# APLICAÇÕES DE DETERMINANTES

24/06/21

## 1. Cálculo de $A^{-1}$.

Se $\det A \neq 0$, então $A$ é invertível e

$$A^{-1} = \frac{1}{\det A} C^t,$$

onde

$$C = \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}$$ é a matriz dos

cofatores de $A$.

**Dem.** Vamos verificar a fórmula:

$$AC^t = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det A \end{pmatrix} = (\det A)\, I$$

1.ª linha & 1.ª coluna:

$$a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} = \det A$$

Esta é exatamente a expansão de Laplace segundo a 1.ª linha de A, que dá $\det A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad C_{11}$$

2.ª linha & 2.ª coluna

$$a_{21}C_{21} + a_{22}C_{22} + \dots + a_{2n}C_{2n} = \det A$$

$$\vdots$$

1.ª linha & 2.ª coluna

$$a_{11}C_{21}^{A} + a_{12}C_{22}^{A} + \dots + a_{1n}C_{2n}^{A} = (*)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad \therefore = 0$$

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{11} & a_{12} & \cdots & a_{1n}} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \xleftarrow{\text{Laplace}}$$

$$\underbrace{\det \tilde{A}}_{\substack{\shortparallel \\ 0}} = a_{11} \underbrace{C^A_{21}}_{= C^{\tilde{A}}_{21}} + a_{12} \underbrace{C^A_{22}}_{= C^{\tilde{A}}_{22}} \cdots + a_{1n} \underbrace{C^A_{2n}}_{= C^{\tilde{A}}_{2n}} = (*)$$

$\leftarrow$ pois $\tilde{A}$ tem 2 linhas iguais.

Agora $AC^t = (\det A) I$. Sendo $\det A \neq 0$,

$$A \left( \frac{1}{\det A} C^t \right) = I$$

$$\therefore \quad A^{-1} = \frac{1}{\det A} C^t \quad //.$$

**Ex.** $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det A = ad - bc \neq 0$ Suponha

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} \cancel{a} & \cancel{b} \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

# 2. A Regra de Cramer

$$Ax = b \qquad A \text{ matriz } n \times n \text{ invertível}$$

$$\Rightarrow x = A^{-1}b = \left(\frac{1}{\det A} C^t\right) b = \frac{1}{\det A} C^t b$$

$$C^t b = \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \boxed{b_1 C_{11} + \dots + b_n C_{n1}} \\ \vdots \\ b_1 C_{1n} + \dots + b_n C_{nn} \end{pmatrix} = \begin{pmatrix} \det B_1 \\ \vdots \\ \det B_n \end{pmatrix}$$

Expansão de Laplace da matriz: segundo 1.ª coluna

$$B_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

j-ésima coluna

$$B_j = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}$$

$$x = \frac{1}{\det A} C^t b = \frac{1}{\det A} \begin{pmatrix} \det B_1 \\ \vdots \\ \det B_n \end{pmatrix}$$

$$\parallel$$

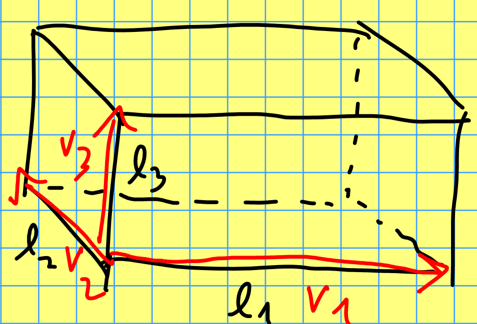
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \therefore \quad x_j = \frac{\det B_j}{\det A} \quad //$$

$$j = 1, \dots, n$$

Ex. $\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 4x_2 = 6 \end{cases}$ $\quad x_2 = \dfrac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{-2} = \dfrac{6}{-2} = -3$

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9$$

## 3. Volumes de Caixas ou Paralelotopos




$$\text{volume} = l_1 l_2 l_3$$

$v_1 \perp v_2$ $\quad v_1 \perp v_3$

$v_2 \perp v_3$

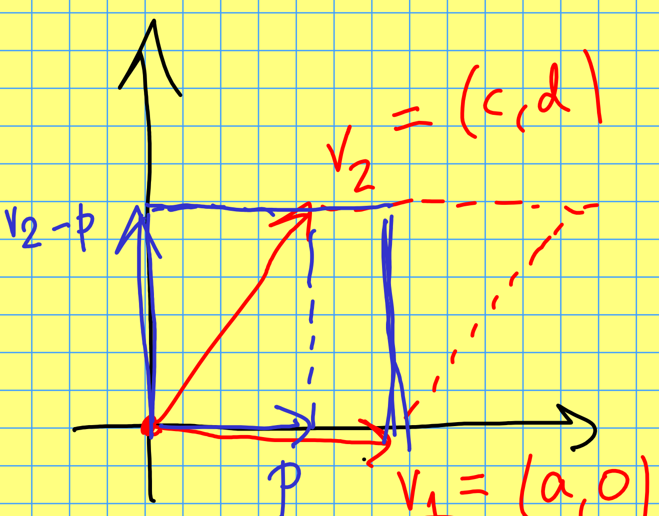
$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \qquad A^t A = \begin{pmatrix} - & v_1^t & - \\ - & v_2^t & - \\ - & v_3^t & - \end{pmatrix} \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} v_1^t v_1 & v_1^t v_2 & v_1^t v_3 \\ v_2^t v_1 & v_2^t v_2 & v_2^t v_3 \\ v_3^t v_1 & v_3^t v_2 & v_3^t v_3 \end{pmatrix} \leftarrow \text{matriz de produtos escalares}$$

$$= \begin{pmatrix} \|v_1\|^2 & 0 & 0 \\ 0 & \|v_2\|^2 & 0 \\ 0 & 0 & \|v_3\|^2 \end{pmatrix}$$

$$\det(A)^2 = \underbrace{\det(A^t A)}_{\underbrace{\det(A^t)}_{=\det(A)} \cdot \det(A)} = \|v_1\|^2 \|v_2\|^2 \|v_3\|^2$$

$$\therefore |\det A| = \|v_1\| \|v_2\| \|v_3\| = \ell_1 \ell_2 \ell_3 = \text{volume}$$




$p = \lambda v_1$

$$\underline{\text{Área (Verm)}} = \text{Área (Azul)} = \left|\det \begin{pmatrix} | & | \\ v_1 & v_2 - p \\ | & | \end{pmatrix}\right|$$

$$= \left|\det \begin{pmatrix} | & | \\ v_1 & v_2 - \lambda v_1 \\ | & | \end{pmatrix}\right|$$

$$= \left| \det \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} - \lambda \cancel{\det \begin{pmatrix} | & | \\ v_1 & v_1 \\ | & | \end{pmatrix}} \right|$$

$$= \left| \det \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \right|$$

<u>Ex.</u> $v_1 = (1,1,1)$ · $P$ paralelepípedo $v_1, v_2, v_3$

$v_2 = (1,1,0)$

$v_3 = (1,0,0)$

$$Vol(P) = \left| \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right|$$

$$= \left| \mp \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$= 1 \; //$$

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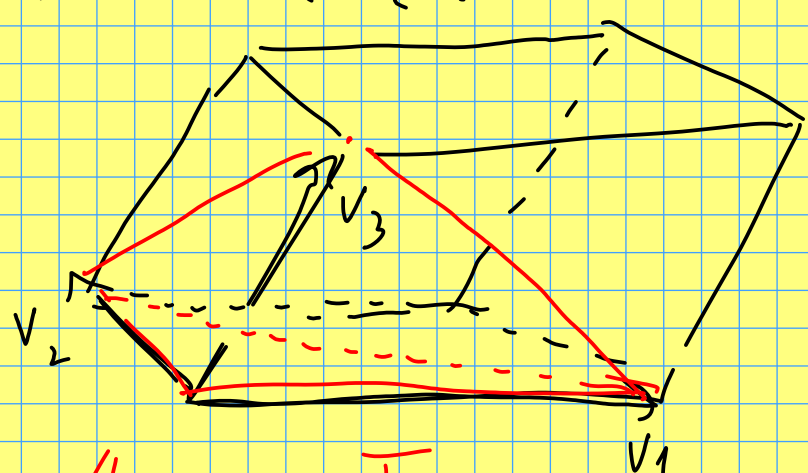
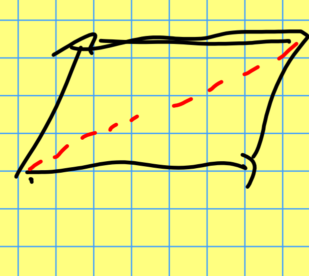
$\mathbb{R}^3$ $v_1, v_2, v_3$ L.I.




Paralelepípedo $P$

$$Vol(P) = \left| \det \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \right|$$

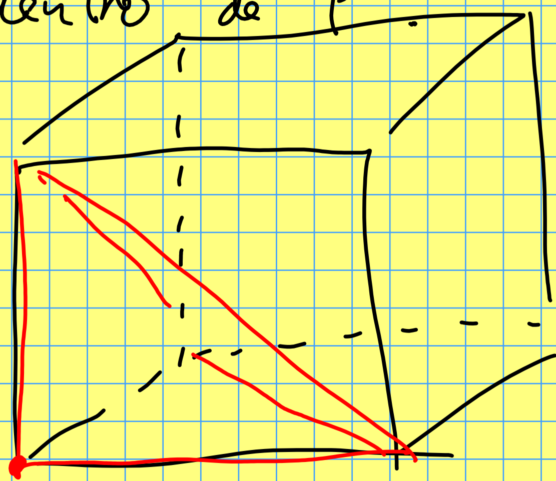


$\frac{6}{\frac{3}{2}} = 4$ $=$

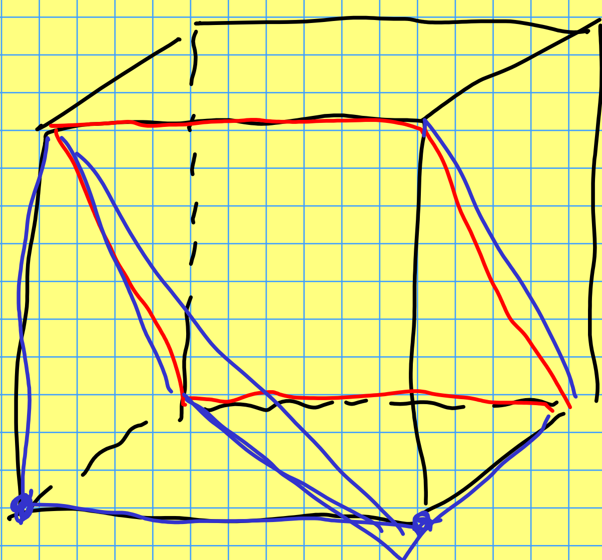
$T$

<u>tetraedro</u>

Cabem exatamente <u>6</u> tetraedros congruentes a $T$ dentro de $P$



$$\mathrm{Vol}(T) = \frac{1}{6}\left|\det\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}\right|$$



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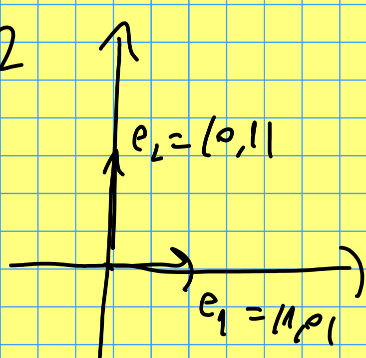
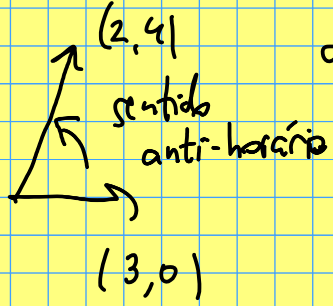
$n=2$



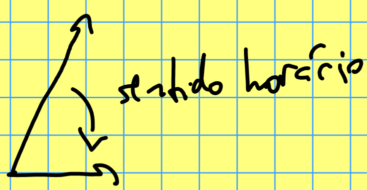

$$\det\begin{pmatrix} | & | \\ e_1 & e_2 \\ | & | \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\det\begin{pmatrix} e_2 & e_1 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

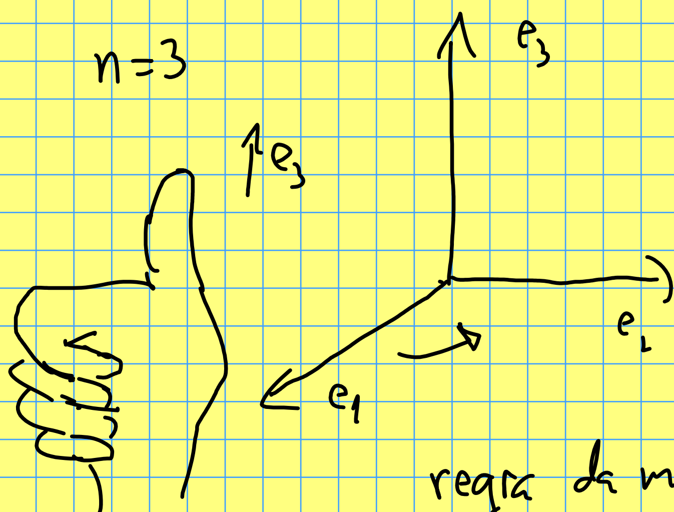



$$\det \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} = 12$$

$$\det \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix} = -12$$

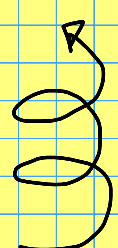



$n=3$




$$\det \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = 1 > 0$$

regra da mão direita



$$\det \begin{array}{c} \begin{array}{ccc} e_2 & e_1 & e_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = -1$$

Bases do $\mathbb{R}^3$

= $\boxed{\text{mão esquerda}}$ $\cup$ $\boxed{\text{mão direita}}$

| | |
|---|---|
| $e_2\, e_1\, e_3$ | $e_1\, e_2\, e_3$ |
| $e_3\, e_2\, e_1$ | $e_2\, e_3\, e_1$ |
| $e_1\, e_3\, e_2$ | $e_3\, e_1\, e_2$ |

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