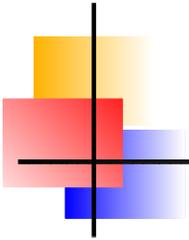


Taut representations of compact simple Lie groups

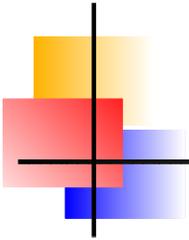
Claudio Gorodski

<http://www.ime.usp.br/~gorodski/>
gorodski@ime.usp.br

II Encuentro de Geometría Diferencial
6 al 11 de junio 2005
La Falda, Sierras de Córdoba
ARGENTINA

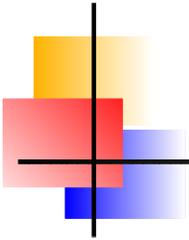


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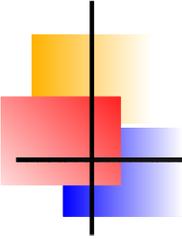
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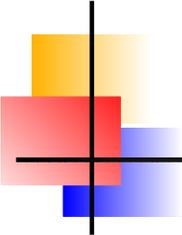
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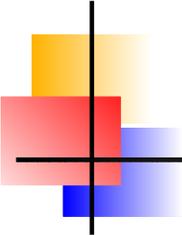
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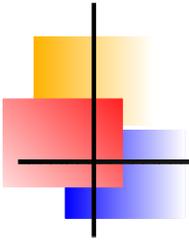
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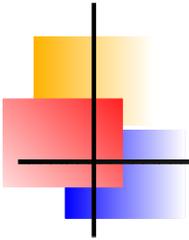
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- Since

$$\chi = b_0 - b_1 + b_2 = \mu_0 - \mu_1 + \mu_2,$$

this condition in fact implies that $\mu_i = b_i$ for all i (use \mathbf{Z}_2 coefficients)

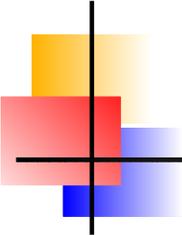


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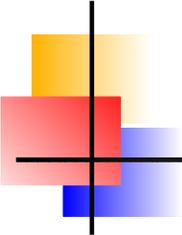
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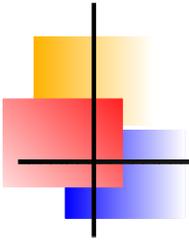


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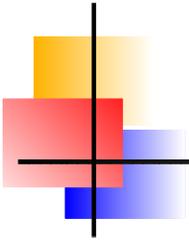
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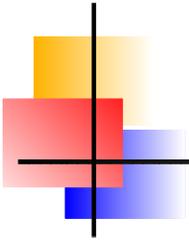


Taut submanifolds



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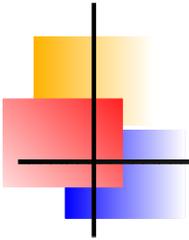
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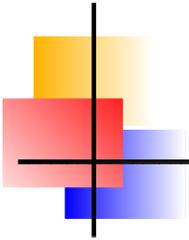
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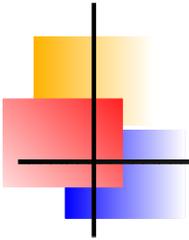
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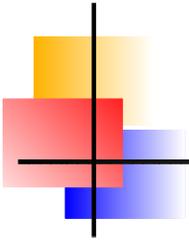
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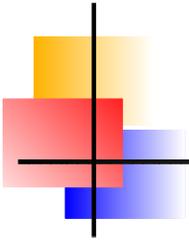


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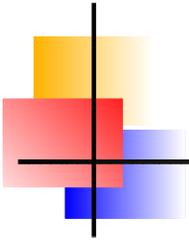
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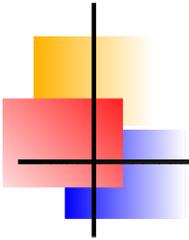


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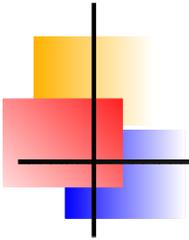
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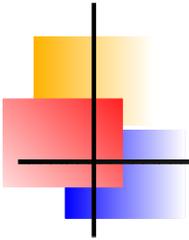
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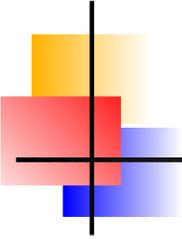
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There is no complete geometrical classification (not even of the possible substantial codimensions).

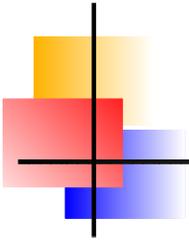


Other results



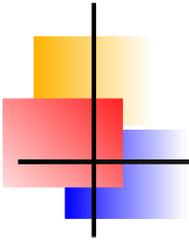
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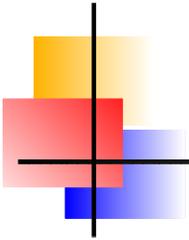
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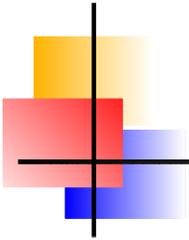
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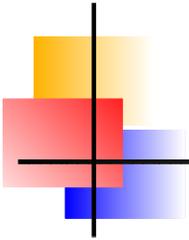
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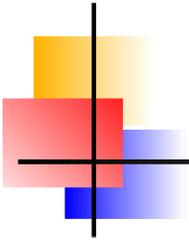


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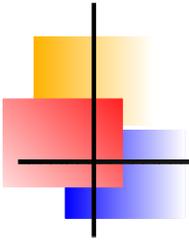


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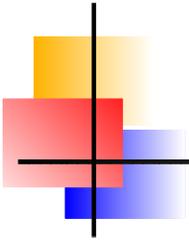
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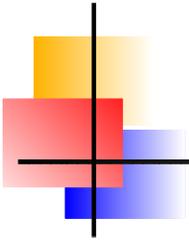


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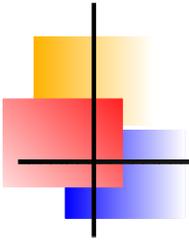
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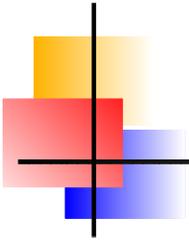
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Also: the irreducible representations of *copolarity* 1 (G., Olmos and Tojeiro, TAMS 2004).



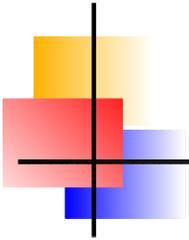
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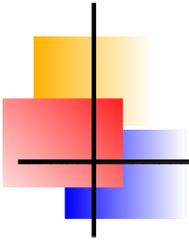
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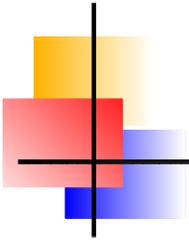
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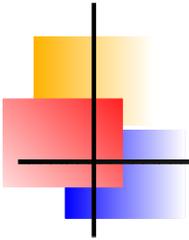
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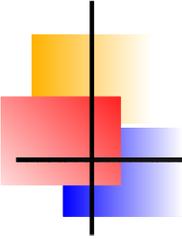
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Taut reducible representations, cont'd

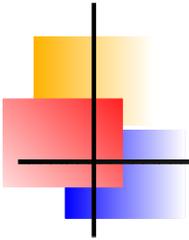
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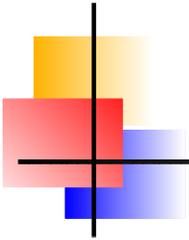
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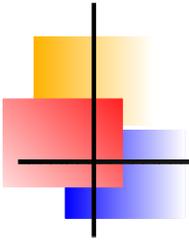


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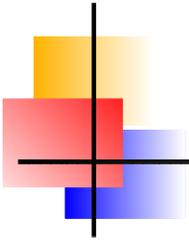
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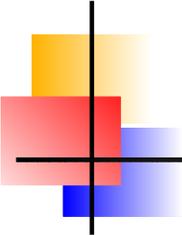
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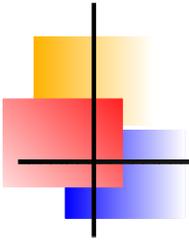
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- In fact, every irreducible summand of a taut reducible representation is the isotropy representation of a symmetric space.



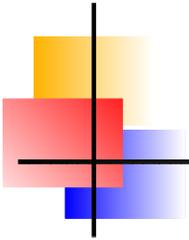
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- Case by case analysis: for each compact simple Lie group, we first discard a number of cases, and then prove that the remaining cases are taut.
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- In fact, every irreducible summand of a taut reducible representation is the isotropy representation of a symmetric space.
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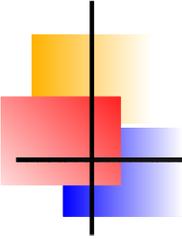


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- There are three important results which are used in the classification.



Fundamental result about taut sums



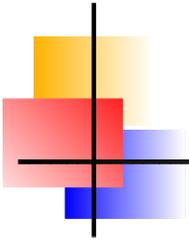
Fundamental result about taut sums

Theorem (G. and Thorbergsson, 2000)

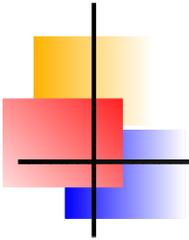
Let ρ_1 and ρ_2 be representations of a compact connected Lie group G on V_1 and V_2 , respectively. Assume that $\rho_1 \oplus \rho_2$ is F -taut. Then the restriction of ρ_2 to the isotropy group G_{v_1} is F -taut for every $v_1 \in V_1$. Furthermore, we have that

$$p(G(v_1, v_2); F) = p(Gv_1; F) p(G_{v_1} v_2; F),$$

where $p(M; F)$ denotes the Poincaré polynomial of M with respect to the field F . In particular, $G_{v_1} v_2$ is connected and $b_1(G(v_1, v_2); F) = b_1(Gv_1; F) + b_1(G_{v_1} v_2; F)$, where $b_1(M; F)$ denotes the first Betti number of M with respect to F .

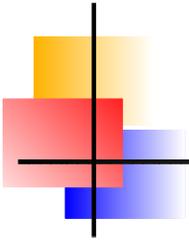


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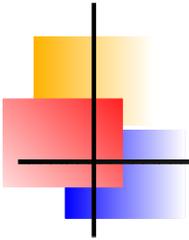
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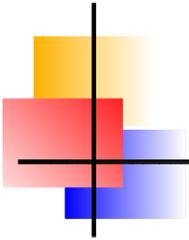
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- The adjoint representation of a Lie group of rank greater than one. (Use b_1 .)

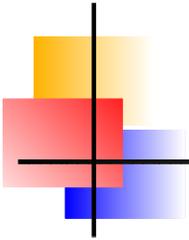


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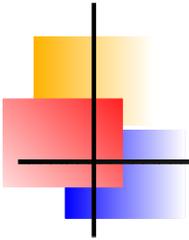
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Arguments involving b_3 are also useful.

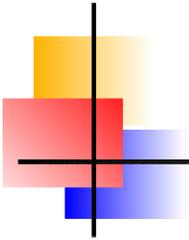


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Recall that the **slice representation** of a representation $\rho : G \rightarrow \mathbf{O}(V)$ at a point $p \in V$ is the representation induced by the isotropy G_p on the normal space to the orbit Gp at p .

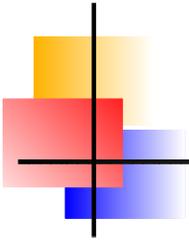


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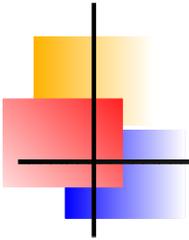
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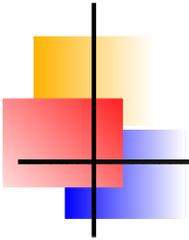
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E.g: This result is used to reduce the proof of the nontautness of $\mathbf{SU}(n)$ acting on $\mathbf{C}^n \oplus \dots \oplus \mathbf{C}^n$ (n summands) to the case $n = 3$.

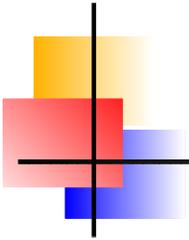


Reduction principle, I



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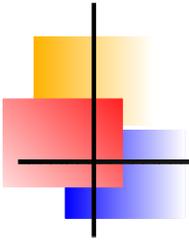
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Moreover, the following result is known.



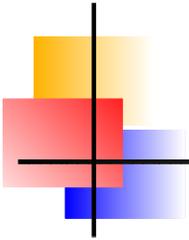
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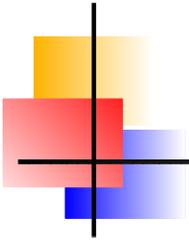
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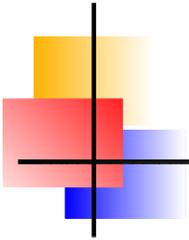
The relation to tautness is expressed by the following result.



Reduction principle, II

Theorem (G. and Thorbergsson 2000)

Suppose there is a subgroup $L \subset H$ which is a finitely iterated \mathbf{Z}_2 -extension of the identity and such that the fixed point sets $V^L = V^H$. Suppose also that the reduced representation $\bar{\rho} : \bar{N}^0 \rightarrow \mathbf{O}(V^H)$ is \mathbf{Z}_2 -taut, where \bar{N}^0 denotes the connected component of the identity of \bar{N} . Then $\rho : G \rightarrow \mathbf{O}(V)$ is \mathbf{Z}_2 -taut.

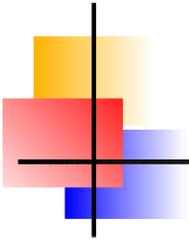


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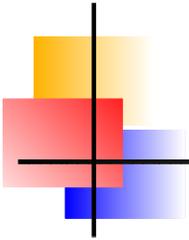
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In some cases, the reduction principle can also be used to prove that certain representations are *not* taut.

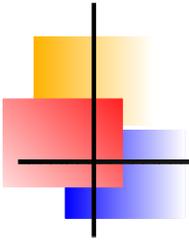


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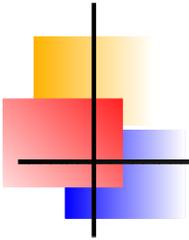
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- Orbits of orthogonal representations are contained in round spheres, so the set of critical points of a distance function also occurs as the set of critical points of a height function (*tightness*).

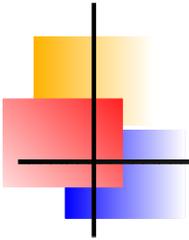


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- Ozawa proved that the set of critical points of a distance function to a taut submanifold decomposes into critical submanifolds which are nondegenerate in the sense of Bott; it follows that the number of critical points of the function equals the sum of the Betti numbers of the critical submanifolds.

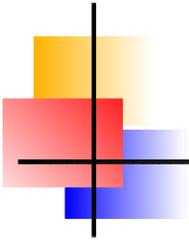


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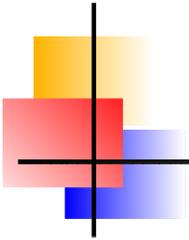
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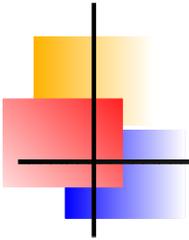
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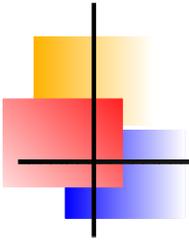
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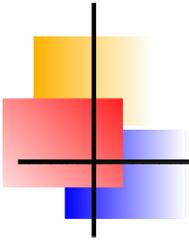
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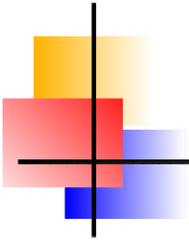
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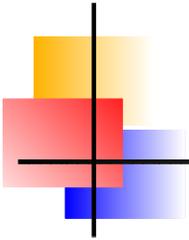
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- For certain choices of M and h , the sum of the Betti numbers of critical set of $h|_{M^H}$ is 12. Hence, M is not taut.

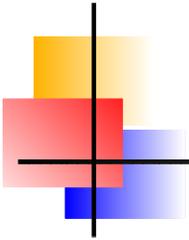


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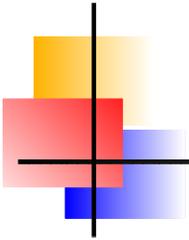
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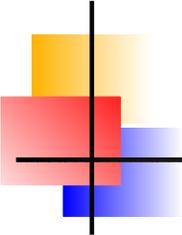
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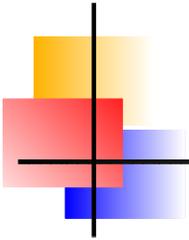
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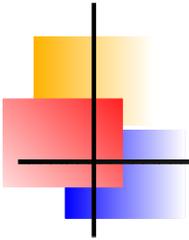
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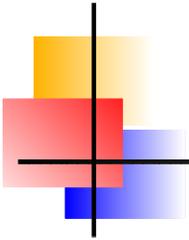
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- The tangent directions of C are principal directions of M . Hence, M is not taut by a result of Pinkall.

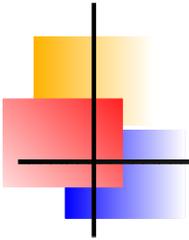


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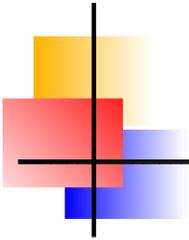
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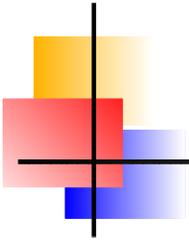
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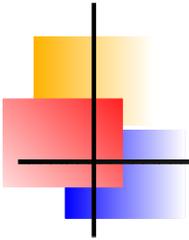
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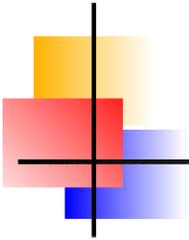
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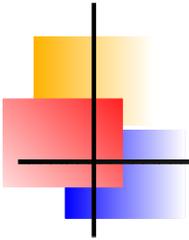
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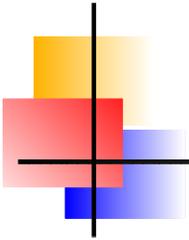
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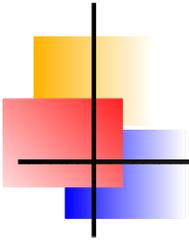
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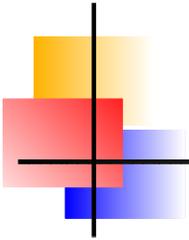
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- As above, $G(qM) = (Gq)M$ is taut.



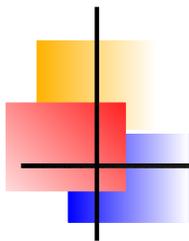
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- As above, $G(qM) = (Gq)M$ is taut.
- A taut submanifold in Euclidean space is tight, and tightness is invariant under linear transformations, so Gq is tight.



Examples, IV

- Suppose now k arbitrary; let $q = (v_1, \dots, v_k) \in V$ be nonzero.
- There exists a nonsingular $k \times k$ matrix M such that $qM = (e_1, \dots, e_l, 0, \dots, 0) \in V$, where $1 \leq l \leq n$.
- As above, $G(qM) = (Gq)M$ is taut.
- A taut submanifold in Euclidean space is tight, and tightness is invariant under linear transformations, so Gq is tight.
- Gq lies in a sphere, and so it is taut.



Thank you!