FÓRMULAS PARA O DETERMINANTE

22/06/21

• Fatoração LDU

Se A é invertível, então $PA = LDU$, onde
L é triangular inferior com 1's na diagonal,
U " " superior " 1's " " ,
D é diagonal, e P é uma matriz de permutação.

Então

$$\det PA = \det LDU$$

$$(\det P)(\det A) = (\det L)(\det D)(\det U) \qquad (1)$$

$$L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \Rightarrow \det L = 1$$

$$U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \Rightarrow \det U = 1$$

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \Rightarrow \det D = d_1 \cdots d_n$$

$d_1, \ldots, d_n$ pivôs de A ($\neq 0$, pois A é invertível)

$$P = \begin{pmatrix} 0 & \cdots 0 & 1 & 0 \\ 1 & & 0 & \\ 0 & 1 & & 0 \\ & \cdots 0 & & 1 \end{pmatrix}$$

Cada linha e cada coluna contém apenas um "1" e as demais posições são "0"

$\rightarrow$ permutando linhas

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\det P = \pm \det I = \pm 1$$

onde o sinal $\pm$ depende do número de transposições ser par ou ímpar.

Substituindo em (1):

$$\pm \det A = 1 \cdot (d_1 \cdots d_n)\, 1$$

$$\therefore \det A = \pm(\text{produto dos pivôs})$$

Exemplo

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \times \tfrac{1}{2}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \times \tfrac{2}{3} \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \times \tfrac{3}{4}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \begin{array}{l} \\ \\ \times 3/4 \\ \times 4/5 \end{array} \qquad \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \times \frac{2}{3}$$

$$A = L \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}}_{=D} U \qquad (P = I)$$

$$\det A = (\cancel{2})\left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{5}{\cancel{4}}\right) = 5$$

Generalização:

$$\det \underbrace{\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & & & & 2 & -1 \\ 0 & & \cdots & & -1 & 2 \end{pmatrix}}_{n} = (\cancel{2})\left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{4}}{\cancel{3}}\right)\cdots\left(\frac{n+1}{\cancel{n}}\right) = (n+1) \quad //.$$

- POLINÔMIO NOS COEFICIENTES

$n=3$ ($n=2$ já foi feito: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$)

$$\det\begin{pmatrix} a_{11}^{+0} & a_{12} & a_{13} \\ a_{21}^{+0} & a_{22} & a_{23} \\ a_{31}^{+0} & a_{32} & a_{33} \end{pmatrix} = \det\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21}^{+0} & a_{22} & a_{23} \\ a_{31}^{+0} & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

$$= \underbrace{\det\begin{pmatrix} 0 & a_{12}^{+0} & a_{13} \\ a_{21} & a_{22}^{+0} & a_{23} \\ 0 & a_{32}^{+0} & a_{33} \end{pmatrix}}_{3\times 3\times 3 = 27 \text{ termos}} + \det\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & a_{13}^{+0} \\ a_{21} & a_{22} & a_{23}^{+0} \\ 0 & a_{32} & a_{33}^{+0} \end{pmatrix} + \dots$$

$$\det\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$$

0

Vão sobrar $\underline{6 = 3!}$ termos:

$$\det\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}_{123} + \det\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}_{132} + \det\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}_{213}$$

$$+ \det\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}_{231} + \det\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}_{312} + \det\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}_{321}$$

$$a_{11}a_{22}a_{33}\underbrace{\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}}_{=1} + a_{11}a_{23}a_{32}\underbrace{\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}}_{=-1} + a_{12}a_{21}a_{33}\underbrace{\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix}}_{=-1}$$

$$+\, a_{12}a_{23}a_{31}\underbrace{\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}}_{=+1} + a_{13}a_{21}a_{32}\underbrace{\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix}}_{=+1} + a_{13}a_{22}a_{31}\underbrace{\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}}_{=-1}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + \dots - a_{13}a_{22}a_{31}$$

$$= \sum_{\substack{(j_1,j_2,j_3) \\ \text{é uma permutação de } (1,2,3)}} a_{1j_1}a_{2j_2}a_{3j_3}\ (\text{sinal da permutação})$$

σ $\quad(\operatorname{sgn}\sigma)$

Em geral

$$\boxed{\det A = \sum_{\sigma \text{ perm de } (1\,2\,\dots\,n)} (\operatorname{sgn}\sigma)\, a_{1\sigma(1)} \dots a_{n\sigma(n)}}$$

- EXPANSÃO DE LAPLACE

$n=3$

$$\begin{vmatrix}\boxed{a_{11} \quad a_{12} \quad a_{13}}\\ a_{21} \quad a_{22} \quad a_{23}\\ a_{31} \quad a_{32} \quad a_{33}\end{vmatrix} =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - \underline{a_{12}}a_{21}a_{33}$$

$$+\underline{a_{12}}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \underline{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \underline{a_{12}} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \underline{a_{13}} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \sum a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

C_{ij} cofator (i,j) de A

M_{12} $\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} : menor (i,j) ; é o determinante da sub-matriz obtida de A eliminando-se a i-ésima linha e a j-ésima coluna.

Em geral:

$$\boxed{\det A = \sum_{j=1}^{n} a_{ij} C_{ij}}$$ onde

$C_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} : menor (i,j) de A

Exemplo:

$$A_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\det A = 2\, C_{11} + (-1)\, C_{12}$$

$$= 2\,(-1)^{1+1} M_{11} - (-1)^{1+2} M_{12}$$

$$= 2 M_{11} + M_{12}$$

$$= 2 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + \left((-1)(-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \right)$$

$$= 2 \underbrace{\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}_{\det A_3} - \underbrace{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}_{\det A_2}$$

$$\det A_4 = 2 \det A_3 - \det A_2$$

$$A_n = \underbrace{\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & & & & 2 & -1 \\ 0 & & \cdots & & -1 & 2 \end{pmatrix}}_{n}$$

$$\boxed{\det A_n = 2 \det A_{n-1} - \det A_{n-2}}$$

Fórmula recursiva

$$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$(n+1) = 2\left((n-1)+1\right) - \left((n-2)+1\right)$$

$$= 2n - (n-1)$$

$$= n+1 \checkmark$$

§4.3, Ex 34

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \text{mas} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |B||C|$$

?

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & 6 \\ \hline 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = A, \quad \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = B, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 7 & 8 \\ 0 & 1 \end{pmatrix} = D$$

(b) Contra-exemplo para o 2.º caso:

$$\left|\begin{array}{cc|c} 1 & 0 & \cdot \\ 0 & 1 & \cdot \\ \hline \cdot & \cdot & \cdot \end{array}\right|$$ não é quadrado

$$\left|\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right| = 0$$

2.ª e 4.ª linhas são iguais!

$$\underbrace{\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}}_{=2} \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1} - \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1} \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1} = 2 \cdot 1 - 1 \cdot 1 = 1 \neq 0$$

(a)
$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & \\ \vdots & \vdots & & \vdots & B \\ a_{m1} & a_{m2} & \cdots & a_{mm} & \\ \hline 0 & 0 & & 0 & \\ \vdots & \vdots & & \vdots & D \\ 0 & 0 & & 0 & \end{array}\right)$$

Expansão de Laplace
pela 1.ª coluna

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$$

$$M_{11} = \det \left(\begin{array}{c|c} \begin{matrix} a_{21} \cdots a_{2m} \\ \vdots \\ a_{m2} \cdots a_{mm} \end{matrix} & B \\ \hline 0 & D \end{array} \right)$$

Por indução $M_{11} = M_{11}^{A} \cdot |D|$

$$\det M = a_{11} C_{11} + \cdots + a_{m1} C_{m1}$$

$$= a_{11}(-1)^{1+1} M_{11} + \cdots + a_{m1}(-1)^{m+1} M_{m1}$$

$$= a_{11} \underbrace{(-1)^{1+1} M_{11}^{A}} |D| + \cdots + a_{m1} \underbrace{(-1)^{m+1} M_{m1}^{A}} |D|$$

$$= (a_{11} C_{11}^{A} + \cdots + a_{m1} C_{m1}^{A}) |D|$$

$$= |A| |D| \quad //.$$

(c) $\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq \det(AD - BC)$

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