AUTOVALORES E AUTOVETORES

29/06/21

Sequência de Fibonacci

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$$\begin{cases} F_n = F_{n-2} + F_{n-1} \\ F_0 = 0 \quad F_1 = 1 \end{cases}$$

fórmula recursiva

$$\underbrace{\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}}_{=x_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}}_{=:x_n}$$

$F_{n+1} = F_n + F_{n-1}$

$F_n = F_n$

$$x_1 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_{n+1} = A x_n$$
$$x_n = A x_{n-1}$$
$$x_{n-1} = A x_{n-2}$$

$$\Rightarrow \quad x_{n+1} = A(A x_{n-1}) = A^2 x_{n-1}$$
$$= A^3 x_{n-2}$$
$$= A^4 x_{n-3}$$
$$\vdots$$
$$= A^n x_1$$

$\therefore\ x_{n+1} = A^n x_1 \Rightarrow \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

?

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

Matrizes diagonais

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & & \\ 0 & b_2 & & 0 \\ \vdots & & \ddots & \\ 0 & & & b_n \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 & \cdots & 0 \\ 0 & a_2 b_2 & 0 & \cdots 0 \\ & & \ddots & \\ 0 & & & a_n b_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}$$

Potenciação de matrizes diagonais ✓

Além disso, se

$$A = MDM^{-1}$$

A é ~~conjugada~~ semelhante a uma matriz diagonal

onde D é diagonal e M é invertível, então

$$A^2 = (MDM^{-1})(MDM^{-1})$$
$$= MD\underbrace{(M^{-1}M)}_{=I}DM^{-1}$$
$$= MDDM^{-1}$$

$$= M D^2 M^{-1}$$

$$A^3 = A A^2 = (M D \cancel{M^{-1}})(\cancel{M} D^2 M^{-1})$$

$$= M D^3 M^{-1}$$

$$A^k = M D^k M^{-1}$$

—||—

Voltando à seq. de Fibonacci:

É $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ conjugada a uma matriz diagonal?

$$= \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x & \lambda_2 z \\ \lambda_1 y & \lambda_2 w \end{pmatrix}$$

$$M^{-1} A M = D \quad ?$$

$$A M = M D \qquad M = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

colunas de M: v_1, v_2

$$A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix}$$

Igualando as colunas:

$$\boxed{\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}}$$

Nosso problema agora é determinar $v_1, v_2, \lambda_1, \lambda_2$.

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{cases} x_1 + y_1 = \lambda_1 x_1 \\ x_1 = \lambda_1 y_1 \end{cases}$$

$$\begin{cases} x_1 (1 - \lambda_1) + y_1 = 0 \\ x_1 - \lambda_1 y_1 = 0 \end{cases}$$

Sistema linear homogêneo em x_1, y_1, para cada λ_1 fixado.

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \underbrace{N}_{\text{espaço-nulo}} \begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} = N(A - \lambda_1 I)$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix}$$

Quando é que $N(A - \lambda_1 I) \neq \{0\}$?

matriz quadrada

Quando $A-\lambda_1 I$ for uma matriz singular, quer dizer, não for invertível, ou seja,

ou A não tem posto máximo (2)

ou A tem colunas LD (linhas)

$$\det(A-\lambda_1 I)=0$$

(também λ_2)

λ_1 deve ser uma raiz do polinômio $\det(A-\lambda I)=0$

polinômio característico de A

$$A-\lambda I = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1-\lambda & 1\\ 1 & -\lambda\end{pmatrix}$$

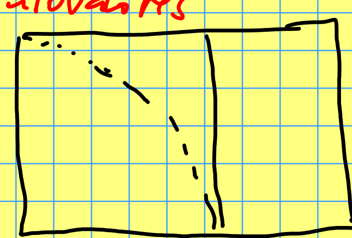
$$\underbrace{\det(A-\lambda I)}_{=\,p_A(\lambda)} = \begin{vmatrix}1-\lambda & 1\\ 1 & -\lambda\end{vmatrix} = (1-\lambda)(-\lambda)-1 = \lambda^2-\lambda-1$$

Raízes: $\boxed{\lambda^2-\lambda-1=0}$ $\qquad \lambda = \dfrac{1\pm\sqrt{5}}{2}$ *

valores característicos ou valores próprios ou autovalores

$\varphi = \dfrac{\sqrt{5}+1}{2}$ "razão áurea"

b.



b

a

a-b

$$\frac{a^{\div b}}{b_{\div b}} = \frac{b^{\div b}}{a-b_{\div b}}$$

$$\frac{a/b}{1} = \frac{1}{a/b-1}$$

$$\left(\frac{a}{b}\right)^2 - \frac{a}{b} = 1 \qquad \boxed{\left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0}$$

$$\varphi^2-\varphi-1=0$$

Agora: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$ $\qquad \lambda_1 - \lambda_2 = \sqrt{5}$

$= \varphi \qquad\qquad = -\varphi^{-1}$

$$\begin{cases} x_1(1-\lambda_1) + y_1 = 0 \\ x_1 - \lambda_1 y_1 = 0 \end{cases}$$

$$1-\lambda_1 = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \lambda_2$$

$$\begin{cases} \cancel{x_1 \lambda_2 + y_1 = 0} \\ x_1 - \lambda_1 y_1 = 0 \end{cases} \qquad x \cdot \lambda_2$$

$x_1 = \lambda_1$, $y_1 = 1$ $\qquad v_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$

Analogamente $\quad v_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

vetores característicos
vetores próprios
autovetores

Então $\quad M = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$M^{-1} A M = D$$

$$\boxed{A = M D M^{-1}}$$

$$A^n = M D^n M^{-1} \qquad \det \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} = \lambda_1 - \lambda_2 = \sqrt{5}$$

$$A^n = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}} \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & -\lambda_1^{n+1}\lambda_2 + \lambda_1 \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n & -\lambda_1^n \lambda_2 + \lambda_1 \lambda_2^n \end{pmatrix}$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}$$

$$\therefore \; F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$\boxed{F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}} \quad n = 0, 1, 2, \ldots$$

$$F_0 = \frac{1-1}{1} = 0 \quad F_1 = \frac{\cancel{1}+\sqrt{5} - (\cancel{1}-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

$$F_2 = \frac{\cancel{1} + 2\sqrt{5} + \cancel{5} - (\cancel{1} - 2\sqrt{5} + \cancel{5})}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1, \ \ldots$$

$$\frac{F_{n+1}}{F_n} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{\cancel{n+1}}\cancel{\sqrt{5}}} \cdot \frac{\cancel{2^n}\cancel{\sqrt{5}}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n}$$

$$= \frac{1}{2} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} \quad \div (1+\sqrt{5})^n$$

$$\div (1+\sqrt{5})^n$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1}{2} \lim_{n\to\infty} \frac{(1+\sqrt{5}) - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n (1-\sqrt{5})}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} \quad \to 0 \quad \to 0$$

$$\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1$$

$$= \frac{1}{2} \frac{1+\sqrt{5}}{1} = \frac{1+\sqrt{5}}{2} = \varphi$$

<u>Sumário</u> Dada uma matriz A $n \times n$, dizemos que λ é um <u>autovalor</u> de A se existe um vetor $v \neq 0$ t.q. $Av = \lambda v$. Nesse caso, v é chamado de <u>autovetor</u> de A associado ao autovalor λ. Mostra-se que os autovalores de A são as

raízes do polinômio característico de A, que é

$$p_A(\lambda) = \det(A - \lambda I).$$

η

$$\therefore S = 1 + 2 + 3 + 4 + 5 + \dots$$