2, v_3

Em particular:

$$7x^2 + 4xy + 6y^2 + 4yz + 5z^2 \geq 0$$

$$\forall\, x, y, z \in \mathbb{R}.$$

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$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$a\left(\underbrace{x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}}_{\downarrow} - \frac{b^2}{4a^2} + \frac{c}{a}\right) = 0$$

$$\underbrace{a}_{\neq 0}\cdot\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \qquad \Delta = b^2-4ac$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$$

Em $\mathbb{C}$:

$$x+\frac{b}{2a} = \frac{\pm\sqrt{\Delta}}{2a} \quad \therefore \quad x = \frac{-b\pm\sqrt{\Delta}}{2a}$$

—11—

1.º exemplo. $5x^2+8xy+5y^2$

$$\begin{cases} \xi = \frac{1}{\sqrt{2}}(x-y) \\ \eta = \frac{1}{\sqrt{2}}(x+y) \end{cases}$$

$$5x^2+8xy+5y^2 = \xi^2+9\eta^2$$
$$= \left[\frac{1}{\sqrt{2}}(x-y)\right]^2 + 9\left[\frac{1}{\sqrt{2}}(x+y)\right]^2.$$

Capítulo 6: Matrizes definidas positivas

(ou positivas definidas)

Elipsóides: § 6.2