Taut representations of compact simple Lie groups

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II Encuentro de Geometría Diferencial

6 al 11 de junio 2005

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B R A S I L

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- Since

$$\chi = b_0 - b_1 + b_2 = \mu_0 - \mu_1 + \mu_2,$$

  this condition in fact implies that $\mu_i = b_i$ for all i (use $\mathbf{Z}_2$ coefficients).

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- A compact surface substantially embedded in S^m with the STPP is a round sphere or a cyclide of Dupin in S^3, or the Veronese embedding of a projective plane in S^4. (Banchoff, 1970)
- The STPP condition is also equivalent to requiring that the induced homomorphism

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in Čech homology is injective for every closed ball B.

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- Tautness is equivalent to the STPP for surfaces.

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There is no complete geometrical classification (not even of the possible substantial codimensions).

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Also: the irreducible representations of *copolarity* 1 (G., Olmos and Tojeiro, TAMS 2004).

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$$\mathbf{Spin}(7) : \begin{cases} \mathbf{R}^7 \oplus \mathbf{R}^8 \\ \mathbf{R}^8 \oplus \mathbf{R}^8 \\ \mathbf{R}^8 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8 \\ \mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^8 \end{cases}$$

Taut reducible representations, cont'd

$$\mathbf{Spin}(8): \begin{cases} \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \\ \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \\ \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \end{cases}$$

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$$\mathbf{Spin}(9) : \mathbf{R}^{16} \oplus \mathbf{R}^{16}$$

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- Since every irreducible summand of a taut reducible representation is taut, we need to decide which sums are allowed.
- In fact, every irreducible summand of a taut reducible representation is the isotropy representation of a symmetric space.
- Each compact simple Lie group admits few such representations (spin groups are a case apart).
- There are three important results which are used in the classification.

Fundamental result about taut sums

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Theorem (G. and Thorbergsson, 2000)
Let ρ_1 and ρ_2 be representations of a compact connected Lie group G on V_1 and V_2, respectively. Assume that $\rho_1 \oplus \rho_2$ is F-taut. Then the restriction of ρ_2 to the isotropy group G_{v_1} is F-taut for every $v_1 \in V_1$. Furthermore, we have that

$$p(G(v_1, v_2); F) = p(Gv_1; F)\, p(G_{v_1}v_2; F),$$

where $p(M; F)$ denotes the Poincaré polynomial of M with respect to the field F. In particular, $G_{v_1}v_2$ is connected and $b_1(G(v_1, v_2); F) = b_1(Gv_1; F) + b_1(G_{v_1}v_2; F)$, where $b_1(M; F)$ denotes the first Betti number of M with respect to F.

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Arguments involving b_3 are also useful.

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The slice representation of a taut representation at any point is taut.

E.g: This result is used to reduce the proof of the nontautness of $\mathbf{SU}(n)$ acting on $\mathbf{C}^n \oplus \cdots \oplus \mathbf{C}^n$ (n summands) to the case $n = 3$.

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Moreover, the following result is known.

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The relation to tautness is expressed by the following result.

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Suppose there is a subgroup $L \subset H$ which is a finitely iterated $\mathbf{Z}_2$-extension of the identity and such that the fixed point sets $V^L = V^H$. Suppose also that the reduced representation $\bar{\rho} : \bar{N}^0 \to \mathbf{O}(V^H)$ is $\mathbf{Z}_2$-taut, where $\bar{N}^0$ denotes the connected component of the identity of $\bar{N}$. Then $\rho : G \to \mathbf{O}(V)$ is $\mathbf{Z}_2$-taut.

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In some cases, the reduction principle can also be used to to prove that certain representations are *not* taut.

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- Ozawa proved that the set of critical points of a distance function to a taut submanifold decomposes into critical submanifolds which are nondegenerate in the sense of Bott; it follows that the number of critical points of the function equals the sum of the Betti numbers of the critical submanifolds.

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- In this case: If M is principal and taut, it has the homology of $S^5 \times S^6 \times S^7$ by the fundamental result.

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- The reduced representation is a certain representation of $\mathbf{SO}(3) \times \mathbf{SO}(4) \times \mathbf{SO}(4)$ on $\mathbf{R}^{11}$.
- For certain choices of M and h, the sum of the Betti numbers of critical set of $h|M^H$ is 12. Hence, M is not taut.

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- M and M^H have the same normal spaces along M^H, and the Weingarten operator of M with respect to a normal vector restricts to the Weingarten operator of M^H.
- The tangent directions of C are principal directions of M. Hence, M is not taut by a result of Pinkall.

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- Suppose first $k \leq n$; set: $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$, $p = (e_1, \ldots, e_k) \in V$.
- View $V = \mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n \cong \mathbf{R}^n \otimes \mathbf{R}^k$.

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- Suppose first $k \leq n$; set: $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$, $p = (e_1, \ldots, e_k) \in V$.
- View $V = \mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n \cong \mathbf{R}^n \otimes \mathbf{R}^k$.
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- Let $\hat{G} = \mathbf{SO}(n) \times \mathbf{SO}(k)$ act on V.
- $\hat{G}p = Gp$ and $(\hat{G}, V)$ is the isotropy representation of $G_k(\mathbf{R}^{n+k})$; hence Gp is taut.

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- A taut submanifold in Euclidean space is tight, and tightness is invariant under linear transformations, so Gq is tight.
- Gq lies in a sphere, and so it is taut.

Thank you!