Ângulos e Projeções

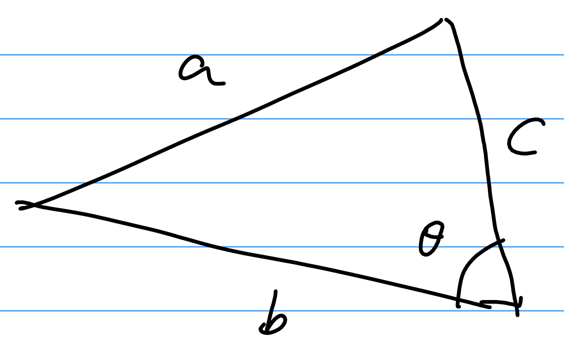
08/06/21

(§3.2)

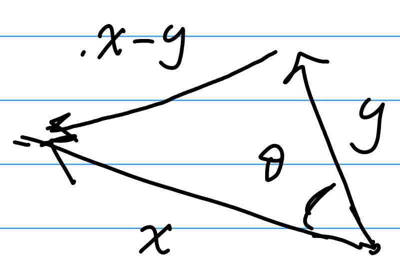
$\mathbb{R}^n \qquad x, y \in \mathbb{R}^n$

$x^t y = 0 \iff x \perp y$

<u>Lei dos cossenos</u>



$$a^2 = b^2 + c^2 - 2bc\cos\theta$$



$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

$$\|x-y\|^2 = (x-y)^t (x-y)$$
$$= (x^t - y^t)(x-y)$$
$$= x^t x - x^t y - y^t x + y^t y$$
$$= \|x\|^2 - 2x^t y + \|y\|^2 \qquad (2)$$

$$\left[\; x^t y = (x^t y)^t = y^t x \;\right]$$

Substituindo (2) em (1) :

$$\cancel{\|x\|^2} \cancel{-} \cancel{2} x^t y + \cancel{\|y\|^2} = \cancel{\|x\|^2} + \cancel{\|y\|^2} \cancel{-} \cancel{2} \|x\| \|y\| \cos\theta$$

$$\boxed{\underbrace{x^t y}_{} = \|x\| \|y\| \cos\theta} \qquad \theta = \measuredangle(x,y)$$

produto escalar de x com y

Suponhamos que $x, y \neq 0$. Então $\|x\|, \|y\| \neq 0$

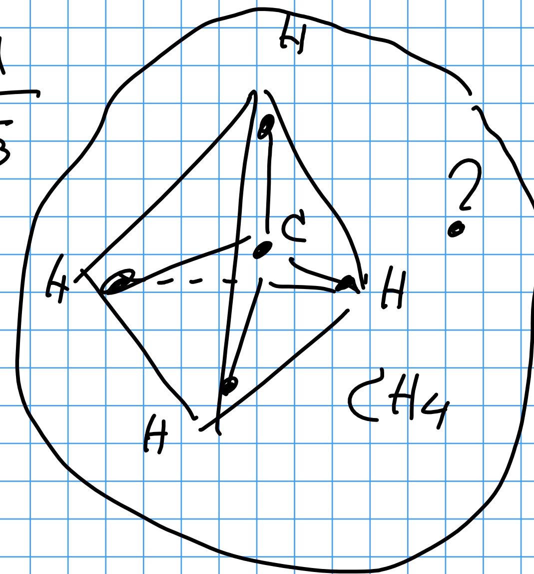
e

$$\boxed{\cos \measuredangle(x,y) = \frac{x^t y}{\|x\| \, \|y\|}}$$

Ex. $x = (1,1,1) \qquad y = (1,0,0)$

$\theta = \measuredangle(x,y)$

$$\cos\theta = \frac{1.1+1.0+1.0}{\sqrt{1+1+1}\,\sqrt{1+0+0}} = \frac{1}{\sqrt{3}}$$



$$\therefore\ \theta = \arccos 1/\sqrt{3}$$

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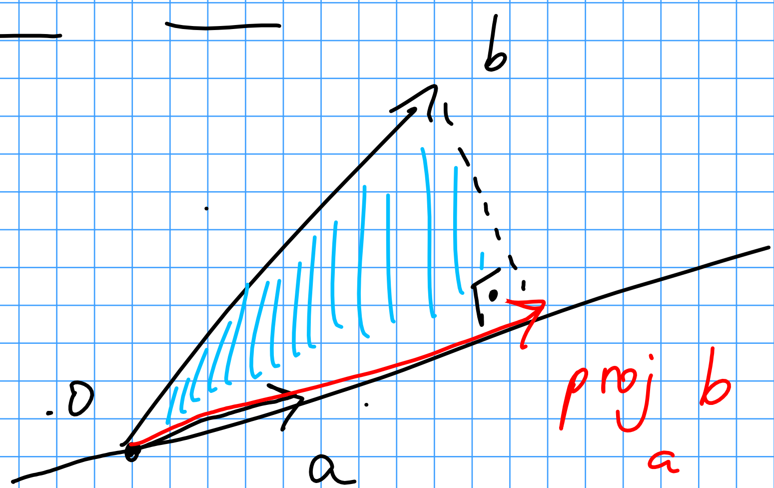
Projeção ortogonal sobre uma reta



$$\operatorname{proj}_a b =: c$$

Dados $a, b \in \mathbb{R}^n$, $a \neq 0$, calcular c.

- $c = \lambda a$ para algum $\lambda \in \mathbb{R}$
- $c - b \perp a$

$$a^t(c-b) = 0 \Rightarrow a^t \underbrace{c}_{=\lambda a} = a^t b$$

$$\Rightarrow \lambda \underbrace{(a^t a)}_{=\|a\|^2 \neq 0} = a^t b \Rightarrow \lambda = \frac{a^t b}{\|a\|^2}$$

$$\therefore \boxed{\operatorname{proj}_a b = \frac{a^t b}{\|a\|^2} a}$$

Ex Calcular $\operatorname{proj}_a b$ onde $b = (1,2,3)$ e $a = (1,1,1)$.

Resolução: $\operatorname{proj}_a b = \dfrac{(1\,1\,1)\begin{pmatrix}1\\2\\3\end{pmatrix}}{1+1+1}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$= \frac{6}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix} //$$

—//—

Em geral: $a \neq 0$

$$\operatorname{proj}_a x = \frac{a^t x}{\|a\|^2} a \quad \in \mathbb{R}$$

$$= a \frac{a^t x}{a^t a}$$

$$= \frac{a a^t}{a^t a} x$$

matriz simétrica $n \times n$

$\mathbb{R} \ni$

<u>Ex.</u> $a = (1,1,1)$

$$a^t a = (1\ 1\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix} = 3$$

$$a a^t = \begin{pmatrix}1\\1\\1\end{pmatrix}(1\ 1\ 1) = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

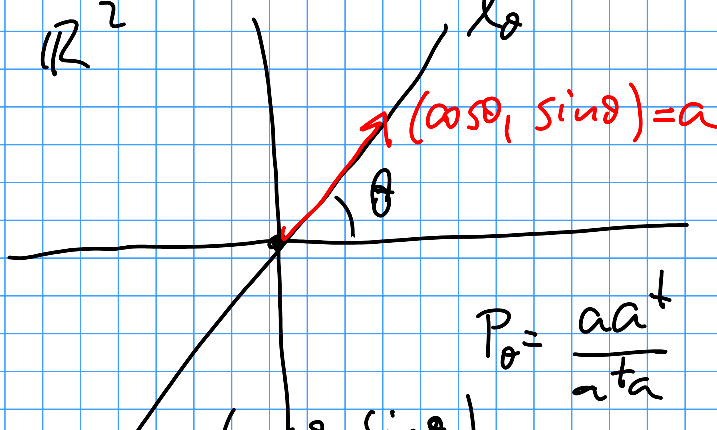
$$\mathrm{proj}_a x = \frac{\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}}{3} x = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}x_1+x_2+x_3\\x_1+x_2+x_3\\x_1+x_2+x_3\end{pmatrix}$$

$P = \mathrm{proj}_a : \mathbb{R}^3 \to \mathbb{R}^3$ transformação linear

representada pela matriz $\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ ←

<u>Ex</u> $\mathbb{R}^2$



P_θ: proj ortogonal sobre a reta l_θ

$$P_\theta = \frac{aa^t}{a^t a}$$

matriz d

$$(P_\theta) = \frac{\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}(\cos\theta\ \sin\theta)}{(\cos\theta\ \sin\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}} = \frac{\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}}{\cos^2\theta + \sin^2\theta}$$

$$\therefore\ (P_\theta) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

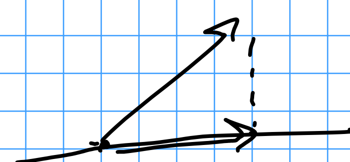
Em part: $\theta = 0$ proj ortog sobre eixo x

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

—n—

Propriedades da projeção ortogonal sobre uma reta



- $P^2 x = P(Px) = Px$

$$(P) = \frac{aa^t}{a^ta} \qquad a = \begin{pmatrix} \vdots \end{pmatrix}$$

$$(P)^2 = \left(\frac{aa^t}{a^ta}\right)\left(\frac{aa^t}{a^ta}\right) = \frac{a(a^ta)\,a^t}{(a^ta)\,(a^ta)}$$

$$= \frac{\cancel{(a^ta)}\,(aa^t)}{\cancel{(a^ta)}\,(a^ta)} = \frac{aa^t}{a^ta} = (P)$$

$$\therefore\ P^2 = P \quad (P \text{ é idempotente})$$

$$N(P) = \,?$$

$$Px = 0 \iff x \perp a$$

$$\therefore\ N(P) = a^{\perp}$$

- (P) é simétrica

$$(P) = \frac{aa^t}{a^t a}$$

$$(P)^t = \frac{1}{a^t a}\,(aa^t)^t = \frac{1}{a^t a}\,\underbrace{(a^t)^t}\,a^t$$

$$= \frac{1}{a^t a}\,aa^t = (P)$$

—n—

$$(aa^t)(aa^t) = \underset{n\times1}{(\ )}\ \underset{1\times n}{(\ )}\ \underset{n\times1}{(\ )}\ \underset{1\times n}{(\ )} \qquad n\times n$$

$$a(a^t a)a^t = \underset{n\times1}{(\ )}\ \underset{1\times1}{(\ )}\ \underset{1\times n}{(\ )}$$

$$= \underset{1\times1}{(\ )}\ \underset{n\times1}{(\ )}\ \underset{1\times n}{(\ )}$$

§ 3.3 Projeções e método dos mínimos quadrados

Ceres Gauss

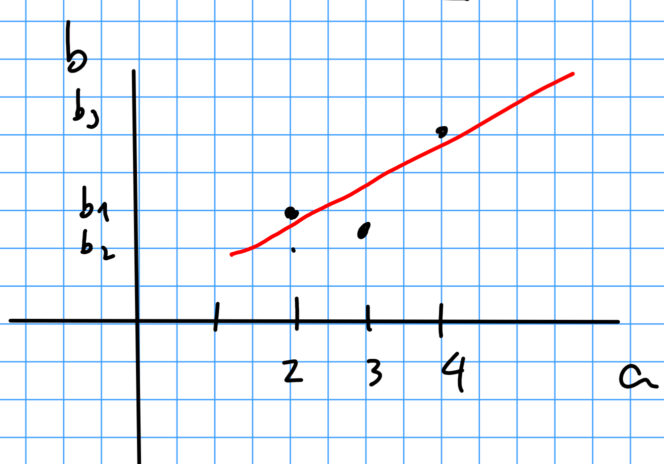
parâmetro x a ser determinado

valor de uma grandeza a	medições de b
2	b_1
3	b_2
4	b_3

Teoria diz que b é proporcional a $\underline{a}$

$$\boxed{b = a x}$$

$$x = \underbrace{\frac{b_1}{2}} \text{ ou } \underline{\frac{b_2}{3}} \text{ ou } \underline{\frac{b_3}{4}}$$



Sistema de eqs. lineares

$$\begin{cases} 2x = b_1 \\ 3x = b_2 \\ 4x = b_3 \end{cases}$$

3 eqs., 1 incógnita

$\underline{\underline{\text{sobredeterminado}}}$

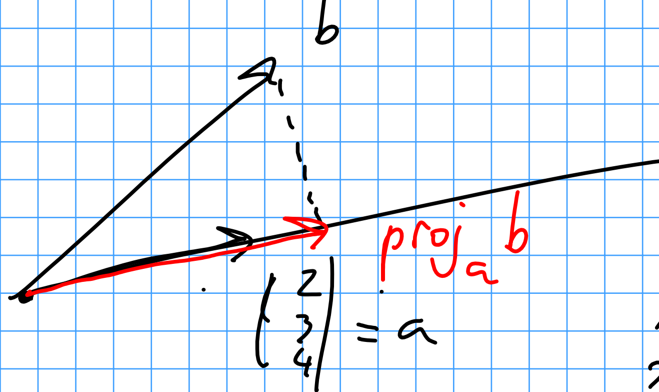
Uma solução exata só existe quando

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in C(A)$$

$$A = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

ou seja, quando b é um múltiplo de a

ou b_1, b_2, b_3 estão na proporção $2:3:4$.



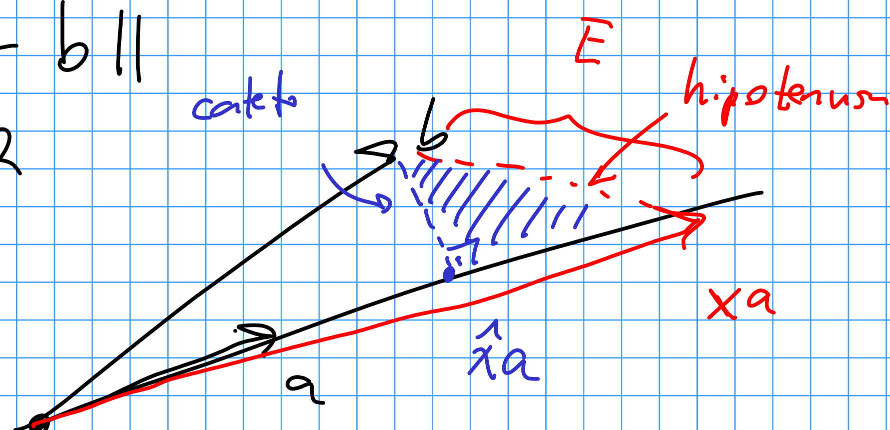
$\text{proj}_a b = \hat{x}\, a$

$\hat{x}$ é uma <u>solução</u> <u>aproximada</u> para o nosso problema.

<u>Erro:</u>

$E = \|xa - b\|$

$x \in \mathbb{R}$



Minimizar E é o mesmo que minimizar E^2.

Isso ocorre quando

$$\hat{x}.a = \text{proj}_a b = \frac{a^t b}{a^t a}.a ,$$

ou seja, $\boxed{\hat{x} = \frac{a^t b}{a^t a}}$

Ex

a	b
2	4.5 = 9/2
3	5.5 = 11/2
4	7.5 = 15/2

$$\begin{cases} 2x = \frac{9}{2} \\ 3x = \frac{11}{2} \\ 4x = \frac{15}{2} \end{cases}$$

$x = 9/4$? 2.25

$x = 11/6$? 1.83...

$x = 15/8$? 1.875

$$\hat{x} = \frac{a^t b}{a^t a} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{a_1^2 + a_2^2 + a_3^2}$$

$$= \frac{2\,\frac{9}{2} + 3\,\frac{11}{2} + 4\,\frac{15}{2}}{2^2 + 3^2 + 4^2} = \frac{9 + \frac{33}{2} + 30}{4 + 9 + 16}$$

$$= \frac{\frac{78}{2} + \frac{33}{2}}{29} = \frac{111}{58}$$ 1,91...

Ex

$$b = C + Dt \qquad C,\ D = ?$$

t_1	b_1
t_2	b_2
$\vdots$	$\vdots$
t_m	b_m

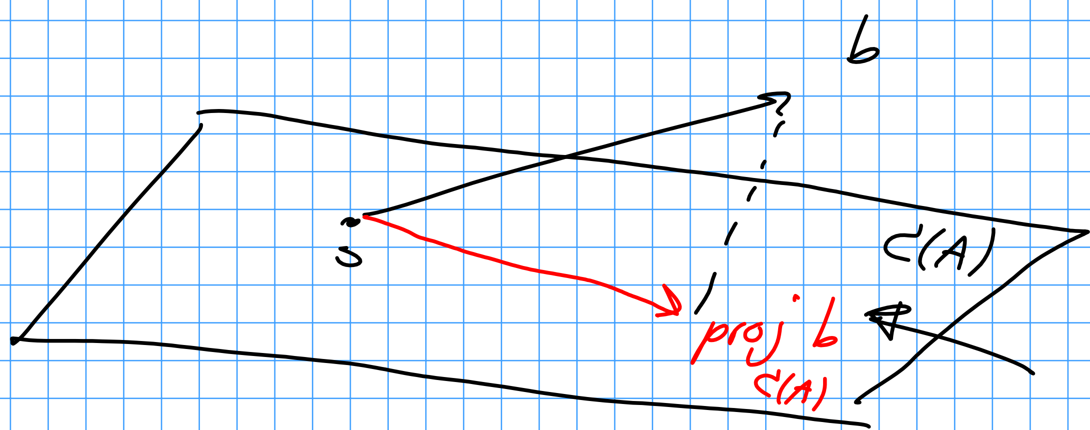
$$C + D t_1 = b_1$$
$$\vdots$$
$$C + D t_m = b_m$$

$$\begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\underset{\vec{x}}{x} = b$$

$A \in M(m \times 2, \mathbb{R})$

$Ax = b$ é solúvel $\Leftrightarrow$ $b \in C(A)$

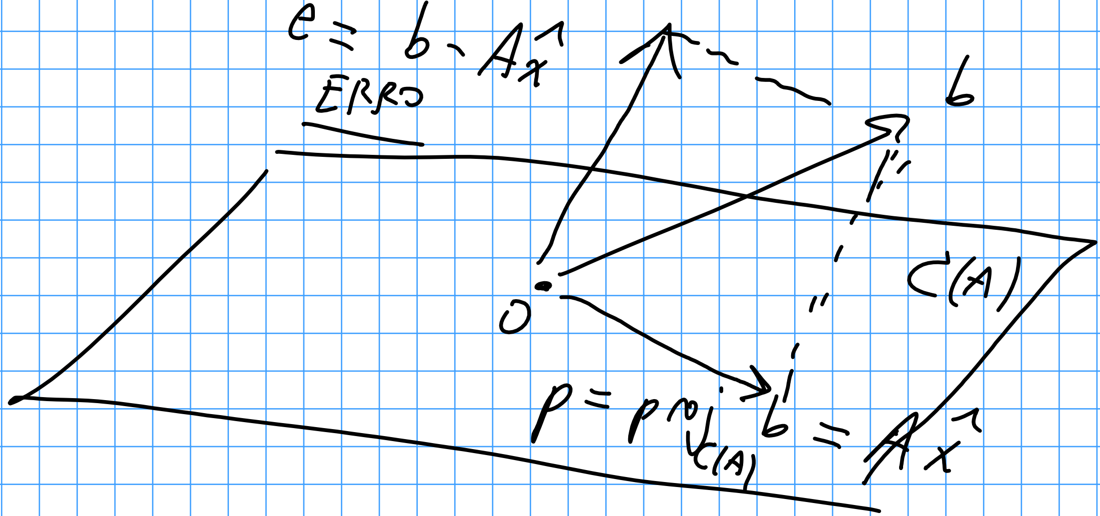


Existe $\hat{x}$ t.q. $A\hat{x} = \text{proj}_{C(A)} b$

$$\hat{x} = \begin{pmatrix} \hat{C} \\ \hat{D} \end{pmatrix}$$

<u>Projeção</u> <u>ortogonal</u> <u>sobre</u> <u>um</u> <u>subspaço</u> <u>vetorial</u>

Suponhamos que queremos projetar ortogonalmente sobre $C(A)$.



$e \perp C(A)$ ou seja $e \in C(A)^{\perp}$

$C(A)^{\perp}$ é o espaço-nulo à esquerda de A

Portanto

$$e^t A = 0 \Rightarrow A^t e = 0$$

$$\Rightarrow A^t (b - A\hat{x}) = 0$$

$$\Rightarrow A^t b = A^t A \hat{x}$$

$$\therefore \boxed{A^t A \hat{x} = A^t b}$$

Fato $A^t A$ é uma matriz inversível se e somente se as colunas de A são LI.

Neste caso:

$$\boxed{\hat{x} = (A^t A)^{-1} A^t b}$$

$$p = \operatorname{proj}_{\underline{\underline{C(A)}}} b = A\hat{x} = A(A^tA)^{-1}A^t \underline{\underline{b}}$$

$$\therefore \left(P_{C(A)}\right) = A(A^tA)^{-1}A^t$$

matriz da proj ortogonal sobre $C(A)$ (se A tem colunas L.I.)

<u>Ex.</u> $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}$ $\quad b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$$C(A) = \Big\langle \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}_{L.I.} \Big\rangle = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = 0 \right\} \quad \text{é o plano } xy$$

$$\operatorname{proj}_{C(A)} b = \operatorname{proj}_{\text{plano } xy} b = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$$

$$A^tA = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}$$

$$(A^tA)^{-1} = \begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix}$$

$$[P_{C(A)}] = A \, (A^tA)^{-1} A^t$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \end{pmatrix} \end{array}$$

$$P_{C(A)} e_1 = e_1$$

$$P_{C(A)} e_2 = e_2$$

$$P_{C(A)} e_3 = 0$$

$$p = P_{C(A)} b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$$

$$\hat{x} = (A^tA)^{-1} A^t b = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$